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ON (N, P_n) AND $(K, 1, \alpha)$ SUMMABILITY METHODS

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1.1 Let $\Sigma a_m{}^1$ be a given infinite series with the sequence of partial sums $\{s_n\}$. The Cesàro transform of order α of Σa_n is defined by

$$s_n^\alpha = S_n^\alpha / A_n^\alpha, \qquad \alpha > -1, \tag{1.1.1}$$

where S_n^α and A_n^α are by the relations;

$$S_n^\alpha = \sum_{v=0}^{n} A_{n-v}^{n} a_v = \sum_{v=0}^{n} A_{n-v}^{\alpha-1} S_v;$$

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}, \qquad (|x| < 1). \tag{1.1.2}$$

The series Σa_n is said to be summable (C, α) to s, if $s_n^\alpha \to s$, as $n \to \infty$, [2].

The series Σa_n an is said to be summable $(K, 1, \alpha)$ to sum s, [5] if the series

$$f(\alpha, t) = B_\alpha^{-1}\ t^{\alpha+1} \sum_{n=1}^{\infty} S_n^\alpha \int_t^\pi \frac{\sin nx}{2 \tan x/2} dx \tag{1.1.3}$$

converges in some interval $0 < t < t_0$ and $\lim_{t\to+0} f(\alpha, t) = s$, where

$$B_\alpha = \begin{cases} \pi/2 & \alpha = -1 \\ (\alpha+1)^{-1} \sin(\alpha+1)\pi/2 & -1 < \alpha < 0 \\ 1 & \alpha = 0 \end{cases}$$

where $\alpha = -1$, the method $(K, 1, \alpha)$ reduces to the method $(K, 1)$ [11].

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[1] unless or otherwise stated Σ denotes Σ_0^∞.

The method $(K, 1, \alpha)$ is not regular when $-1 \leq \alpha \leq 0$ [5].

Let $\{p_n\}$ be a sequence of constants, real or complex, such that

$$P_n = \sum_{v=0}^{n} P_v \neq 0, \quad P_{-1} = p_{-1} = 0,$$

and let us write

$$t_n = \frac{T_n}{P_n} = \sum_{v=0}^{n} \frac{P_{n-v} s_v}{P_n}. \tag{1.1.4}$$

The series Σa_n is said to be summable (N, P_n) to sum s, if $\lim_{u \to \infty} t_n$ exists and is equal to s ([7], [10]).

In the special cases in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha > -1); \tag{1.1.5}$$

$$\begin{cases} p_n = (n+1)^{-1} & (\alpha > -1); \\ p_n \log n, & \text{as } n \to \infty, \end{cases} \tag{1.1.6}$$

The (N, p_n) summability reduces to (C, α) summability, $\alpha > -1$, [3] § 5.13 and harmonic summability methods [3], § 5.13 respectively.

The conditions for the regularity of the method of summation (N, p_n) defined by (1.1.4), are

$$\lim_{n \to \infty} \frac{p_n}{P_n} = 0, \tag{1.1.7}$$

and

$$\sum_{v=0}^{n} |p_v| = O(p_n), \text{ as } n \to \infty, \quad \text{(see [3])}. \tag{1.1.8}$$

If p_n is real, non-negative and monotonic non-increasing, the conditions of regularity (1.1.7) and (1.1.8) are autcmatically satisfied and the method (N, p_n) is then regular and hence the harmonic summability method is also regular. It is known that summability $(N, {}^1/_{(n+1)})$ implies summability (C, α) for every $\alpha > 0$.

1.2. We set

$$(\Sigma p_n x_n)^{-1} = \Sigma c_n x^n \qquad (|x| < 1;\ C_0 = 1) \tag{1.2.1}$$

Then from (1.1.4) and (1.2.1), we get

$$s_n = \sum_{v=1}^{n} c_{n-v} T_v \tag{1.2.2}$$

$$a_n = \sum_{v=1}^{n} c_{n-v}(T_v - T_{v-1}). \tag{1.2.3}$$

In what follows we take $a_0 = 0$, so that $T_0 = 0$.

2.1. Giving relation between (N,p_n) and $(R,1,\alpha)$ summabilties recently the authors [2] have proved the following theorem:

THEOREM A. *Σa_n is (N,p) summable and if*

$$\sigma_n = \sum_{k=1}^{n} |T_k - T_{k-1}| = O(P_n), \tag{2.1.1}$$

then the series Σa_n is summable $(R,1,\alpha)$ for $-1 \le \alpha \le 0$, provided that p_n is a non-negative, non-increasing sequence such that $P_n \to \infty$, and

$$\sum_{k=n+1}^{\infty} |C_k| = O\left(\frac{1}{P_n}\right), \qquad n \ge 0; \tag{2.1.2}$$

$$\sum_{k=n}^{\infty} \frac{P_{k-n}}{k(k+1)} = O\left(\frac{P_n}{n}\right), \qquad n \ge 1; \tag{2.1.3}$$

$$\sum_{k=0}^{n} \frac{1}{P_k} = O\left(\frac{n}{P_n}\right); \tag{2.1.4}$$

(2.1.5) *for a positive number μ and $n = [\mu t^{-1}], \tau = [t^{-1}]$*

$$P_n = O(P_\mu P_\tau).$$

It has been proved by Izumi [6] that for Fourier series, summability $(K,1)$ is equivalent to summability (R_1). Since it is known that for Fourier series summability $(R,1)$ and (R_1) are mutually exclusive [4], it follows that in general, summability $(K,1)$ and $(R,1)$ are also independent of each other. Therefore, the object of this paper is to show that this Theorem A also holds for summability $(K,1,\alpha)$.

2.2. Our Main theorem is:

THEOREM 1. *Let $\{p_n\}$ be a non-negative, non-increasing sequence, such that $P_n \to \infty$, and the conditions* (2.1.2) *through* (2.1.5) *hold. If Σa_n is (N, p_n)-summable and if* (2.1.1) *holds, then Σa_n is also summable $(K, 1, \alpha)$, for $-1 \le \alpha \le 0$.*

Combining Theorem 1 with Lemma 5 below, we also get the following interesting and simple result.

THEOREM 2. *Let $\{p_n\}$ be a positive, non-increasing sequence, such that $p_0 = 1$, $P_n \to \infty$ and $\{p_{n+1}/p_n\}$ is non-decreasing sequence and the conditions* (2.1.3) *through* (2.1.5) *hold. If Σa_n is (N, p_n) summable and if* (2.1.1) *holds, then Σa_n is also sumable $(K, 1, \alpha)$, for $-1 \le \alpha \le 0$.*

2.3. The following lemmas are pertinent for the proof of our theorems.

LEMMA 1. ([1], Lema 1). *If $\{p_n\}$ is a non-negative, non-increasing sequence such that the series $\sum_{v=n}^{\infty} P_{v-n}/v(v+1)$ converges, then $\frac{P_n}{n} \to 0$, as $n \to 0$.*

LEMMA 2. ([1], Lemma 2). *Let $\{p_n\}$ be a non-negative, non-increasing sequence such that, for $n \ge 1$,*

$$\sum_{v=n}^{\infty} \frac{P_{v-n}}{v(v+1)} = O\left(\frac{P_n}{n}\right). \tag{2.3.1}$$

Then for $n \ge 1$,

$$\sum_{v=n}^{\infty} \frac{P}{f(v+1)} = O\left(\frac{P_n}{n}\right). \tag{2.3.2}$$

LEMMA 3. ([1], Lemma 3). *Let $\{p_n\}$ be a non-negative, non-increasing sequence such that $\{P_n/n\}$ is a null sequence. If Σa_k is summable (N, p_n) then*

(i) $$W_n = \sum_{v=n}^{\infty} \frac{T_v - T_{v-1}}{v} = o\left(\frac{P_n}{n}\right),$$

(ii) $$W_n{}' = \sum_{v=1}^{\infty} W_v = o(P_n).$$

LEMMA 4. ([3], Theorem 22). *If $p(x) = \Sigma p_n x^n$ is convergent for $|x| < 1$ and*

$$p_0 = 1, \quad p_n > 0, \quad \frac{p_{n+1}}{p_n} \ge \frac{p_n}{p_{n-1}} \qquad (n > 0), \text{ then} \tag{2.3.3}$$

$$\{p(x)\}^{-1} = 1 + C_1 x + C_2 x^2 + \cdots \tag{2.3.4}$$

where $C_n \geq 0$, *for* $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} |C_n| \leq 1$. *If* $\Sigma p_n = \infty$, *then* $\sum_{n=1}^{\infty} |C_n| = 1$.

LEMMA 5. ([9], Lemma 2). *If* $\{p_n\}$ *is a positive and non-increasing sequence such that* $p_0 = 1$, $P_n \to \infty$, *and* $\{p_{n+1}/p_n\}$ *in non-decreasing sequence, then for* $n \geq 0$,

$$d_n = \sum_{v=n+1}^{\infty} |C_n| = \sum_{v=0}^{n} C_v = O\left(\frac{1}{P_n}\right). \tag{2.3.5}$$

REMARK. The identity

$$d_n = \sum_{v=n+1}^{\infty} |C_v| = \sum_{v=0}^{n} C_v \tag{2.3.6}$$

is obtained by virtue of the Lemma 4.

LEMMA. ([2], Lemma 9) *Let* $\{p_n\}$ *be a non-negative sequence such that* $P_n \to \infty$, *and the conditions* (2.1.2) *to* (2.1.4) *of Theorem A hold. Then* (N, p_n) – *summability of the series* Σa_n *to the sum* s *implies its* $(C, 1)$ – *summability to the same sum. In particular, if* $T_n = o(P_n)$, *then* $S_n^1 = o(n)$.

LEMMA 7. *Let* $\Phi(n, t) = \int_t^{\pi} \frac{\pi \sin nu}{2 \tan u/2} du$. *Then*

$$\Phi(n, t) = O(1/nt) \tag{2.3.7}$$

and

$$\Delta^m \Phi(n, t) = 0\left(\frac{t^{m-1}}{n}\right) \tag{2.3.8}$$

where $\Delta^m \Phi(n, t)$ *denotes the m-th difference of* $\Phi(n, t)$ *with respect to* n *and* m *is a non-negative number.*

PROOF. $\Phi(n, t) = \int_t^{\pi} \frac{\sin nu}{2 \tan u/2} du = 2(\tan t/2)^{-1} \int_t^{\xi} \sin nu \ du, \quad t < \xi < \pi$

$$= (2 \tan t/2)^{-1} - \left[-\frac{\cos nu}{n}\right]_t^{\xi} = O(1/nt).$$

Again

$$\begin{aligned}\Phi(n,t) &= \Delta\left[\int_t^\pi \frac{\sin nu}{2\tan u/2}du\right]\\ &= \int_t^\pi \frac{\sin nu - \sin(n+1)u}{2\tan u/2}du\\ &= -\int_t^\pi \frac{2\cos(n+1/2)u\sin u/2}{2\tan u/2}du,\\ &= -\frac{1}{2}\int_t^\pi \cos(n+1)u+\cos nu)du,\\ &= \frac{t}{2}\left[\frac{\sin(n+1)t}{(n+1)t}+\frac{\sin nt}{nt}\right].\end{aligned}$$

Hence

$$\begin{aligned}\Delta^m\Phi(n,t) = \Delta^{m-1}\Phi(n,t) = t/2\Delta^{m-1}\left[\frac{\sin(n+1)t}{(n+1)t}+\frac{\sin nt}{nt}\right]\\ = O\left(\frac{t^{m-1}}{n}\right),\end{aligned}$$

by virtue of

$$\Delta^m\left(\frac{\sin nt}{nt}\right)^p = O(n^{-p}t^{m-p}), \text{(see Obrechkoff [8] Lemma 1).}$$

This completes the proof.

LEMMA 8. *Let* $G_v(t) = t^{\alpha+1}\sum_{n=0}^{\infty} A_{n-v}^{\alpha-1}\Phi(n,t)$, $-1 \le \alpha \le 0$. *Then*

$$G_v(t) = O(1/v), \tag{2.3.9}$$

and for positive integer k,

$$\Delta^k G_v(t) = O\left(\frac{t^k}{v}\right). \tag{2.3.10}$$

PROOF. Let $G(t) = t_{\alpha+1}\left(\sum_{n=0}^{v+\rho}+\sum_{n=v+\rho+1}^{\infty}\right) = U_1 + U_2$, say, where $\rho = [1/t]$.

Now by (2.3.8) we have for $-1 < \alpha < 0$,

$$\begin{aligned} U_2 &= t^{\alpha+1} \sum_{n=v+\rho+1}^{\infty} A_{n-v}^{\alpha-1} \Phi(n,t) \\ &= O\left(t^{\alpha}(v+\rho+1)^{-1} \sum_{n=v+\rho+1}^{\infty} (n-v)^{\alpha-1} \right) \\ &= O(t^{\alpha} v^{-1} \rho^{\alpha}) = O(v^{-1}), \end{aligned}$$

and applying Abel's transformation to U_1 we have

$$\begin{aligned} U_1 &= t^{\alpha+1} \sum_{n=0}^{\rho} A_n^{\alpha-1} \Phi(n+v,t), \\ &= t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n^{\alpha} \Phi(n+v,t) + t^{\alpha+1} A_{\rho}^{\alpha} \Phi(\rho+v,t) \\ &= O\left(t^{\alpha+1} \sum_{n=2}^{\rho-1} A_n^{\alpha} (n+v)^{-a} \right) + O(v^{-1}) \\ &+ O(t^{\alpha+1} \rho^{\alpha+1} v^{-1}) + O(v^{-1}) = O(v^{-1}). \end{aligned}$$

Hence, $G_v(t) = O(v^{-1})$, for $-1 < \alpha < 0$. When $\alpha = 0$ $G_v(t) = t\ \Phi(v,t) = O(v^{-1})$. Similarly, we hawe the result when $\alpha = -1$.

Now

$$\begin{aligned} G_v(t) &= t^{\alpha+1} \sum_{n=v}^{\infty} A_{n-v}^{\alpha-1} \Phi(n,t) \\ &= t^{\alpha+1} \sum_{n=0}^{\infty} A_n^{\alpha-1} \Phi(n+v,t), \end{aligned}$$

hence by using the method of proof of (2.3.9), we have

$$\Delta^k G_v(t) = t^{\alpha+1} \sum_{n=0}^{\infty} A_n^{\alpha-1} \Delta^k(n+v,t) = O\left(\frac{t^k}{v}\right),$$

Hence, the Lemma.

Lemma 9. *Let $K_v(t) = \sum_{n=v}^{\infty} G_n(t)$, then*

(2.3.11) $$K_v(t) = O(v^{-1} t^{-1}).$$

PROOF. We have

$$\begin{aligned} K_v(t) = \sum_{n=v}^{\infty} G_n(t) &= t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{k=n}^{\infty} A_{k-n}^{\alpha-} \Phi(k,t) \\ &= t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{k=0}^{\infty} A_k^{\alpha-1} \Phi(n+k,t) \\ &= t^{\alpha+1} \sum_{k=0}^{\infty} A_k^{\alpha-1} \sum_{n=v}^{\infty} \Phi(n+k,t), \end{aligned}$$

the change of order of summation can be easily justified. To prove the lemma we just show that

$$\gamma_{v+k}(t) = \sum_{n=v}^{\infty} \Phi(n+k,t) = O((v+k)^{-1}t^{-2})$$

We have,

$$\begin{aligned} \sum_{n=v}^{\infty} \Phi(n+k,t) &= \sum_{n=v}^{\infty} \int_t^{\pi} \frac{\sin(n+k)x}{\tan x/2} dx \\ &= \sum_{n=v}^{\infty} \frac{1}{2\tan t/2} \int_t^{\xi} \sin(n+k)x dx, \quad t < \xi < \pi, \\ &= (2\tan t/2)^{-1} \sum_{n=v}^{\infty} \left[-\frac{\cos(n+k)x}{n+k} \right]_t^{\xi} = O\left(\frac{t^{-2}}{(v+k)} \right), \end{aligned}$$

since $\sum_{n=v}^{\infty} \frac{\cos nt}{n} = O\left(\frac{1}{nt}\right)$.

Now for $-1 < \alpha < 0$, we write

$$t^{\alpha+1} \sum_{k=0}^{\infty} A_k^{\alpha-1} \gamma_{v+k}(t) = \sum_{k=0}^{\rho} + \sum_{\rho+1}^{\infty} = V_1 + V_2, \text{ say.}$$

We have

$$\begin{aligned} V_2 &= O\left(t^{\alpha+1} \sum_{k=\rho+1}^{\infty} k^{\alpha-1}(v+k)^{-1} t^{-2} \right) \\ &= O((v+\rho+1)^{-1} t^{\alpha-1} \rho^{\alpha}) = O(v^{-1}t^{-1}), \text{ and} \\ V_1 &= t^{\alpha+1} \sum_{k=0}^{\rho-1} A_k^{\alpha} \Delta_k (\gamma_{k+v}(t) + t^{\alpha-1} A_{\rho}^{\alpha} \gamma_{\rho+v}(t)) = O(v^{-1}t^{-1}). \end{aligned}$$

Hence, (2.3.11) follows for $-1 < \alpha < 0$. The result for $\alpha = 0$ is quite obvious. This completes the proof.

LEMMA 10. *If $S_n^1 = o(n)$, then we have*

$$t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} \Phi(n,t) = \sum_{n=1}^{\infty} s_n G_n(t),$$

where

$$G_n(t) = t^{\alpha+1} \sum_{v=n}^{\infty} A_{v-n}^{\alpha-1} \Phi(v,t) \qquad (-1 < \alpha \le 0).$$

PROOF. We have

$$\begin{aligned} t^{\alpha+1} \sum_{n=1}^{\infty} S_n^{\alpha} \Phi(n,t) &= t^{\alpha+1} \sum_{n+1}^{\infty} \Phi(n,t) \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k, \\ &= t^{\alpha+1} \sum_{k=1}^{\infty} s_k \sum_{n=k}^{\infty} A_{n-1}^{\alpha-1} \Phi(n,t), \\ &= \sum_{k=1}^{\infty} s_k G_k(t). \end{aligned}$$

Here we shall prove the change of order of summation is justified. For this purpose it is sufficient to prove that, for fixed $t > 0$,

$$I_n = \sum_{k=1}^{N} s_k \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-1} \Phi(n,t) = o(1), \qquad \text{as} \qquad N \to \infty.$$

Using Abel's transformation, we have

$$\begin{aligned} I_n &= \sum_{k=1}^{N-1} S_k^1 \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-2} \Phi(n,t) + S_N^1 \sum_{N=n+1}^{\infty} A_{n-N}^{\alpha-1} \Phi(n,t) \\ &= \left(\sum_{k=1}^{N-1} |s_k^1| N^{-1} (N-K)^{\alpha-1} \right) + o(NN^{-1}) = o(1), \quad \text{as } N \to \infty. \end{aligned}$$

This proves the lemma.

LEMMA 11. *Let $G_n(t)$ and $K_n(t)$ be the same as defined in lemmas 8 and 9 respectively. If $s_n K_{n+1}(t) = o(1)$, $n \to \infty$, then the convergence of $\sum_{n=1}^{\infty} a_n K_n(t)$ implies the convergence of $\sum_{n=1}^{\infty} s_n G_n(t)$ and*

$$\sum_{n=1}^{\infty} a_n (K_n(t) = \sum_{n=1}^{\infty} s_n G_n(t).$$

The proof of this lemma follows from the identity

$$\sum_{v=1}^{m} s_v G_v(t) = \sum_{v=1}^{m} a_v K_v(t) = s_m K_{m+1}(t).$$

LEMMA 12. *If p_n is such that it satisfies all the conditions of the theorem (2.1.3), then the series*

$$\sum_{n=0}^{\infty} c_n K_{n+v}(t) = H_v(t), \tag{2.3.12}$$

is absolutely convergent and for $m = 0, 1, 2,$

$$\Delta^m H_v(t) = O\left(\frac{t^{-m-1}}{vP_\tau}\right). \tag{2.2.13}$$

where $\Delta^m H_v(t)$ denote the mth difference of $H_v(t)$, with respect to v.

Absolute convergence of (2.3.12) follows from the hypotheses (2.1.2) since $\Sigma c_n < \infty$. To prove (2.3.13), we have

$$\begin{aligned}\Delta^m H_v(t) = \Delta^m\left(\sum_{n=0}^{\infty} c_n K_{n+v}(t)\right) &= \Delta^{m-1}\left(\sum_{n=0}^{\infty} c_n K_{n+v}(t)\right)\\ &= \Delta^{m-1}\sum_{n=0}^{\infty} c_n G_{n+v}(t) = \sum_{n=0}^{\infty} c_n \Delta^{m-1} G_{n+v}(t)\\ &= \left(\sum_{n=0}^{\tau} + \sum_{n=\tau+1}^{\infty}\right) c_n \Delta^{m-1} G_{n+v}(t) = H_v^{(1)}(t) + H_v^{(2)}(t),\end{aligned}$$

say.

Now, by hypotheses and lemmas 8 and 11, we have, for $m = 1, 2, \ldots,$

$$\begin{aligned}H^{(2)}(t) &= \sum_{n=\tau+1}^{\infty} c_n \Delta^{m-1} G_{n+v}(t) = \left(\sum_{n=\tau+1}^{\infty} |c_n| \frac{t^{m-1}}{(n+v)}\right)\\ &= O\left(\frac{t^{m-1}}{v+\tau+1}\right) \sum_{n=\tau+1}^{\infty} |c_n| = O\left(\frac{t^{m-1}}{vP_\tau}\right).\end{aligned}$$

And by applying Abel's transformation and lemmas 5 and 8, we have, $m = 1, 2, \ldots,$

$$\begin{aligned}H_v^{(1)}(t) &= \sum_{n=0}^{\tau-1} d_n \Delta^m G_{n+v}(t) + d\Delta^{m-1} G_{\tau+v}(t)\\ &= O\left(\sum_{n=0}^{\tau-1} \frac{1}{P_n} \frac{t^m}{(n+v)}\right) + O\left(\frac{t^{m-1}}{vP_\tau}\right) = O\left(\frac{t^{m-1}}{vP_\tau}\right)\end{aligned}$$

By hypothesis, and for $m=0$,

$$\begin{aligned} {}^mH_v(t) = H_v &= \sum_{n=0}^{\infty} c_n K_{n+v}(t) \\ &= \sum_{n=0}^{\tau} c_n K_{n+v}(t) + \sum_{n=\tau+1}^{\infty} c_n K_{n+v}(t), \\ &= \sum_{n=0}^{\tau} d_n H_{n+v}(t) + d_\tau K_{n+v+1}(t) + O\left(\frac{1}{vt}\sum_{n=\tau+1}^{\infty}|c_n|\right) \\ &= O\left(\frac{1}{v}\sum_{n=0}\frac{1}{P_n}\right) + O\left(\frac{1}{vtP_\tau}\right) + O\left(\frac{1}{vtP_\tau}\right) = O\left(\frac{1}{vtP_\tau}\right), \end{aligned}$$

by hypotheses and lemmas 5 and 9.

2.4. *Proof of theorem* 1. We may assume, without loss of generality that $T_n = (P_n)$, as $n \to \infty$. By virtue of Lemmas 6 and 10, we have

$$t^{\alpha+1}\sum_{n=1}^{\infty} S_n^{\alpha}\Phi(n,t) = \sum_{n=1}^{\infty} s_n G_n(t).$$

Again, by (1.2.2) and lemma 6, we have, as $n \to \infty$

$$\begin{aligned} S_n K_{n+1}(t) &= K_{n+1}(t)\sum_{v=1}^{n}(t) c_{n-v} T_v \\ &= O\left(\frac{P_n}{(n+1)t}\right)\sum_{v=1}^{n-1}|c_{n-v}| + O\left(\frac{p_n}{nt}\right) = o(1) \end{aligned}$$

for fixed $t>0$ and by hypothesis (2.1.2) and (2.1.3) and Lemma 1.

Therefore, by virtue of lemma 11, it is sufficient to prove that $\Sigma a_n K_n(t)$ converges in $0<t<t_0$ and tends to zero as $t \to +0$.

Employing (1.2.3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n K_n(t) &= \sum_{n=1}^{\infty} K_n(t) \sum_{v=1}^{\infty} c_{n-v}(T_v - T_{v-1}) \\ &= \sum_{v=1}^{\infty}(T_v - T_{v-1})\sum_{n=v}^{\infty} c_{n-v} K_n(t), \end{aligned}$$

the interchange of order of summations being legitimate, since the double series is absolutely convergent.

Since by hypothesis and the fact that $\sum_{n=0}^{\infty}|c_n| < \infty$ for every fixed $t>0$, we have

$$\sum_{v=1}^{\infty}|T_v - T_{v-1}|\sum_{n=0}^{\infty}|c_n K_{n+v}(t)| = \left(\sum_{v=1}^{\infty}\frac{1}{v}|T - T_{v-1}|\right).$$

Now, as $n \to \infty$,

$$\sum_{v=1}^{n} \frac{1}{v}|T_v - T_{v-1}| = O(n^{-1}\sigma_n) + O\left(\sum_{v=1}^{n-1} \frac{\sigma v}{v(v+1)}\right)$$
$$= 0(1)\frac{P_n}{n} + 0(1)\sum_{v=1}^{n-1} \frac{P_v}{v(v+1)} = 0(1),$$

by hypotheses and lemmas 1 and 2.
Thus

$$\begin{aligned} f(\alpha, t) &= \sum_{v=1}^{\infty}(T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} K_n(t) = \sum_{v=1}^{\infty}(T_v - T_{v-1}) H_v(t) \\ &= \left(\sum_{v=1}^{n} \sum_{v=n+1}^{\infty}\right) T_v - T_{v-1}) H_v(t) \\ &= \Sigma_1 + \Sigma_2, \text{ say,} \end{aligned} \tag{2.4.2}$$

Now

$$\begin{aligned} |\Sigma_2| &= \left|\sum_{v=n+1}^{\infty}(T_v - T_{v-1})H_v(t)\right| = \left(\sum_{v=n+1}^{\infty} |T_v - T_{v-1}| \frac{1}{vtP_\tau}\right) \\ &= O\left[\frac{1}{tP_\tau}\left(\sum_{v=n+1}^{\infty} \frac{\sigma v}{v(v+1)} - \frac{\sigma_n}{n+1}\right)\right] \\ &= O\left[\frac{\tau}{P_\tau}\frac{P_n}{n}\right] = O\left(\frac{P_\mu}{\mu}\right), \\ &= O(1)\frac{P_\mu}{\mu} \end{aligned} \tag{2.4.3}$$

by hypotheses and lemmas 2 and 12.

Next by lemma 2, we hawe

$$\begin{aligned} \Sigma_1 &= \sum_{v=1}^{n}(T_v - T_{v-1})H_v(t) = \sum_{v=1}^{n}(W_v - W_{v+1})vH_v(t) \\ &= \sum_{v=1}^{n} W_v[vH_v(t) - (v-1)H_{v-1}(t)] - nW_{n+1}H_n(t) \\ &= -\sum_{v=1}^{n} H_v v[H_{v-1}(t) - H_v(t)] + \sum_{v=1}^{n} W_v H_{v-1}(t) - nW_{n+1}H_n(t) \\ &= -\Sigma_{1,1} + \Sigma_{1,2} - nW_{n+1}H_n(t), \text{ where, by Lemma 3 (ii) and 12,} \end{aligned}$$

$$
\begin{aligned}
\Sigma_{1,1} &= \sum_{v=1}^{n} vW_v \Delta H_{v-1}(t) = \sum_{v=1}^{n} \left\{ \sum_{\mu=1}^{n} \mu W_\mu \right\} \Delta^2 H_{v-1}(t) \Delta H_n(t) \sum_{v=1}^{n} vW_v \\
&= o\left(\sum_{v=1}^{n} vP_v \frac{t}{vP_\tau} \right) + o\left(\frac{1}{nP_\tau} nP_n \right) = o\left(nt \frac{P_n}{P_\tau} \right) + \left(\frac{P_n}{P_\tau} \right) \\
&= o(1)\mu P_\mu + o(1) p_\mu = o(1)
\end{aligned}
$$

since $\sum_{v=1}^{n} vW_v = o\left(\sum_{v=1}^{n} v\frac{P_v}{v}\right) = 0(n, P_n)$, and by applying Abel's transforrmation twice, writng $W'_m = \sum_{\mu=1}^{m} W_\mu$ and by virtue of Lemma 1, 3 (ii) and 11, we hawe

$$
\begin{aligned}
\Sigma_{1,2} &= \sum_{v=1}^{n} \left(\sum_{m=1}^{v} W'_m \right) \Delta^2 H_{v-1}(t) + \Delta H_n(t) \sum_{v=1}^{n} W'_v + H_n(t) W'_n \\
&= o\left(\sum_{v=1}^{n} vP_v \frac{t}{vP_\tau} \right) + o\left(\frac{1}{nP_\tau} \sum_{v=1}^{n} P_v \right) + o\left(\frac{P_n}{ntP_\tau} \right) \\
&= o(1)\mu P_\mu + o(1) P_\mu + o(1) \frac{P_\mu}{\mu} = o(1).
\end{aligned}
$$

Hence,

$$\Sigma_1 = o(1) \qquad (2.4.4)$$

Therefore, from (2.4.2), (2.4.3) and (2.4.4), we have

$$f(\alpha, t) = o(1) + O(1)\frac{P_\mu}{\mu}, \text{ as } t \to 0.$$

Consequently, $\lim_{t\to 0} \sup f(\alpha t) \leq O(1)\frac{P_\mu}{\mu}$, being arbitrary large and O (1) independent of μ we get finally

$$f(\alpha, t) \to 0, \text{ as } t \to 0.$$

This completes the proof of our theorem.

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