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INEQUIVALENT REGULAR FACTORS OF REGULAR GRAPHS ON 8 VERTICES

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In this paper we shall find all nonisomorphic factorizations of all regular graphs on 8 vertices into two regular factors without the use of a computer (as a contrast to [1]). These factorizations are significant since they produce regular graphs with the least eigenvalue -2 which are neither line-graphs nor cocktail-party graphs but which are cospectral to line-graphs (cf [1]).

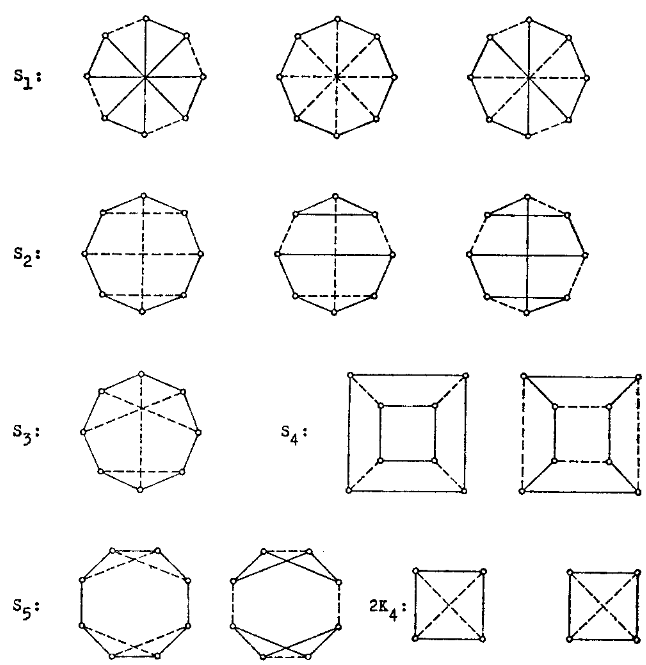


Fig. 1

Regular graphs of degrees 0, 1, 2 are not interesting. Nonisomorphic factorizations of the six cubic graphs were found in [1] (Fig. 1), the connected cubic graphs being denoted by $S_1, \ldots, S_5$. In sections 1 and 2 we shall solve the problem for regular graphs of degrees 4 and 5 respectively. The remaining cases are mentioned in section 3.

1. Graphs of Degree 4

Let us denote by $\bar{G}$ the complement of a graph $\bar{G}$ and by $L(G)$ the linegraph of G. In order to determine the automorphism groups of the graphs $\bar{S}_1, \ldots, \bar{S}_5$, we will rather consider $S_1, \ldots, S_5$, since their automorphisms can be described easier (because of the less number of edges). On the other hand, any automorphism of a graph G generates an automorphism of $L(G)$ and we will often refer to automorphisms of $L(\bar{S}_i)$ induced by the group of $\bar{S}_i$.

Graps $\bar{S}_1$: Graphs S_1 and $\bar{S}_1$ are displayed in Fig. 2.

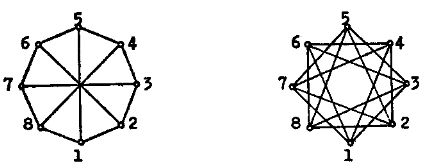


Fig. 2

THEOREM 1. *All nonisomorphic regular factorizations of* $\bar{S}$ *are determined by three inequivalent 1-factors in Fig. 3 and 2-factors (a), (c), (d) and (f) in Fig. 4.*

PROOF: The automorphism group is the dihedral group D_8. Graph S_1 has only one orbite, but $L(\bar{S}_1)$ has two, since the edges between the vertices being at distance 2 in the characteristic octogon $12 \ldots 8$ of S_1 cannot be mapped into those pairs of vertices which are at distance 3. Let us denote those sets of "shorter" and "longer" edges by O_1 and O_2. Now, there are only 4 1-factors which consist only of the edges of O_1; they are all equivalent since a suitable rotation[1] sends them one into another (Fig. 3. (a)). There are also two 1-factors whose edges are only of O_2 and they are equivalent, too (Fig. 3. (b)). It is not difficult to see that three edges from one orbite and one from another cannot be taken and that also would be impossible to make a 1-factor by taking two edges from O_2 and two edges from the same quadrangle of O_1. But if we take one edge from the quadrangle 1357 and one from 2468, then we can construct a 1-factor in case we didn't take those two edges in such way that they span four successive vertices of the octogon $12 \ldots 8$ of S_1. Thus we get eight new 1-factors and they are all equivalent because rotating any one of them gives the rest (Fig. 3. (c)). The necessary condition for two factors to be equivalent is that they have the same number of edges taken from every orbite; hence the three groups of 1-factors are different equivalence classes and $\bar{S}_1$ has 3 inequivalent 1-factors.

[1] Geometric notions will be used rather then abstract isomorphism mappings.

Since 2-factors of the form C_8 and 2 C_4 are sums of 1-factors, all such 2-factors can be found by mutual combining of 1-factors. Thus, 1-factors in the first equivalence class give only one 2-factor having all eight edges in O_1 (Fig. 4 (*a*)), and similarly 1-factors in the second class generate only one 2-factor taking all edges from O_2 (Fig. 4 (*b*)). Since all 1-factors in the third class can be obtained by a rotation of any of these factors, it is sufficient to take the sums of, say, factor (7) and all other factors in this class. The only possibilities are to combine (7) with (8), (11) and (14). Among these three 2-factors ((7, (14)) is mapped into ((7), 11)) (Fig. 4 (d)) by a rotation, while ((7), (8)) is neither as a graph isomorphic with (d)

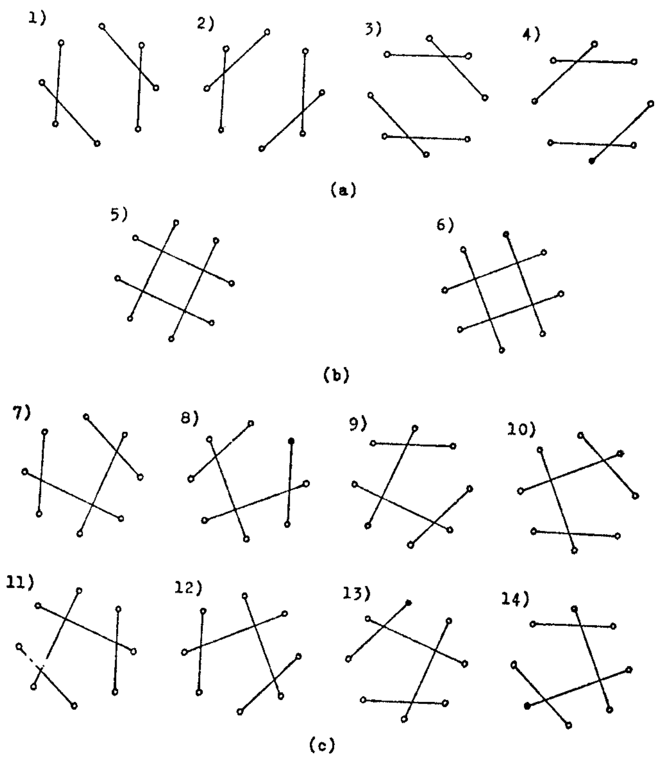


Fig. 3

(Fig. 4 (c)); in fact, (c) is C_8 and (d) is 2 C_4. These 2-factors take four edges from every orbite. When combining 1-factors of the first and the second class, we see that in both classes the successive rotation of any factor gives the rest. That is why we may only combine 1-factor (1) with the factors in the second class. The pair ((1), (5)) is (d) again, but ((1), (6)) yet cannot be mapped into (c) in spite of having four edges in every orbite because it has no two adjacent edges in 0_1 (while (c) obviously has) - Fig. 4 (e). When combining the 1-factors of the first class with those of the third one we may again consider only the factor (1) in the first class and get the only case in Fig. 4(f) which takes 6 edges from O_1 and 2 edges from

O_2. The second and the third class (2 edges from O_1 and 6 from 0_2) give rise to the 2-factor in Fig 4 (g). Now, in order to find 2-factors of the form $C_3 \cup C_5$ we should only notice all inequivalent triangles. Since all eight triangles of $\bar{S}_1$ are equivalent, let us take, say, 146. The remaining five vertices induce only one pentagon in $\bar{S}_1$. The obtained 2-factor is shown in Fig 4 (h).

Finally, one can easily find the complementary pairs of inequivalent 2-factors: (a) and (b), (c) and (h), (d) and (e), (f) and (g). This completes the proof.

Graph $\bar{S}_2$: Graphs S_2 and $\bar{S}_2$ are shown in Fig. 5.

THEOREM 2. *All nonisomorphic regular factorizations of* $\bar{S}_2$ *are determined by five ineguivalent 1-factors in Fig. 9 and 2-factors (a), (b), (c), (d), (g), (h), (k), (m), (n) and (g) in Fig. 7.*

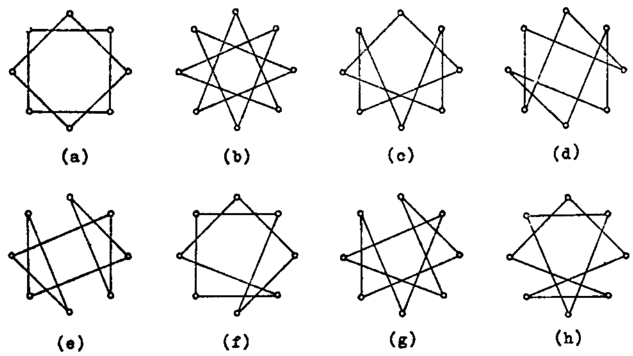


Fig. 4

PROOF: The automorphism group is the Klein four-group generated by the horizontal and vertical reflection. Vertices 2, 4, 6 and 8 constitute the orbite O_1, vertices 1 and 5 the orbite O_2 and 3 and 7 the orbite O_3; however, $L(\bar{S}_2)$ has five orbites: 1) $\{13,35,57,71\}$; 2) $\{36,38,72,74\}$; 3) $\{14,16,52,58\}$; 4) $\{24,68\}$; 5) $\{26,48\}$. The number of orbites in $\bar{S}_2$ and $L(\bar{S}_2)$ can easily be vertfied by means of the Burnside Lemma.

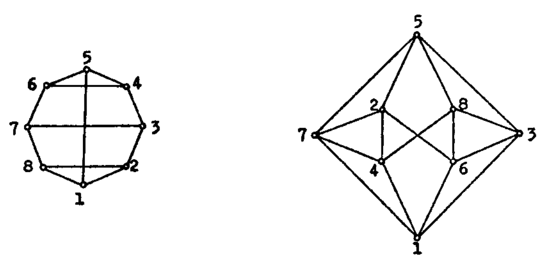


Fig. 5

Burnside Lemma: If $\mathcal{G}$ is a permutation group acting on a set A and $\eta(g) =$

$|\{x \mid g(x) = x\}|$, then the number of orbites of $\mathcal{G}$ is $\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\eta(g)$.

In order to find all 1-factors we first take two edges of the subgraph of $\bar{S}_2$ induced by 0_1. There are four such factors (Fig. 6. (a) and (b)), the cases (a) and (b) being clearly in different equivalence classes. Then, there are two 1-factors having no edges in the subgraph induced by 0_1 (Fig. 6 (c)). If we take now one edge of that subgraph, there are two possibilities: if we take 24 or 68, we get four 1-factors (the fourth equivalence class – Fig. 6 (d)), while taking 26 or 48 we obtain the last, fifth class (Fig. 6 (e)).

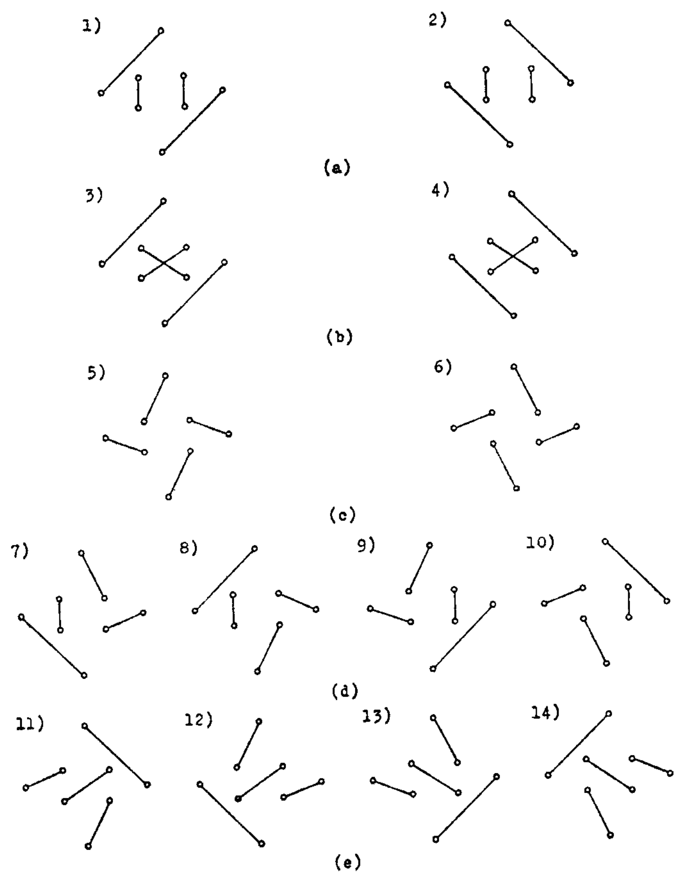


Fig. 6

We might again use the Burnside Lemma to verify that among fourteen 1-factors there are five equivalence classes. If 1-factors of $\bar{S}_2$ are elements of the set A, then the equivalence classes with respect to the mappings of factors induced by the automorphisms of $\bar{S}_2$ are the orbites of A. All fourteen 1-factors are fixed by the identity, no one is fixed by the two reflexions and six 1-factors are fixed by the composition of reflexions; hence the number of orbites is $\frac{1}{4}\cdot 20 = 5$.

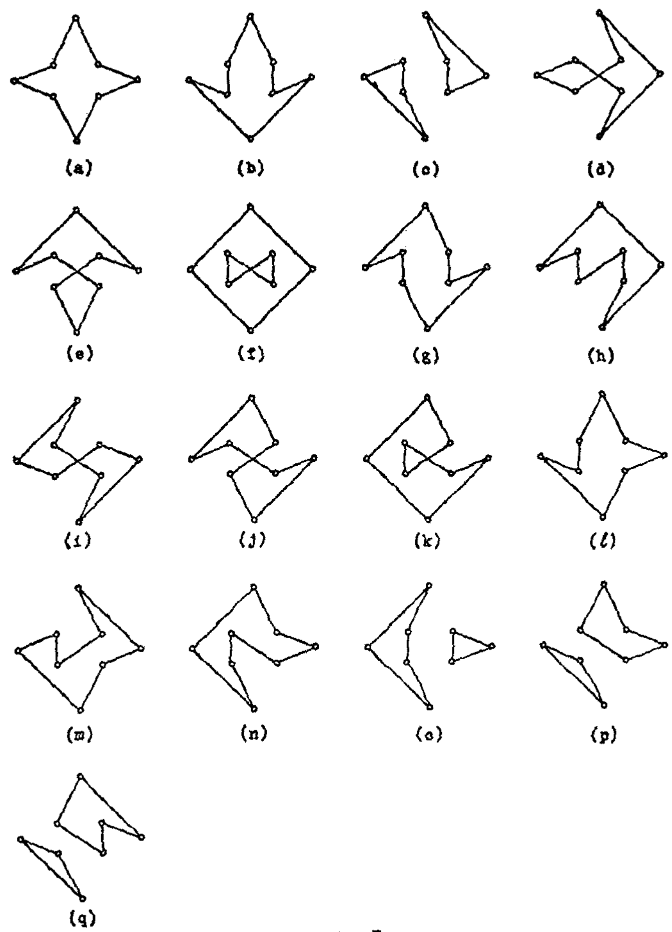


Fig. 7

All 2-factors of $\bar{S}_2$ of the form C_8 or 2 C_4 (those which we get by means of 1-factors) are displayed in Fig. 7. (a)–(n) and we'll give a shortened survey of their construction. Thus, 1-factors in the first or the second equivalence class cannot form any sum because of the common edges, while the third class gives (a). In the fourth class (7) and (8) (as well as (9) and (10)) are in the horizontal reflexion and therefore ((7), (9)) and ((8), (10)) are equivalent (case (b)). Similarly, only one of the two pairs ((7), (10)) and ((8), (9)) can be taken (case (c)), leaving no other sum in that class. The fifth class gives two inequivalent 2-factors – (d) and (e). The pairs formed by one factor of each of the first two classes give rise only to (f). Taking the first and the third class we see that we should consider only ((1), (5)) and (1), (6)). The first case already appeared and the second is (g). Continuing in the similar way we find that the first and the fifth class generate only (h), the second ane the third class only (i) and (j), while the second and the fourth give (k). The case (q) is obtained by 1-factors taken from the third and the forth class. Now, the only remaining possibility is that of combining factors of the last two classes. Notice that horizontal reflexion sends (7) into (8), vertical reflexion maps it into (9) and their composition maps (7) into (10), while (11) is mapped by the same automorphisms into (12), (14) and (13) respectively. Thus we take only two new cases – (m) and (n). Now, we can easily find that $\bar{S}_2$ contains only two inequivalent triangles. The first one is, say, 368, and the remaining five vertices determine a unique pentagon (case (o)). Let the second triangle be 174, and now the remaining vertices induce two inequivalent pentagons (case (p) and (q)).

To make sure that all obtained 2-factors are inequivalent we should notice that in the majority of cases two 2-factors which are isomorphic as graphs (e. g. two octogons) do not have the same number of edges in the same orbites; but if they do, then a 2-factor containing two adjacent edges of an orbite cannot be mapped into 2-factor in which the edges of the same orbite are not adjacent ((m) and (n) is such pair); finally, in rare occasions when it is necessary we may very easily directly check out the inequivalence (e. g. (i) and (j)).

At the end we see that inequivalent complementary pairs are: ((a), (f)), ((b), (e)), ((c), (i)), ((d), (o)), ((g), (j)), ((h), (p)), ((k), (l)), while (m), (n) and (q) have equivalent complements with respect to $\bar{S}_2$, and that completes the proof.

Graph $\bar{S}_3$: Fig. 8 shows this graph together with its complement.

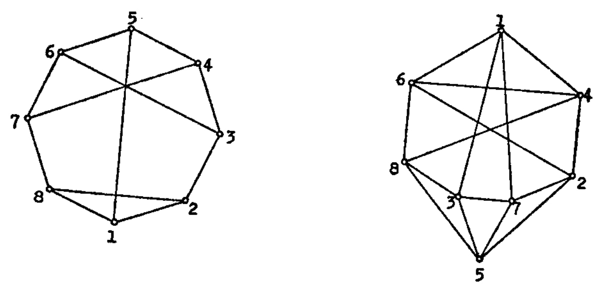


Fig. 8

THEOREM 3. *All nonisomorphic regular factorizations of* $\bar{S}_3$ *are determined by two inequivalent 1-factors in Fig. 9 and 2-factors (a), (c) and (d) in Fig. 10.*

PROOF: Let us first determine the automorphism group. There is only one triangle in S_3 – 128, and those vertices constitute an orbite. Besides the obvious vertical reflexion (with respect to the axis 15), it is clear that the permutation 123(46)578 is also an automorphism. Those mappings generate the Klein group and fix the vertices of the triangle. Trying with the other permutations in the set $\{1,2,8\}$ we come to a new automorphism (128)(375)(46) denoted by A_6, while $A_5^2 = A_9$ gives the permutation (182)(357)(46) and $A_5^3 = A_1$ (A_1 denotes the identity). Those automorphisms generate all the rest, the total number being twelve, and all of them are presented in the next table:

A_1: Identity	A_7: (128)(375)(46)
A_2: 1(28)(37)(46)5,	A_8: (18)234(57)6,
A_3: 123(46)578,	A_8: (182)(357)46,
A_4: 1(28)(37)456,	A_{10}: (12)(35)(46)78,
A_5: (128)(375)46,	A_{11}: (182)(357)(46),
A_6: (18)23(46)(57),	A_{12}: (12)(35)4678.

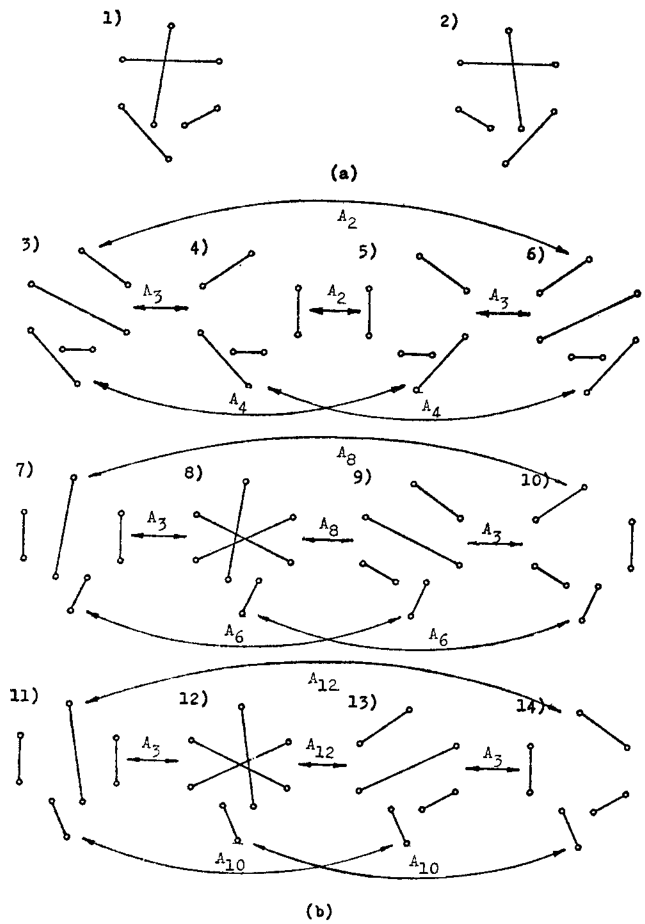


Fig. 9

Here is, for example, $A_6 = A_5 \circ A_2$, $A_7 = A_5 \circ A_3$, $A_8 = A_5 \circ A_4$ and similarly for A_{10} – A_{12}. The orbites are: $O_1 = \{1,2,8\}$, $O_2 = \{4,6\}$, $O_3 = \{3,5,7\}$; the consequences of that are, for example, that the edge 46 in $\bar{S}$ is always fixed and that the edges of the triangle 357 form an orbite in $L(\bar{S}_3)$.

Looking for 1-factors we get only two inequivalent cases. In fact, there are two equivalent 1-factors containing the edge 46 (Fig. 9 (a)) and twelve factors without that edge (Fig. 9 (b)). These twelve 1-factors are separated into three groups according to the edge of the triangle 357 they contain and the corresponding automorphisms in every group are indicated (of course, we find them among those which fix the considered edge of the triangle). It is sufficient now to notice that A_5 maps (3) into (7) and A_2 maps (7) into (11) to be sure that the three groups are in the same equivalence class.

Consider now the first class as the first group of 1-factors and the three groups of the second class as the second, the third and the fourth group. In order to mahe 2-factors we can take 1-factors only from different groups. The factors in the first group are fixed by A_i if i is odd, and mapped into each other if i is even. Thus, taking 1-factors from the first two groups we can reduce the number of cases and get only the 2-factor in Fig. 10(a). Now, notice that every factor from the third group has a symmetric mate in the fourth group. As (1) and (2) are also symmetric, combining the first group with the third and the fourth one we see that (1) combined with (9), (10), (11) and (12) gives just the same as (2) with (7), (8), (13) and (14). But (9) and (10) and also (11) and (12) are equivalent under A_3, while A_5 sends (9) into (11). Since all odd automorphisms fix (1), all those 2-factors are equivalent and since also A_7 maps (5) into (9) and fixes (1) again, it arises that they are equivalent to (a). If we continue in the similar way, we find that combining the second group with the third and the fourth we first get the pair ((3), (7)), (Fig. 10(b)) which has three equivalent mates and, of course, is not equivalent to (a) because of the edge 46. Then, we get ((3), (10)) (Fig. 10 (c)) of the form 2 C_4 (equivalent cases are ((4), (9)), ((5), (13)), (6), (14))). The next 2-factor is ((3), (11)) (Fig. 10(d)); to see that it is equivalent to (b) we should notice its edges in the triangle 357 and make sure that neither A_5 nor A_7 map it into (b). But the factor ((3), (13)) is equivalent to (b), because A_9 maps (b) into it. (The interesting detail is that A_9 does not fix (3), but maps it into (13), and (7) into (3)). Taking 1-factors from the last two groups we cannot get any new case. We have eight possibilities and see, for example, that ((7), (12)) (which is equivalent to ((8), (11))) is of the form 2 C_4 and can be obta ned from (c) by A_8, that we can get (7), (13)) from (b) by A_5, etc. Now, it remains only to look for the 2-factors of the form $C_3 \cup C_5$. We first see that 357 cannot be taken and that two triangles having one edge in the triangle 357 are symmetric, while 137 can be mapped, say, by A_5 into 275. Let us take 137. The two possible pentagons in the subgraph induced by the remaining vertices are equivalent under A_3 which fixes chosen triangle and we take only the case in Fig. 10(e). Finally, the remaining three triangles are also equivalent: let us take, say, 146, getting the case in Fig. 10(f).

It can easily be verified that the complementary pairs are: (a) and (b); (c) and (f), (d) and (e). This completes the proof.

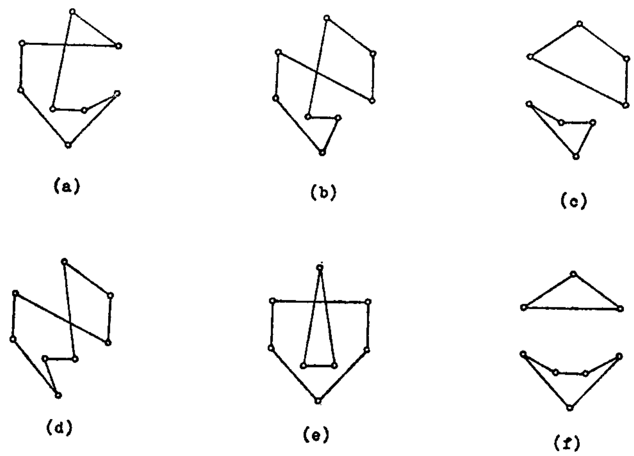


Fig. 10

Graph $\bar{S}_4$: We may imagine S_4 as a cube and the edges of $\bar{S}_4$ as "big" and "small" diagonals of the cube. The automorphism group of $\bar{S}_4$ is transitive on vertices, but $L(\bar{S}_4)$ has two orbites since a small diagonal cannot be mapped into a big one.

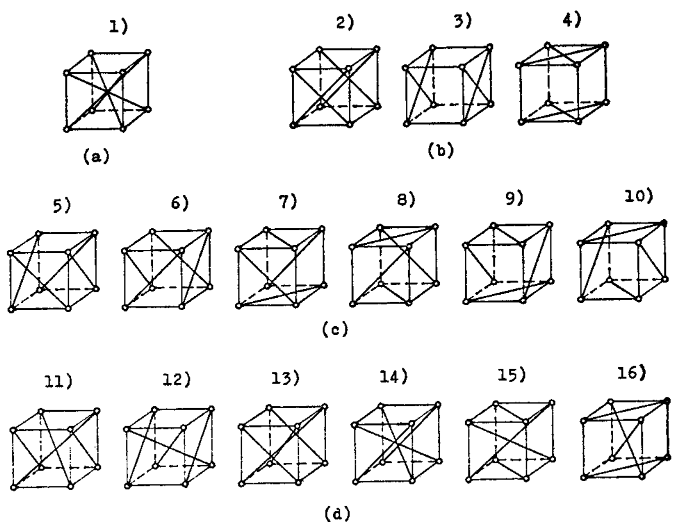
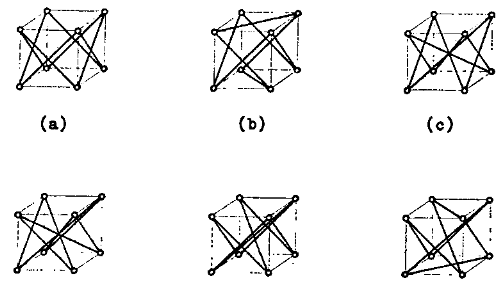


Fig. 11

THEOREM 4. *All nonisomorphic regular factorizations of* $\bar{S}_4$ *are determined by 1-factors in Fig. 11 and 2-factors (a), (b), (c) and (d) in Fig. 12.*



(a) (b) (c) (d) (e) (f)

Fig. 12

PROOF: In order to find the inequivalent 1-factors (Fig. 11) we may first take that which consists only of big diagonals and thus cannot be equivalent to any other one (case (a)). Factors consisting only of small diagonals can be constructed in two inequivalent ways: by taking both diagonals from each of two opposite faces of the cube (case (b)) or by choosing diagonals from four different faces (case (c)). The last two possibilities are obviously inequivalent since the automorphisms of cube preserve the belonging of a diagonal to a face. The last possible inequivalent 1-factor is that which has two big and two small diagonals (case (d)).

We can exceptionally find the inequivalent 2-factors of $\bar{S}_4$ directly (Fig. 12). We should only notice that all 1-factors have an even number of edges in every orbite and that 2-factors must satisfy the same condition. If a 2-factor contains four big diagonals, the four small diagonals can be put in two or in four faces (case (a) and (b)); clearly, such 2-factors are not equivalent. If a 2-factor has two big diagonals, again two cases arise. Since every such 2-factor must contain two small diagonals between those vertices which are not already on the big diagonals, we can choose the last four small diagonals from four different faces (case (c)), or from some two faces (case (d)). Finally, 2-factors containing only small diagonals can also be constructed in two inequivalent ways: by taking both diagonals from each of some two faces and one from every remaining face (case (e)), or by taking all diagonals from four faces (case (f)).

It is easy to check out that after taking any triangle of $\bar{S}_4$ we cannot obtain a pentagon and that $\{(a),(f)\}$ and $\{(b),(e)\}$ are complementary paris, while (c) and (d) are selfcomplementary. This completes the proof.

Graph $\bar{S}_5$: Graphs S_5 and $\bar{S}_5$ are displayed in Fig. 13.

THEOREM 5. *All nonisoinorphic regular factorizations of* $\bar{S}_5$ *are determined by four ineguivalent 1-factors in Fig. 14 and 2-factors (a), (b), (c), (f) and (g) Fig. 15.*

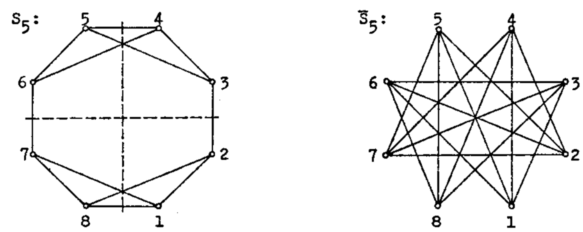


Fig. 13

PROOF: In order to find the automorphism group we should first notice the two pairs of triangles of S_5 with the common edges 18 and 45. Every automorphism maps any such pair into the same or the other pair. The subgraph induced by the vertices 1, 2, 8, 7 has four automorphisms, for every of them the position of vertices 3 and 6 is uniquely determined and there are two possible positions for the vertices 4 and 5; that means that eight automorphisms fix the edge 18. Taking also those eight automorphims which map the edge 18 into 45 we get the total number of sixteen automorpshims. In the next table we present some of them, actually those which we need in the proof.

A_1: Identity

A_2: (18)(27)(36)(45), (vertical reflexion)

A_3: (14)(23)(58)(67), (horizontal reflexion)

A_4: (15)(48)(26)(37), ($A_3 \circ A_2$)

A_5: (18)234567,

A_6: 123(45) 678,

A_7: (18)23(45)67.

It is obvious that $\{1,4,5,8\}$ and $\{2,3,6,7\}$ are the orbites; on the other hand, the four edges joining the vertices of the first orbite of $\bar{S}_5$ constitute an orbite O_1 in $L(\bar{S}_5)$. We also find that the sets of edges $\{27,36\}$ and $\{26,37\}$ are orbites O_2 and O_3 and that the remaining eight edges determine the orbite O_4.

Taking both edges of O_2 and a pair of edges of O_1 we get two equivalent 1-factors (the first equivalence class – Fig. 14 (a)). Edges of O_3 together with the same pairs in O_1 give the second class (Fig. 14 (b)). If we take only edges of O_4, we see that there are four such 1-factors, which are equivalent under the indicated automorphisms (Fig. 14 (c)). Finally, after checking that there is no other possibility, we can construct the rest of 1-factors by taking exactly one edge of O_3. There are eight such 1-factors and they are given in Fig. 14 (d) together with the automorphisms that map factors (9)–(12) and (13)–(16) into each other; we should only notice, say, that (9) and (13) are equivalent to be sure that all those eight factors are equivalent.

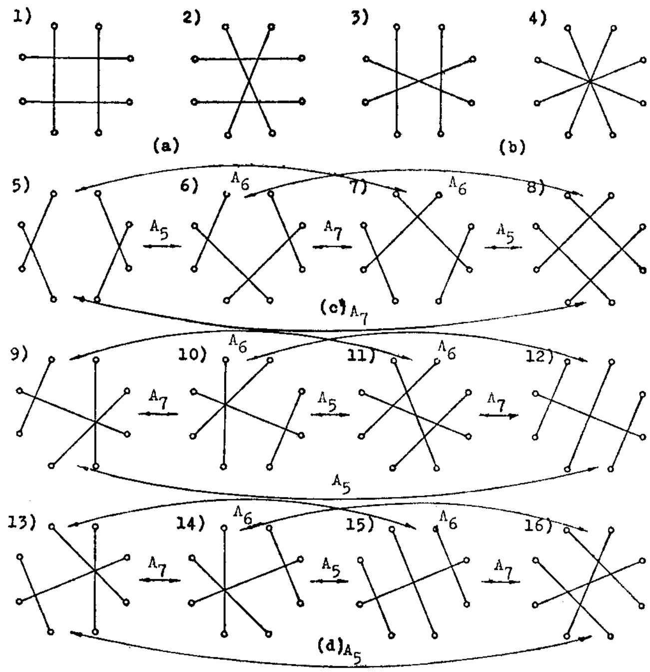


Fig. 14

In order to find 2-factors (displayed in Fig. 15) we start with the 1-factors in the third equivalence class and get (a). As for fourth class, it will be suitable to regard it as two different groups of 1-factors (two rows in Fig. 14), because 1-factors in the same group cannot be combined. The indicated automorphisms tell us that, for example, ((9), (14)), ((10), (13)), ((11), (16)) and ((12), (15)) are equivalent and we take ((9), (14)) as case (b) and ((9), (15)) as (c). We must also clarify what happens with ((9), (16)): it is equivalent to ((9), (15)) because A_5 maps (9) into (12) and horizontal reflexion maps (12) into (15), while the same automorphisms map (16) through (13) into (9). If we now combine 1-factors of the first two classes, we get (d). When considering the first and the third class we should notice that A_6 fixes (1) and (2), while A_5 and A_7 map them mutually; taking care of that we see that there are only two new cases – (e) and (f). The first and the fourth class give 2-factors which are not equivalent to previous cases because of the different number of edges in the orbites. But they are all mutually equivalent because A_7 and A_3 fix (1) which then, combined with (11), (12), (15) or (16), gives equivalent cases; since A_5 and A_6 map (1) into (2), it is clear that the remaining four pairs give nothing new (the pair ((1), (15)) is presented as the case (g)). Taking now the second and the third equivalence class we have again two edges in O_1, two in O_3 and four edges in O_4, just as in cases (b) and (c). Since A_7 fixes (3) and (4), while A_5 and A_6 map them into each other, the pairs ((5), (6)), ((3), (7)), ((4), (5)) and ((4), (8)) are equivalent, but they are also equivalent to (b). The remaining four pairs generate 2-factors which, being not equivalent to those just mentioned, also cannot be equivalent to (c) because otherwise two nonadjacent edges in O_1 would have to be mapped into a pair of adjacent edges. We take the pair ((3), (5)) as the case (h). Since we cannot combine the second and the fourth equivalence class, we next take the third and the fourth. There are eight possible pairs of 1-factors. The pairs ((5), (11)) and ((5), (16)) are equivalent because A_3 mutually maps (11) and (16) and fixes (5). But A_5 maps these pairs into ((6), (10)) and ((6), (13)) and similarly A_6 and A_7 map them into remaining possible 2-factors; we may choose, say, ((8), (12)) as the case (i). At the end, we see that all triangles in $\bar{S}_5$ are equivalent and if we take, for example, 136, the remaining vertices induce two pentagons; but since $A_7 \circ A_2$ maps one of them into another and fixes the triangle, we have only one case of the form $C_3 \cup C_5$ (case (j)).

The coplementary pairs are: $\{(a),(d)\}$, $\{(b),(e)\}$, $\{(c),(j)\}$, $\{(f),(h)\}$, $\{(g),(i)\}$, and this completes the proof.

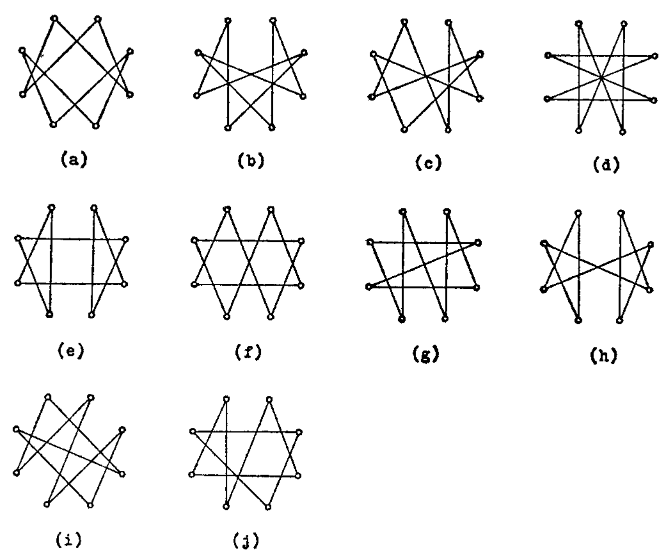


Fig. 15

Graph $\overline{2K_4}$: The automorphisms of that graph allow arbitrary permutations of vertices inside each of two sets of mutually nonadjacent vertices. Therefore, the following theorem is obvious.

THEOREM 6. *The only three nonisomorpfic regular factorizations of* $2\bar{K}_4$ *are those determined by any 1-factor, any 2-factor which is* C_8 *and any 2-factor which is* $2C_4$.

2. Graphs of Degree 5

These graphs are the complements of the graphs of degree 2. We should remember that the automorphism group of C_n is dihedral group D_n.

Graph C_8: Edges of this graph can be regarded as diagonals of an octogon.

THEOREM 7. *All nonisoynorphic regular factorizations of* $\bar{C}_8$ *are determined by its 1-factors in Fig. 16 and 2 factors in Fig. 17.*

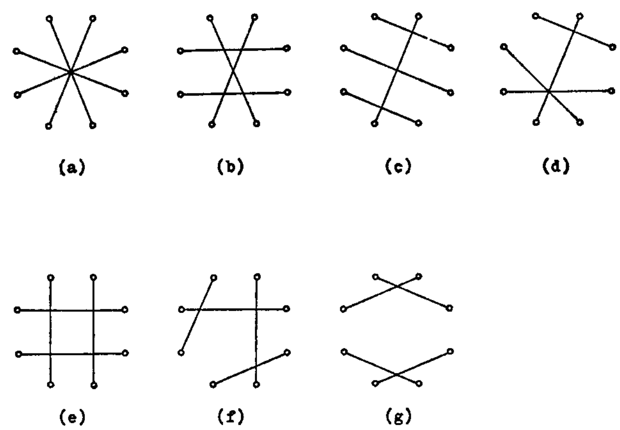


Fig. 16

PROOF: Since the diagonals join the vertices which are at distance 2, 3 or 4 in the octogon (we will call them small, middle and big diagonals), $L(\bar{C}_8)$ has three orbites. We start with 1-factors that have four big diagonals (Fig. 16 (a)). If we take two big diagonals whose endpoints are adjacent in the octogon, we get the case (b), but if those endpoints are not adjacent we obtain 1-factor (c). Only one big diagonal gives rise to 1-factor (d). The remaining cases, all presented in Fig. 16 are: 1-factor consisting completely of middle diagonals (case (e)), 1-factor having two middle and two small diagonals (case (f)), and that which consists only of small diagonals (case (g)).

Now, we can find 2-factors following an idea mentioned in [3]; the procedure is based upon the next fact:

LEMMA: If G_1 and G_2 have complements $\bar{G}_1$ and $\bar{G}_2$, then $G_1 \subseteq G_2$ if and only if $\bar{G}_1 \supseteq \bar{G}_2$.

Since every graph has the same automorphism group as its complement, we may conclude that, instead of looking for all inequivalent 2-factors (and hence 3-factors) of $\bar{C}_8$, we should only find all inequivalent ways in which C_8 can be embedded into the complements of all regular graphs of degree 3 on 8 vertices. That means that we are to collect all cases when the octogon is a 2-factor of a regular graph of degree 4 on 8 vertices. If we want to find, say, 3-factors of the form S_i, we will first determine all inequivalent 2-factors of the form C_8 in $\bar{S}_i$. The edges of the graph A which is the complement of C_8 with respect to $\bar{S}_i$ and the edges of S_i together constitute $\bar{C}_8$, where the graphs A and S_i determine the position of 2-factor, and therefore also the 3-factor S_i, in $\bar{C}_8$. Since C_8 appears thirty times as a 2-factor of the graphs of degree 4, $\bar{C}_8$ has thirty inequivalent pairs of 2-factors and 3-factors (Fig. 17). The construction of all such 2-factors completes the proof.

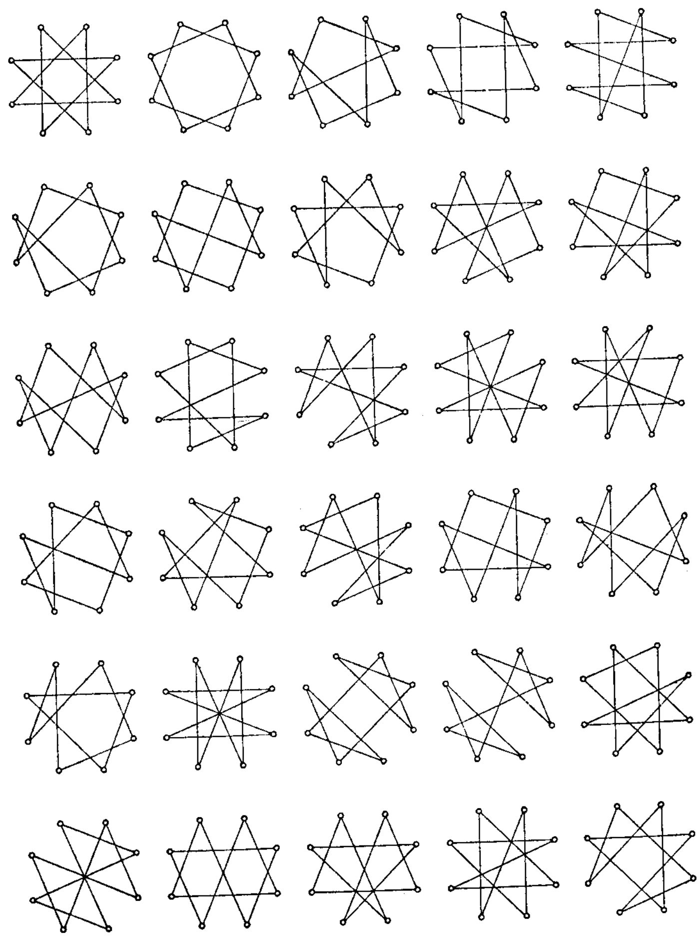


Fig. 17

Graph $\overline{2C_4}$: The diagonals of the quadrangles of 2 C_4 induce an orbite in $L(\overline{2C_4})$ – let us denote it by O_1, while O_2 consists of the remaining edges.

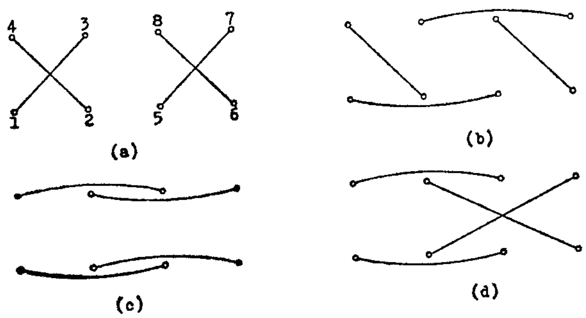


Fig. 18

THEOREM 8. *All nonisomorphic regular factorizations of* $\overline{2C_4}$ *are determined by 1-factors in Fig. 18 and 2-factors in Fig. 19.*

PROOF: There is only one 1-factor whose edges are in O_1 Fig. 18 (a)). The second equivalence class is that consisting of 1-factors which take two edges from O_1 (case (b)). In order to find inequivalent 1-factors with all edges in O_2, denote by 1, 2, 3, 4 and 5, 6, 7, 8 the vertices of the two quadrangles of 2 C_4. Edges of O_2 can map $\{1,2,3,4\}$ into any permutation of the set of vertices $\{5,6,7,8\}$ and hence there are 24 such 1-factors. Consider the factor in Fig. 18 (c). Any two of its edges join a pair of vertices of the set $\{1,2,3,4\}$ which are not adjacent in $\overline{2C_4}$ to a pair of nonadjacent vertices of the set $\{5,6,7,8\}$. All eight such factors, as one can easily see, are equivalent. Another possibility is to join a pair of nonadjacent vertices of the first set to a pair of adjacent vertices of the second set (such case is (d)). To see that (c) and (d) are not equivalent we should observe that any mapping of a pair of edges in the factor (c), for example 15 and 37, into a pair of edges of an equivalent 1-factor, would have required the mapping of the quadrangle 1573 into a quadrangle which also contains two edges of O_1; but it is obvious that no pair of edges in 1-factor (d) belongs to such a quadrangle. On the other hand, it can easily be checked that all sixteen 1-factors of type (d) are equivalent.

Of course, the inequivalent 2-factors (or 3-factors) can be found in the same way as in the case of $\bar{C}_8$ and they are presented in Fig. 19. This completes the proof.

Graph $\overline{C_3 \cup C_5}$: Let us denote the vertices of the triangle of $C_3 \cup C_5$ by 1, 2, 3 and the vertices of the pentagon by 4, 5, 6, 7, 8.

THEOREM 9. *All nonisomorphic regular factorizations of* $\overline{C_3 \cup C_5}$ *are determined by its arbitrary 1-factor and 2-factors in Fig. 21.*

PROOF: A 1-factor of $\overline{C_3 \cup C_5}$ must contain three edges which join vertices 1, 2, 3 to those of the set $\{4,5,6,7,8\}$ and that is why any 1-factor has exactly one diagonal of the pentagon 45678. The diagonals are equivalent under the automorphism group of the pentagon and we take one of them, say 57. The remaining six cases of choosing the edges are also equivalent because the permutations of the triangle map then into each other while the taken diagonal remains fixed. There are thirty such equivalent 1-factors and one of them is given in Fig. 2b.

Of course, the 2-factors could be found in the same way as those of $\bar{C}_8$ and $\overline{2C_4}$; they are displayed in Fig. 21, completing the proof.

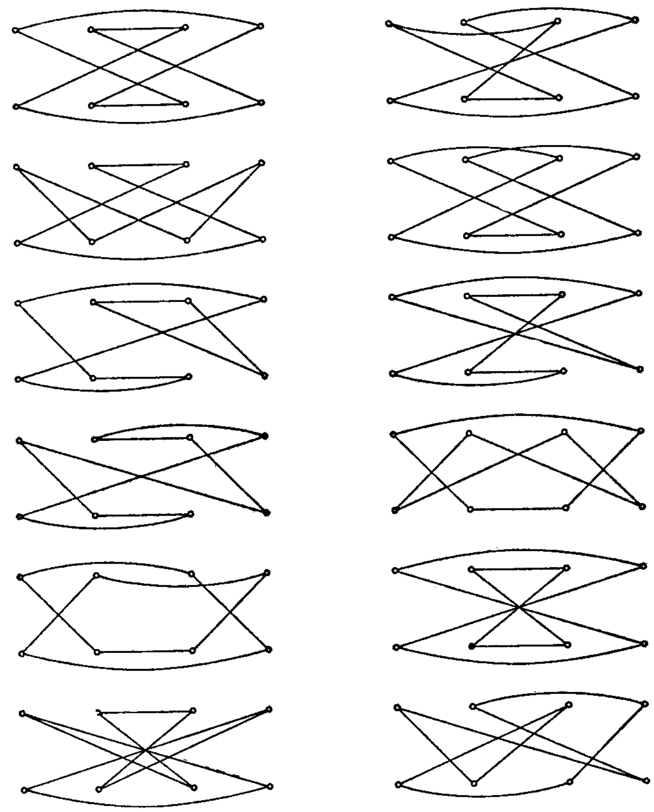


Fig. 19.

3. Other Regular Graphs on 8 vertices and the Summary of Results

There is only one regular graph of degree 6 on 8 vertices – the socalled cocktail-party graph denoted by CP (4). Its inequivalent regular factors can be found by embedding its complement into the complements of those regular factors in all inequivalent ways: that was done in [3] and Fig. 22 shows all such factors. Those 3-factors that determine all nonisomorpic factorizations are marked by asterisk.

The case of K_8 is trivial.

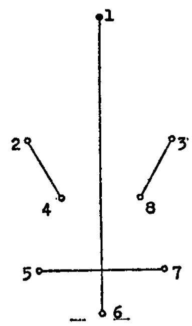


Fig. 20

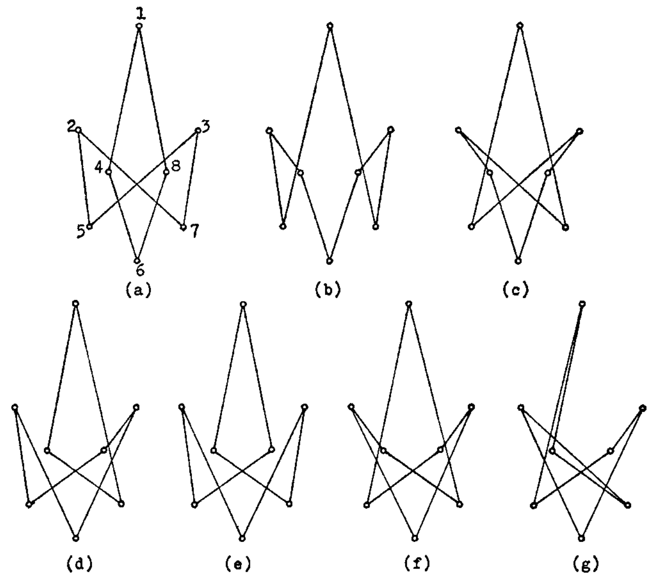


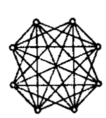
Fig. 21

The following table indicates the number of nonisomorphic factorizations of the graphs of degrees 3, 4, 5 and 6. The first column gives the total number, the second one shows the number of factorizations generated by 1-factors, the third column indicates the number of $2-2$, $2-3$ or $2-4$ factorizations and the fourth column gives the number of $3-3$ factorizations of CP (4).

S_1 :	3	3	
S_2 :	3	3	
S_3 :	1	1	
S_4 :	2	2	
S_5 :	2	2	
$2K_4$:	2	2	
$\overline{C_8}$:	37	7	30
$\overline{2C_4}$:	16	4	12
$\overline{C_3 \cup C_5}$:	8	1	7

$\bar{S}_1$:	7	3	4	
$\bar{S}_2$:	15	5	10	
$\bar{S}_3$:	5	2	3	
$\bar{S}_4$:	8	4	4	
$\bar{S}_5$:	9	4	5	
$\overline{2K_4}$:	3	1	2	
$CP(4)$:	26	2	12	12

CP(4)



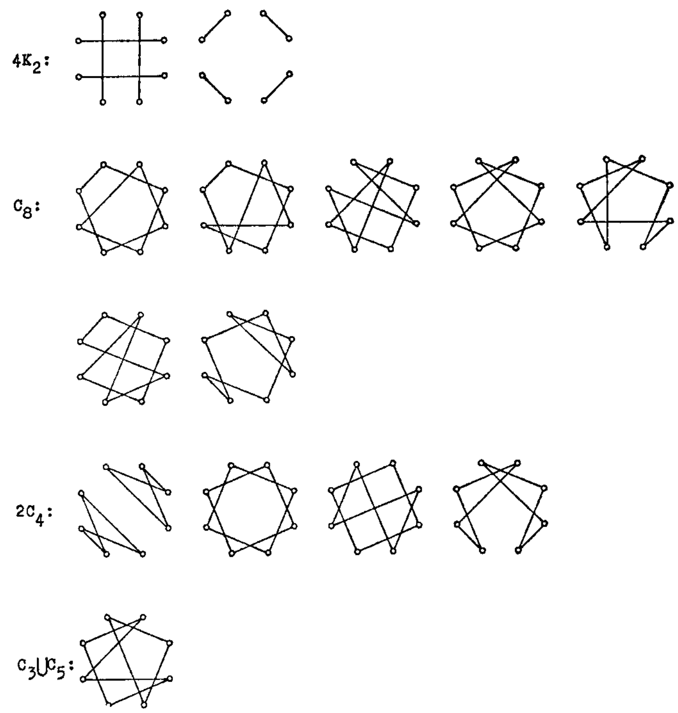


Fig. 22—1

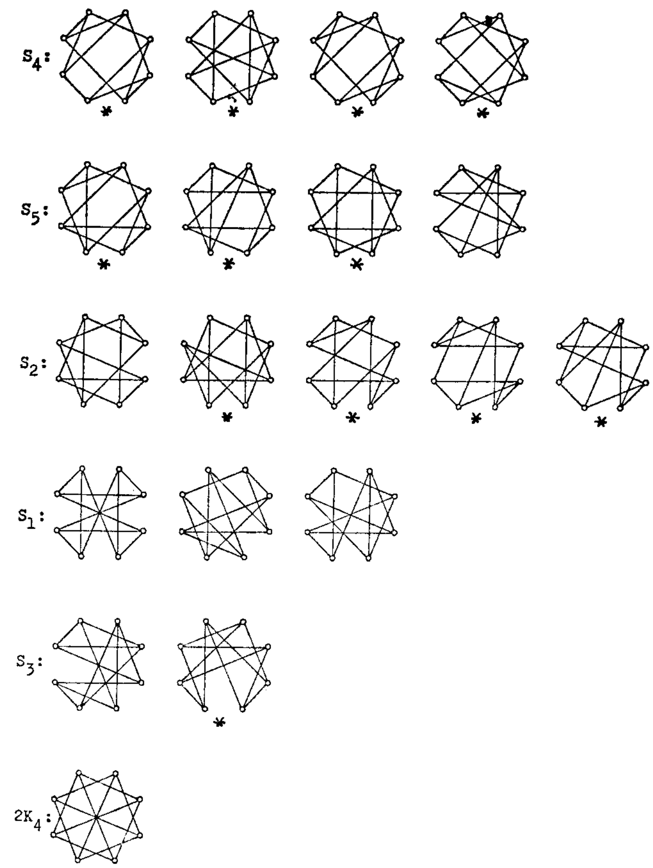


Fig. 22−2

REFERENCES

[1] F. C. Bussemaker, D. M. Cvetković *and* J. J. Seidel, *Graphs related to exceptional root systems. In: Combinatorics* (Coll. Mat. Soc. J. Bolyai 18, ed. A Hajnal and V. Sós). North-Holland Publ. Co., Amsterdam-Oxford-New York 1978, Vol. I, pp. 185–191; Technological University Eindhoven, T- H. -Report 76-WSK-05, 1–91.

[2] F. S. Bussemaker, S. Čobeljić, D. M. Cvetković *and* J. J. Seidel, *Computer investigation of cubic graphs.* Technological University Eindhoven, T. H. -Report 76-WSK-01; Cubic graphs on $\leq$ 14 vertices, J. Comb. Theory (B) 23 (1977), 234–235.

[3] M. Doob, *Graphs with a small number of distinct eigenvalues*, II (manuscript).

[4] F. Harary *and* E. M. Palmer, *Graphical Enumeration.* Academic Press, New York and London 1973. Russian translation, Moscow 1977.

[5] Z. Radosavljević, *Factorization of graphs with its aplication to the spectral theory of graphs* (in Serbo-Croatian), Master thesis, University of Beograd, Beograd 1980.