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ALMOST CONTINUITY AND NEARLY (ALMOST) PARACOMPACTNEES

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Abstract. The purpose of the present paper is to investigate some properties of nearly (almost) paracompactness under almost continuous mappings.

Notation is standard except that $\alpha(A)$ will be used to denote interior of the closure of A. The topology τ^* is the semi-regularization of τ and has as a base the regularly open sets from τ.

1. Preliminaries

Definition 1.1. A subset of a space is said to be *regular open* iff it is the interior of some closed set or equivalenty iff it is the interior of its own closure. A set is said to be *regularly closed* iff it is the closure of some open set or equivalenty iff it is the closure of its own interior ([**1**]).

A subset is regularly open iff its complement is regularly closed.

Definition 1.2. A space X is said to be *almost regular* iff for any regularly closed set F and any point $x \notin F$, there exists disjoint open sets containing F and x respectively ([**14**]).

A space X is almost regular iff for each point $x \in X$ and each regularly open set V containing x there exists a regularly open set U such that $x \in U \subset \overline{U} \subset V$ ([**14**]), Theorem 2.2).

Definition 1.3. A space X is *nearly paracompact* iff every regularly open cover of X has a locally finite open refinement ([**18**]).

Definition 1.4. A space X is *nearly strongly paracompact* iff every regularly open cover of X has a star finite open refinement ([**8**]).

Definition 1.5. A space X is said to be *almost paracompact* iff for every open covering $\mathcal{U}$ of X there exists a locally finite family of open sets $\mathcal{V}$ which refines $\mathcal{U}$ and is such that the family of closures of members of $\mathcal{V}$ forms a covering of X ([**15**]).

Definition 1.6. A space X is said to be *almost compact* iff each open covering of X has a finite subfamily the closures of whose members cover X ([**15**]).

Definition 1.7. A space X is said to be *almost strongly paracompact* iff for every open covering $\mathcal{U}$ of X there exists a star finite family of open sets $\mathcal{V}$ which refines $\mathcal{U}$ and is such that the family of closures of members of $\mathcal{V}$ forms a covering of X ([**8**]).

Definition 1.8. Let X be a topological space and A a subset of X. The set A is a *α-almost paracompact* iff for every X-open covering $\mathcal{U}$ of A there exists an X-locally finite family of X-open sets $\mathcal{V}$ which refines $\mathcal{U}$ and is such that the family of X-closures of members of $\mathcal{V}$ forms a covering of A ([**4**]).

Definition 1.9. A space X is *locally almost paracompact* iff each point of X has an open neighbourhood U, such that $\overline{U}$ is α-almost paracompact ([**4**]).

Definition 1.10. Let X be a topological space, and A a subset of X. The set A is *α-nearly paracompact* iff every X-regularly open cover of A has an X-open X-locally finite refinement which covers A ([**6**]).

Definition 1.11. A space X is *locally nearly paracompact* iff each point of X has an open neighbourhood U such that $\overline{U}$ is α-nearly paracompact ([**5**]).

Definition 1.12. Let X be a topological space and A, a subset of X. The set A is *α-nearly strongly paracompact* iff every X-regularly open cover of A has an X-open star finite refinement which covers A ([**7**]).

Definition 1.13. A topological space X is called *locally nearly strongly paracompact* iff each point of X has an open neighbourhood U such that $\overline{U}$ is an α-nearly strongly paracompact subset of X ([**7**]).

Definition 1.14. A subset A of a space X is *α-nearly compact (N-closed)* iff every X-regular open cover of A has a finite subcovering ([**17**]).

Definition 1.15. A topological space X is *locally nearly compact* iff each point has an open neighbourhood U such that $\overline{U}$ is an α-nearly compact subset of X ([**2**]).

Definition 1.16. A subset A of a space X is said to be *H-closed* iff for every X-open cover $\{U_\alpha : \alpha \in I\}$ of A, there exists a finite subset I_0 of I such that

$$A \subset \cup\{\overline{U}_\alpha : \alpha \in I_0\}.$$

Definition 1.17. A function $f : X \to Y$ is said to be *almost continous* iff for each point $x \in X$ and each open neighbourhood V of $f(x)$ in Y there exists an open neighbourhood U of x in X such that $f(U) \subset \alpha(V)$ ([**13**]).

A function is almost continous iff the inverse image of every regularly open set is open ([**13**], Theorem 2.2).

Definition 1.18. A function $f : X \to Y$ is said to be *almost closed (resp. almost open)* iff for every regularly closed (resp. regularly open) set F of $X, f(F)$ is closed (resp. open) in Y ([**13**]).

Definition 1.19. A closed set F of (X,τ) is said to be *star closed* iff F is closed in (X,τ^*), A function $f:X\to Y$ is said to be *star closed* iff for every star closed set F of X, $f(F)$ is closed in Y ([**11**]).

LEMMA 1.1. *If a mapping $f:X\to Y$ is almost continuous and almost closed, then*

1) *For each regularly closed set F of Y, $f^{-1}(F)$ is regularly closed in X,*
2) *For each regularly open set V of Y, $f^{-1}(V)$ is regularly open in X.*

Proof. 1. Let F be any regularly closed subset of Y. Then $f^{-1}(F)$ is closed, since f is almost continuous. Hence we have

$$\overline{[f^{-1}(F)]^0}\subset f^{-1}(F).$$

On the other hand, since f is almost continuous and F^0 is a non empty regularly open subset of Y, $f^{-1}(F^0)$ is non empty open and hence we have

$$f^{-1}(F^0)\subset [f^{-1}(F)]^0\subset \overline{[f^{-1}(F)]^0}.$$

Moreover, since f is almost closed and $\overline{[f^{-1}(F)]^0}$ is a regularly closed subset of X, $f\overline{(f^{-1}(F)]^0)}$ is closed. Hence we have

$$F=\overline{F^0}\subset f(\overline{[f^{-1}(F)]^0}).\ \text{Thus we obtain}\ f^{-1}(F)\subset \overline{[f^{-1}(F)]^0}.$$

This completes the proof of 1.

2) The proof of 2) follows easily from 1) and the following two facts:

a) $f^{-1}(Y\backslash F)=X\backslash f^{-1}(F)$ for each subset F of V.

b) F is regularly closed iff $Y\backslash F$ is regularly open subset of Y.

2. Nearly (almost) paracompactness

THEOREM 2.1. *Let X be a nearly paracompact amost regular space. If $f:X\to Y$ in an almost continuous, almost closed surjection, such that $f^{-1}(y)$ is an α-nearly compact subset of X for each point $y\in Y$, then Y is nearly paracompact almost regular.*

Proof. Since $f:F\to Y$ is almost continous and almost closed surjection such that $f^{-1}(y)$ is α-nearly compact for each $y\in Y$, Y is almost regular ([**5**]. Lemma 5).

Next, we shall shown that Y is nearly paracompact. Let $\mathcal{U}=\{U_\alpha:\alpha\in I\}$ be any regularly open cover of Y. Then by Lemma 1.1 $f^{-1}(\mathcal{U})=\{f^{-1}(U_\alpha):\alpha\in I\}$ is a regularly open cover of X. Since X is nearly paracompact, there exists a locally finite regularly open refinement $\mathcal{V}=\{V_\beta:\beta\in J\}$ or $f^{-1}(\mathcal{U})=\{f^{-1}(U_\alpha):\alpha\in I\}$. Since f is almost closed and $f^{-1}(y)$ is α-nearly compact for each $y\in Y$, $f(\mathcal{V})=\{f(V_\beta):\beta\in J\}$ is locally finite ([**11**], Lemma 2). Also, $f(\mathcal{V})$ covers Y and is a refinement of $\mathcal{U}$.

Hence $f(\mathcal{V})$ is a locally finite refinement of $\mathcal{U}$ and thus Y is nearly paracompact ([**18**], Theorem 1.5).

THEOREM 2.2. *Let f be any almost closed, almost continuous and almost open mapping of a space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$. Then, the image of an α-almost paracompact subset of X is an α-almost paracompact subset of Y.*

Proof. Let A be any α-almost paracompact subset of X. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any Y-regularly open cover of a subset $f(A)$. Sice f is almost continuous and almost open,

$$f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \alpha \in I\}$$

is an X-regularly open cover of A. Since A is α-almost paracompact, then there exists an X-open X-locally finite family $\mathcal{V} = \{V_\beta : \beta \in J\}$ which refines $f^{-1}(\mathcal{U})$ and is such that $\{\overline{V}_\beta : \beta \in J\}$ forms a covering of A. Consider the family

$$\mathcal{W} = \{\alpha(V_\beta) : \beta \in J\}.$$

Then $\mathcal{W}$ is an X-locally finite X-regularly open family which refines $f^{-1}(\mathcal{U})$ and is such that $\{\overline{\alpha(V_\beta)} : \beta \in J)$ forms a covering of A.

Since f is almot continuous, almost open and almost closed such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, $\{f(\alpha(V_\beta)) : \beta \in J\}$ is a Y-locally finite family which refines $\mathcal{U}$ ([**11**], Lemma 2). Since f is almost closed and almost continuous, therefore $f(\overline{\alpha(V_\beta)}) = \overline{f(\alpha(V_\beta))}$. Hence $\{f(\alpha(V_\beta)) : \beta \in J\}$ is a Y-locally finite family of Y-open subsets refining $\{U_\alpha : \alpha \in I\}$ and such that $\{f(\alpha(V_\beta)) : \beta \in J\}$ is a covering of $f(A)$. Hence $f(A)$ is α-almost paracompact ([**4**] Lemma 1.3).

COROLLARY 2.1. *If $f : X \to Y$ is any almost closed, almost continous and almost open mapping of an almost paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is almost paracompact.*

Proof. X is an α-almost paracompact subset of X. Therefore $f(X) = Y$ is an α-almost paracompact subset of Y, i.e. Y is almost paracompact.

COROLLARY 2.2. ([**15**]. *Theorem* 6.3.1). *If f is a closed, continuous, open mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is almost paracompact if X is almost paracompact.*

THEOREM 2.3. *If f is an almost closed, almost continuous, almost open mapping of a locally almost paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is locally almost paracompact.*

Proof. Let $y \in Y$. Then, there exists $x \in X$ such that $f(x) = y$. Since X is locally almost paracompact, there exists an open neighbourhood U of x such that $\overline{U}$ is α-almost paracompact subset of X. Then $\alpha(U)$ is a regularly open neighbourhood of x such that $\overline{\alpha(U)} = \overline{U}$ is α-almost paracompact subset of X.

Then, $f(\alpha(U))$ is a Y-open neighbourhood of y such that $f(\overline{\alpha(U)}) = \overline{f(\alpha(U))}$ is α-almost paracompact subset of Y, therefore Y is locally almost paracompact.

COROLLARY 2.3. ([**4**], *Theorem* 2.2.) *If f is any closed, continuous, open mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is locally almost paracompact if X is locally almost paracompact.*

LEMMA 2.1. *Let f be any almost open and almost continuous mapping of a space X onto a space Y such that $f^{-1}(G)$ is H-closed for each proper open subset $G \subset Y$. Then the image of any α-nearly paracompact subset of X is α nearly strongly paracompact subset of Y.*

Proof. Let A be any α-nearly paracompact subset of X. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any Y-regularly open cover of a subset $f(A)$. Since f is almost continuous and almost open,

$$f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \ \alpha \in I\}$$

is an X-regularly open cover of A. Since A is α-nearly paracompact, then there exists an X-regularly open X-locally finite family $\mathcal{V} = \{V_\beta : \ \beta \in J\}$ which refines $f^{-1}(\mathcal{U})$ and is such that

$$A \subset \cup\{V_\beta : \ \beta \in J\}.$$

Since f is almot continous, almost open that $f^{-1}(G)$ is H-closed for each proper open subset $G \subset Y$, $\{f(V_\beta) : \ \beta \in J\}$ is a Y-open star finite family which refines U, and is such that

$$f(A) \subset \cup\{f(U_\beta) : \beta \in J\}$$

([**9**]. Corollary 4.1). This implies that $f(A)$ is α-nearly strongly paracompact.

THEOREM 2.4. *If f is an almost open, almost closed and almost continuous mapping of a locally nearly paracompact space X onto a space Y such that $f^{-1}(G)$ is H-closed for every proper open set $G \subset Y$, then Y is locally nearly strongly paracompact.*

Proof. Let $y \in Y$ be any point. Then there exists $x \in X$ such that $f(x) = y$. Since X is locally nearly paracompact, there exists an open neighbourhood U of x such that $\overline{U}$ is α-nearly paracompact subset of X. Then, $\alpha(U)$ is a regularly open neighbourhood of x such that $\overline{\alpha(U)} = \overline{U}$ is α-nearly paracompact subset of X. Then, $f(\alpha(U))$ is a Y-open neighbourhood of y such that $\overline{f(\alpha(U))} = \overline{f(\alpha(U))}$ is α-nearly strongly paracompact, therefore Y is locally strongly paracompact.

LEMMA 2.2. *A space X is almost strongly paracompact iff for every regularly open covering of X there exists a star finite family of open sets which refines it and the closures of whose members cover the space X.*

Proof. Only the "if" partneeds to be proved.

Let $\{U_\lambda : \lambda \in I\}$ be any open covering of X. Then $\{\alpha(U_\lambda) : \lambda \in I\}$ is a regularly open covering of X. There exists an open star finite family $\{H_\beta : \beta \in J\}$

which refines $\{\alpha(U_\lambda) : \lambda \in I\}$ such that $X \cup \{\overline{H}_\beta : \beta \in J\}$. For each $\beta \in J$ there exists $\lambda(\beta) \in I$ such that $H_\beta \subset \alpha(U_{\lambda(\beta)})$. For each $\beta \in J$, let

$$M_\beta = H_\beta \backslash [\overline{U_{\lambda(\beta)}} \backslash U_{\lambda(\beta)}].$$

Since $H_\beta \subset \alpha(U_{\lambda(\beta)}) \subset \overline{U_{\lambda(\beta)}}$, therefore $M_\beta = H_\beta \cap U_{\lambda(\beta)}$.
Thus $\{M_\beta : \beta \in J\}$ is a star finite family of open sets which refines $\mathcal{U}$. We shall prove that

$$X = \cup \{\overline{M}_\beta : \beta \in J\}.$$

Let $x \in X$. Then $x \in \overline{H}_\beta$ for some $\beta \in J$. Now

$$\overline{M}_\beta = \overline{H_\beta \cap U_{\lambda(\beta)}} = \overline{H_\beta \cap \overline{U_{\lambda(\beta)}}} = \overline{H}_\beta.$$

Thus $x \in \overline{M}_\beta$. Hence $\{M_\beta : \beta \in J\}$ is an open star finite family which refines $\mathcal{U}$ and the closures of whose members cover the space X, therefore X is almost strongly paracompact.

THEOREM 2.3. *If f is an almost continous, almost open mapping of an almost paracompact space X onto a space Y such that $f^{-1}(G)$ in H-closed for each proper open set $G \subset X$, then Y is almost strongly paracompact.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any regularly open cover of Y. Then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a regularly open cover of X. There exists a locally finite family $\{V_\beta : \beta \in J\}$ of open sets refining $\{f^{-1}(U_\alpha) : \alpha \in I\}$ such that $X = \cup\{\overline{V}_\beta : \beta \in J\}$. Now, $\{\alpha(V_\beta) : \beta \in J\}$ is a locally finite family of regularly open sets refining $\{f^{-1}(U_\alpha : \alpha \in I\}$ and is such that $X = \cup\{\overline{\alpha(V_\beta)} : \beta \in I\}$. Since f is almost open and $f^{-1}(G)$ is H-closed for each proper open set $G \subset Y$, $\{f(\alpha(V_\beta)) : \beta \in J\}$ is a star finite family of open sets refining $\{U_\alpha : \alpha \in I\}$ ([**9**]), Lemma 4.2). Since $f(\overline{(\alpha(V_\beta))} \subset \overline{f(\alpha(V_\beta))}$, therefore $\{f(\alpha(V_\beta)) : \beta \in J\}$ is a star finite family of open sets refining $\{U_\alpha : \alpha \in I\}$ and is such that $Y = \cup\{\overline{f_\alpha(V_\beta))} : \beta \in J\}$.
Hence Y is almost strongly paracompact.

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