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INVERTIBLE AND WEAKLY INVERTIBLE SINGULAR INNER FUNCTIONS IN THE SPACES D^p

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Abstract. We show that singular inner function S_μ whose associated singular measure μ has the modulus of continuity $\omega_\mu(t) = 0\,(t\log 1/t)$ is weakly invertible in D^p, $0 < p < 2$, and that there exists a positive integer m such that $S_\mu^{1/m}$ is invertible D^p provided in $0 < p < 2$.

1. Introduction. In this paper, as the title suggests, we investigate the question of invertibility and weak invertibility of singular inner functions in the spaces D^p, $0 < p < 2$. In the third section we prove the following theorem.

THEOREM 1. *Let S_μ be a singular inner function whose associated singular measure μ, has the modulus of continuity $\omega_\mu(t) = 0\,(t\log 1/t)$. Then for each p, $0 < p < 2$, there exists a positive integer m such that $S_\mu^{1/m}$ is invertible in D^p.*

Conversely, if S_μ^m is invertible in D^p $0 < p < 2$, for some $m > 0$, then $\omega_\mu(t) = 0\,(t\log 1/t)$.

If $2 \leq p < \infty$, singular inner functions are not invertible in D^p, because $D^p \subset H^p$ (see [**1**] and [**2**], yet if S_μ is a singular inner function with $\omega_\mu(t) = 0\,(t\log 1/t)$, then, for each p $0 < p < \infty$, there exists a positive integer m such that $S_\mu^{1/m}$ is invertible in A^p (see [**6**]). As $D^p \subset A^p$ (see [**4**]) from the invertibility of a function $S_\mu^{1/m}$ in D^p, $0 < p < 2$, follows its invertibility in A^p.

A function $f \in D^p$ is weakly invertible in D^p if there exists a sequence of polynomials p_n such that $\lim_{n\leq\infty} p_n f = 1$, convergence being in the topology of D^p.

In **4.** we prove another theorem.

THEOREM 2. *Let S_μ be singular inner function with $\omega_\mu(t) = 0\,(t\log 1/t)$. Then S_μ is weakly invertible in D^p, $0 < p < 2$.*

As in the case of invertibility, if $2 \leq p < \infty$, singular inner functions are not weakly invertible in D^p; yet if S_μ is a singular inner function with $\omega_\mu(t) =$

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$0\,(t\log 1/t)$, then S_μ is weakly invertible in A^p, for each p, $0 < p < \infty$ (see [**6**]). It is clear that from the weak invertibility in D^p, $0 < p < 2$, follows the weak invertibility in A^p.

The complete characterization of weak invertible singular inner functions in spaces D^p, $0 < p < 2$, and A^p, $0 < p > \infty$, is still an open problem.

2. Preliminaries. For f analytic in $|z| < 1$ and $0 < r < 1$, let

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_o^{2\pi} |f(re^{it}|^p dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \max_t |f(re^{it})|.$$

Then for $0 < p < \infty$ H^p denotes the linear space of analytic functions for which $\sup\limits_r M_p(r,f) < \infty$.

For each $0 < p < \infty$, we denote by A^p the linear space of analytic functions on $|z| < 1$ for which

$$\|f\|_{A^p}^p = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} |f(re^{it})|^p r\,dr\,dt < \infty.$$

For $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ analytic in $|z| < 1$ and $0 < r < 1$, we define the area function $A(r)$ by

$$A(r) = A(r,f) = \int_0^r\int_0^{2\pi} |f'(se^{it})|^2 s\,ds\,dt = \pi\sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

so that $A(r)$ is the area of the image of $|z| < r$ under f, multiply covered points being counted multiply.

For each $0 < p < \infty$, we denote by D^p the linear space of analytic functions of f on $|z| < 1$ for which

$$\|f\|_{D^p}^p = \int_0^1 (A(r,f)/r)^{p/2} dr < \infty.$$

LEMMA 1. *Let f be analytic in $|z| < 1$ with $f(0) = 0$. Then if*

$$2 \le p, \ \ \|f\|_{A^p} \le \|f\|_{H^p} \le c_p\|f\|_{D^p},$$

where $c_p = p\pi^{-1/p}2^{-1/2}$, while if

$$0 < p \le 2, \ \ \|f\|_{H_p} \ge c_p\|f\|_{D_p} \ge k_p\|f\|_{A_p},$$

where $c_p = p\pi^{-1/p}2^{-1/2}$, $k_p = p2^{-1/2}\pi^{(p-2)/2p}$.

Proof. Let $2 \le p < \infty$. Then

$$\|f\|_{A_p} = \left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi}|f(se^{it})|^p s\,ds\,dt\right)^{1/p} \le \left(\int_0^1 M_p^p(s,f)ds\right)^{1/p}$$
$$\le M_p(1,f) = \|f\|_{H_p}.$$

The second inequality is proved in [**2**] (Theorem 2, p. 278).

If $0 < p \le 2$, then

$$\|f\|_{D^p} = \left(\int_0^1 (A(t,f)/t)^{p/2}dt\right)^{1/p} \ge \sqrt{\pi}\left(\int_0^1 M_2^p(t,f)dt\right)^{1/p}$$
$$\ge \sqrt{\pi}\left(\int_0^1 M_p^p(t,f)dt\right)^{1/p} \ge \sqrt{\pi}\|f\|_{A^p}.$$

The first inequality is proved in [**2**] (Theorem 2, p. 278).

Corollary 1. *If $2 \le p < \infty$, then $D^p \subset H^p \subset A^p$. If $0 < p \le 2$, then $H^p \subset D^p \subset A^p$. All inclusions are proper, unless $H^2 = D^2$.*

The functions $f(z) = (1-z)^{-1/p}$, $0 < p < \infty$, belong to $A^p \setminus D^p$ and $A^p \setminus H^p$.

There is a real sequence (ε_n), satisfying $|\varepsilon_n| = 1$ for all $n \ge 1$, such that

$$f(z) = \sum_1^\infty \varepsilon_n n^{-1/2}(\log(n+1))^{-1}z^n \in H^p,$$

for each $p > 0$, while $f(z) \notin \bigcup_{2<p<\infty} D^p$.

There is a real sequence (δ_n), satisfying $|\delta_n| = 1$ for all $n \ge 1$, such that

$$g(z) = \sum_1^\infty \delta_n^{-1/2} z^n \notin H^p,$$

for each $p > 0$, although $g(z) \in D^p$, for each $0 < p < 2$.

An analytic function $f(z)$ in $|z| < 1$ satisfying $|f(z)| \le 1$ for all z, $|z| < 1$ and $|f(e^{it})| = 1$ a. e. is said to be an inner function. Every inner function $f(z)$ has a factorization $e^{i\gamma}B(z)S_\mu(z)$ (see [**1**]) where $B(z)$ is Blaschke product and S_μ is a singular inner function given by

$$S_\mu(z) = \exp\left[-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right] \tag{2.1}$$

where $\mu(t)$ is a bounded nondecreasing singular function, i. e. $\mu'(t) = 0$ a. e.. The modulus of continuity of μ is denoted by ω_μ and is defined by

$$\omega_\mu(\delta) = \sup_{x,|t|\le\delta}|\mu(x+t)-\mu(x)|, \text{ for } \delta > 0.$$

We shall need the following two lemmas.

Lemma 2. (see [**S**]). *Let S_μ be a singular inner function given by* (2.1). *Then*

$$|S_\mu(re^{it})| > \exp[-c\omega_\mu(1-r)/(1-r)]$$

for some positive constant c.

Lemma 3. (see [**1**, p. 109]) *Let $P(z, e^{it}) = (1-|z|^2)|z-e^{it}|^{-2}$ be the Poisson kernel on $|z|<1$. If $\mu(t)$ is a bounded nondecreasing singular function on $[0, 2\pi]$ with $\omega_\mu(t) = 0\,(t \log 1/t)$, then the positive constant k_1 and k_2 exist such that*

$$\int_0^{2\pi} P(z, e^{it}) d\mu(t) \le k_1 + k_2 \log 1/(1-|z|).$$

3. Proof of Theorem 1. By hypothesis there exists a constant $k_3 > 0$ such that

$$\omega_\mu(1-r) \le k_3(1-r)\log 1/(1-r). \tag{3.1}$$

By using Lemma 2 and (3.1) we obtain

$$A(r, S_\mu^{-1/m}) = \int_o^r \int_0^{2\pi} \Bigg\{ |S_\mu(se^{it})|^{-2/m} \cdot$$

$$\cdot\, m^{-2} \left| \int_0^{2\pi} 2e^{i\theta}(e^{i\theta} - se^{it})^{-2} d\mu(\theta) \right|^2 \Bigg\} sdsdt \le$$

$$\le 4rm^{-2} \int_0^r \int_0^{2\pi} \Bigg\{ \exp[2cm^{-1}\omega_\mu(1-s)/(1-s)] \Bigg[(1- \tag{3.2}$$

$$- s^2)^{-1} \int_0^{2\pi} P(se^{it}, e^{i\theta}) d\mu(\theta) \Bigg]^2 \Bigg\} dsdt \le$$

$$\le 4m^{-2} \int_0^r (k_1 + k_2 \log 1/(1-s))^2 (1-s)^{-2sk_3m^{-1}-2} ds,$$

where c is the constant of Lemma 2 in (2.2).

By (3.2),

$$A(r, S_\mu^{-1/m}) \le kr(1-r)^{-2ck_3m^{-1}-\varepsilon-1} \tag{3.3}$$

for some $k > 0$ and for all $\varepsilon > 0$. Let us choose a positive integer m large enough and $\varepsilon > 0$ such that $p(2ck_3 + \varepsilon m + m) < 2m$.

Then by (3.3) $S_\mu^{-1/m} \in D^p$.

Conversely, if S_μ^m is invertible in D^p, $0 < p < 2$, for some $m > 0$, then S_μ^m is invertible in A^p. By Theorem 2 ([**6**, p. 504]) $\omega_\mu(t) = 0\,(t \log 1/t)$.

4. Weakly invertible singular inner functions in D^p. If E is a subspace of D^p we denote by $[E]$ the closure of E in D^p. Also if E is a subspace of D^p and $f \in H^\infty$, $fE = \{fg/g \in E\}$.

LEMMA 4. (see [**3**]) *If $f \in D^p$, $0 < p < \infty$, then $f_r \to f$ in D^p, where for $0 < r < 1$, $f_r(z) = f(rz)$.*

LEMMA 5. *The set of polynomials in z is dense in D^p.*

Proof. Let $f \in D^p$, $0 < p < \infty$. Given $\varepsilon > 0$, there exists a r_0 such that $\|f - f_{r_0}\|_{D^p} < \varepsilon/2$ (by Lemma 4). Let $p_n(z)$ denote the n^{th} partial sum of the Taylor series of $f_{r_0}(z)$. Then $p_n \to f_{r_0}$ uniformly on $|z| \leq 1$, and thus $\|p_n - f_{r_0}\|_{D^p} \to 0$. Hence there exists an integer n such that $\|p_n - f\|_{D^p} < \varepsilon$, which proves the Lemma.

LEMMA 6. *Let $f \in H^\infty$, $g \in D^p$, $0 < p \leq 2$, $g(0) = 0$. Then*

$$\|fg\|_{D^p} \leq C_p \|f\|_{H^\infty} \|g\|_{D^p}.$$

Proof. Case $p = 2$.

$$\|fg\|_{D^2}^2 \leq \pi/2 M_2^2(1, fg) \leq \pi/2 \|f\|_{H^\infty}^2 \|g\|_{H^2}^2 \leq \|f\|_{H^\infty}^2 \|g\|_{D^2}^2.$$

Therefore $\|fg\|_{D^2} \leq \|f\|_{H^\infty} \|g\|_{D^2}$.

Let $0 < p < 2$. Then

$$\begin{aligned}
\|fg\|_{D^p}^p &= \int_0^1 \left(\int_0^r \int_0^{2\pi} |f'(se^{it})g(se^{it}) + f(se^{it})g'(se^{it})|^2 s\,ds\,dt \right)^{p/2} r^{-p/2} dr \\
&\leq \int_0^1 \left\{ \left[\int_0^r \int_0^{2\pi} |f'(se^{it})g(se^{it})|^2 s\,ds\,dt \right]^{1/2} + \right. \\
&\qquad \left. + \left[\int_0^r \int_0^{2\pi} |f(se^{it})g'(se^{it})|^2 s\,ds\,dt \right]^{1/2} \right\}^p r^{-p/2} dr \qquad (4.1) \\
&\leq c_p \left\{ \int_0^1 \left[\int_0^r \int_0^{2\pi} |f'(se^{it})g(se^{it})|^2 s\,ds\,dt \right]^{p/2} r^{-p/2} dr + \right. \\
&\qquad \left. + \int_0^1 \left[\int_0^r \int_0^{2\pi} |f(se^{it})g'(se^{it})|^2 s\,ds\,dt \right]^{p/2} r^{-p/2} dr \right\}
\end{aligned}$$

where $c = 1$ if $0 < p \leq 1$ and $c_p = 2^{p-1}$, if $1 \leq p < 2$. It is clear that

$$\int_0^1 \left[\int_0^r \int_0^{2\pi} |f(se^{it})g'(se^{it})|^2 s\,ds\,dt \right]^{p/2} r^{-p/2} dr \leq \|f\|_{H^\infty}^p \|g\|_{D^p}^p \qquad (4.2)$$

Using Cauch's integral formula we obtain

$$|f'(se^{it})| \le 4\|f\|_{H^\infty}/(1-s).$$

Therefore

$$\int_0^r \int_0^{2\pi} |f'(se^{it})g(se^{it})|^2 sdsdt \le 32\pi\|f\|_{H^\infty} \int_0^r M_2^2(s,g)(1-s)^{-2}sds$$

$$\le 96\|f\|^2_{H^\infty} \int_0^r \left[(1-s)^{-2}\left(\int_0^s A(\rho,g)d\rho\right)\right]ds.$$

Let $F(s) = (1-s)^{-2}\int_0^s A(r,g)dr$. Clearly, the function $F(s)$ is non-decreasing on $(0,1)$. Now let $r_n = (1-2^{-n})r$, $n = 0,1,2,\dots$. Then $r - r_n = r_n - r_{n-1} = r2^{-n}$. So we have

$$\left[\int_0^r F(s)ds\right]^{p/2} = \left(\sum_{n=1}^{\infty}\int_{r_{n-1}}^{r_n} F(s)ds\right)^{p/2} \le \sum_{n=1}^{\infty}[F(r_n)]^{p/2}r^{p/2}2^{-np/2}$$

$$\le 2\sum_{n=1}^{\infty}\int_{r_n}^{r_{n+1}}[F(s)]^{p/2}(r-s)^{p/2-1}ds$$

$$= 2\int_0^r [F(s)]^{p/2}(r-s)^{p/2-1}ds \tag{4.4}$$

$$= 2\int_0^r \left\{(1-s)^{-p}(r-s)^{p/2-1}\left(\int_0^s A(\rho,g)d\rho\right)^{p/2}\right\}ds$$

In the same way we obtain

$$\left[\int_0^s A(\rho,g)d\rho\right]^{p/2} \le 2\int_0^s (s-\rho)^{p/2-1}[A(\rho,g)]^{p/2}d\rho. \tag{4.5}$$

By (4.3), (4.4) and (4.5), we have

$$\int_0^1 \left[\int_0^r \int_0^{2\pi} \left|f'(s^{it})g(se^{it})\right|^2 sdsdt\right]^{p/2} r^{-p/2}dr \tag{4.6}$$

$$\le k_p^{(1)}\|f\|^p_{H^\infty}\int_0^1\left\{\int_0^r\left[(1-s)^{-p}(r-s)^{p/2-1}\right.\right.$$

$$\left.\left.\left(\int_0^s (s-\rho)^{p/2-1}[A(\rho,g)]^{p/2}d\rho\right)\right]ds\right\}r^{-p/2}dr$$

$$= k_p\|f\|^p_{H^\infty}\int\limits_0^1 ds\int\limits_s^1\left\{(1-s)^{-p}(r-s)^{p/2-1}\left(\int\limits_0^s(s-\rho)^{p/2-1}[A(\rho,g)]^{p/2}d\rho\right)r^{-p/2}\right\}dr$$

$$\le k_p\|f\|^p_{H^\infty}\int\limits_0^1\left\{(1-s)^{-p/2}s^{-p/2}\left(\int\limits_0^s(s-\rho)^{p/2-1}[A(\rho,g)]^{p/2}d\rho\right)\right\}ds$$

$$= k_p\|f\|^p_{H^\infty}\int\limits_0^1 d\rho\int\limits_\rho^1(1-s)^{-p/2}s^{-p/2}(s-\rho)^{p/2-1}[A(\rho,g)]^{p/2}ds$$

$$\le k_p\|f\|^p_{H^\infty}\int\limits_0^1\left(\int\limits_\rho^1(1-s)^{-p/2}(s-\rho)^{p/2-1}ds\right)[A(\rho,g)/\rho]^{p/2}d\rho$$

As

$$\int\limits_\rho^1(1-s)^{-p/2}(s-\rho)^{p/2-1}ds=\pi(\sin p\pi/2)^{-1},$$

by (4.1), (4.2) and (4.6)

$$\|fg\|_{D^p}\le C_p\|f\|_{H^\infty}\|g\|_{D^p}.$$

Using Lemma 5 and Lemma 6 one can easily prove the following two lemmas.

LEMMA 7. *Let $f\in H^\infty$. Then f is weakly invertible in D^p, $0<p\le 2$, if and only if $[fD^p]=D^p$.*

LEMMA 8. *If $f_1,f_2\in H^\infty$ and E is a closed subspace of D^p, $0<p\le 2$, then $[f_1f_2E]=[f_1[f_2E]]$.*

Theorem 2 is now a consequence of Theorem 1, Lemma 7 and Lemma 8.

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