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THE REGULATION NUMBER OF A GRAPH

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Abstract. The regulation number $r(G)$ of a graph G with maximum degree d is defined as the smallest number of new points in a d-regular supergraph. It is shown that for $d \geq 3$, every possible value of $r(G)$ between zero and the maximum established by Akiyama, Era and Harary, namely, $d(\text{mod } 2){+}1 + d$, is realized by some graph. Also, a characterization is given for G to have $r(G) = n$.

1. Introduction. The *regulation number* $r(G)$ of a graph G with maximum degree d is the maximum number of new point needed to get a d-regular supergraph. Akiyama, Era and Harary [**1**] determined the following bounds.

THEOREM A. *For a graph G with maximum degree $d \geq 3$,*

$$r(G) \leq d + 2 \text{ when } d \text{ is odd,} \tag{1}$$

$$r(G) \leq d + 1 \text{ when } d \text{ is even} \tag{2}$$

Our first purpose is to demonstrate the interpolation theorem that for each n between 0 and the upper bounds in (1) and (2), there exists a graph with regulation number n. This is accomplished by constructing such a graph.

A necessary and sufficient condition is then derived for a graph to have regulation number n, using the notion of an "f-factor" due to Tutte, [**5**].

In general we follow the notation and terminology of [bf 4].

2. Interpolation. We shall show that for each $d \geq 3$, every integer n between zero , the smallest possible value of $r(G)$, and the maximum value $d+1$ od $d+2$ depending on the parity of d, n is realized as the regulation number of some graph. In the construction of such a graph, it is convenient to use the notation $G_1 + G_2 + G_3$ of [**2**] for the iterated join of three disjoint graphs G_i defined as the union $(G_1+G_2) \cup (G_2+G_3)$. Similarly, the iterated join of $n \geq 3$ disjoints graphs is written $G_1+G_2+G_3+\cdots+G_n$ and is defined as $(G_1+G_2) \cup (G_2+G_3) \cup \cdots \cup (G_{n-1}+G_n)$. We shall encounter the special case $K_1 + K_1 + \cdots + K_1 + G_{k+1} + \cdots + G_n$ where $G_{k+1} \neq K_1$ and will abbreviate it by $P_k + G_{k+1} + \cdots + G_n$ (as i this case the join of the first k copies of K_1 gives the path P_k).

THEOREM 1. *Let* $d \geq 3$.

1. *If* d *is odd and* $0 \leq n \leq d+2$, *then there is a graph* H_n *with maximum degree* d *and* $r(H_n) = n$.

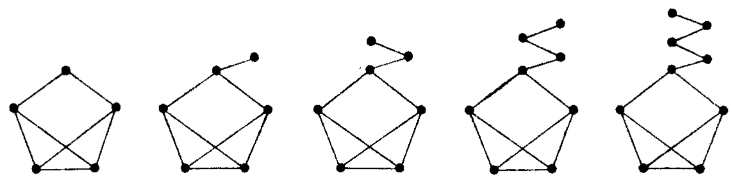
2. *If* d *is even and* $0 \leq n \leq d+1$, *then there is a graph* J_n *with maximum degree* d *and* $r(J_n) = n$.

Proof. When $n = 0$ and $d \geq 3$ is odd, one can take H_0 as a d-regular graph or as a spanning supgraph of such a graph. For $n = d+2$, we have $H_n = K_1 + \bar{K}_2 + K_{d-1}$. (The case $d = 3$ was illustrated in [**1**]). Now for any positive integer n properly between 0 and $d+2$, one possible choice is

$$H_n = P_{d-n+3} + \bar{K}_2 + K_{d-1}.$$

The proof when d is even is analogous, with

$$J_n = P_{d-n+2} + \bar{K}_2 + K_{d-1}. \ \square$$



$\mathbf{H_5}$ $\mathbf{H_4}$ $\mathbf{H_3}$ $\mathbf{H_2}$ $\mathbf{H_1}$

Figure 1. Realization graphs for regulation number interpolation

Figure 1 shows the graphs H_1 to H_5 when $d = 3$. The smallest 3-regular graph containing these H_n is shown in Figure 2. As noted in [bf 4], this is the smallest cubic graph with a bridge.

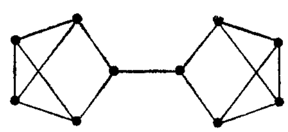


Figure 2. The smallest cubic graph with a bridge

3. Characterization, Let G be any graph with p points $V = \{1, 2, \ldots p\}$. Let $f = (f_1, \ldots, f_p)$ be a vector of p non-negative integers. Then an *f-factor* is a spanning subgraph F of G such that the degree of point i in F is f_i. We recall the following result of Tutte [**5**] giving a criterion for the existence of an f-factor.

THEOREM B. *A graph* G *has an f-factor if and only if for any two disjoint subsets* X *and* Y *of* V, *with* $o(X,Y)$ *the number of odd components of* $G - X - Y$, *and* $d(i, G-X)$ *the degree of* i *in* $G - X$ *we have*

$$o(X,Y) + \sum_{i \in Y} \{f_i - d(i, G-X)\} \leq \sum_{i \in D} f_i. \tag{3}$$

Let $d_i = d(i, G)$ and let the *deficiency* of v_i in G be $f_i = d - d_i$. Then it can easily be verified that G has regulation number 0 if and only if $\bar{G}$ has an f-factor, where $f = (f_1, \ldots, f_p)$ is the vector of deficiencies. We will extend this observation to obtain a criterion for a graph to have regulation number n. Fix n properly between 0 and $d + 2$ and define the join $I_n = \bar{G} + P_n$, with the additional points labelled $p + 1, \ldots, p + n$. Set $I_0 = G$. If $n > 0$, let $f_j = d$ for $j = p + 1, \ldots, p + n$ and set $f = (f_1, \ldots, f_{p+n})$.

THEOREM 2. *Let $0 \leq n \leq d + 2$ and let G be a graph with maximum degree d. Then $r(G) = n$ if and only if n is smallest integer such that I_n has an f-factor.*

Proof. Suppose $r(G) = n$ and consider the set of lines added to $G + \bar{K}_n$ to form a d-regular graph. These edges form an f-factor in I_n. Suppose there is some integer $j < n$ such that I_j contains an f-factor. Then it is easily verified that these edges would regularize $G + K_j$, contradicting the fact that $r(G) = n$. The converse holds by a similar argument. □

Theorem A and Theorem 2 together yield an algorithm which can be used to determine $r(G)$ for a given graph G. However, the paper [**3**] by Erdoös and Kelly implicitly contains an $0(n)$ algorithm for this purpose even though they studied and determined the induced regulation number of a graph.

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REFERENCES

[1] J. Akiyama, H. Era, F. Harary, *Regular graphs containing a given graph*, Elem. Math. (to appear)

[2] J. Akiyama, G. Exoo, F, Harary, *A graph and its complement with specified properties V: The self-complement index* Mathematika **27** (1980), 64–68.

[3] P. Erdoös, P.J. Kelly, *The minimal regular graph containing a given graph*, Amer. Math. Monthly **70** (1963), 1074–1075.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[5] W. T. Tutte, *The factors of graphs* Canad. J. Math. **4** (1952), 314–328.

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