

ONE OF THE POSSIBLE FORMAL DESCRIPTIONS OF DEDUCIBILITY

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Abstract. Having in mind different investigations of implication, i. e., of the logical consequence relation, we will try to point out a general kernel of formal systems in which the deducibility relation is stated in the system itself. In connection with any formal theory θ we observe a formal theory $\theta(\rightarrow)$ which is able to define the fundamental factor of θ -deducibility. By showing that the basic binary relation of $\theta(\rightarrow)$ is just a formal description of the metatheoretic deducibility relation of θ , the essential statement, the assertion 2.9., justifies contemplation of a formal theory like $\theta(\rightarrow)$. Furthermore, by the assertions 3.3 and 3.4. an interesting connection between formal theories $\theta(\sim)$ (cf. [1]) and $\theta(\rightarrow)$ is given.

1. Gentzen's idea of sequent calculi and the paper [1] of S. B. Prešić have influenced immediately on this contemplation. We will try to describe a procedure by which we can assign an *inequational formal theory*, i. e., a formal theory in which some binary predicate is a preordering, to any formal theory. As things stand, a deduction relation (denoted by \vdash and defined in the usual way) is a preordering, relation and so every logical system is in connection with some preordered systems.

2. Let θ be a formal theory. By $\theta(\rightarrow)$ we will denote a formal theory defined as follows:

2.1. The set of basic symbols of the initial formal theory θ will be extended by three **new** symbols: \top , $\&$ and \rightarrow ; \top is a new constant, $\&$ is a binary operational symbol and \rightarrow is a binary predicate symbol.

The fundamental notions are defined as usual:

2.2. *Definition.* (i) \top and all formulas of θ are *preformulas* of $\theta(\rightarrow)$.

(ii) If A and B are *preformulas* of $\theta(\rightarrow)$, then $\&AB$ is a *preformula* of $\theta(\rightarrow)$.

(iii) *Preformulas* are only those expressions obtained by (i) and (ii)

2.3. *Definition.* If A and B are preformulas of $\theta(\rightarrow)$, then $A \rightarrow B$ is a *formula* of $\theta(\rightarrow)$.

The set of all formulas of the formal theory θ will be denoted by $\text{For}(\theta)$. Henceforth, A, B, C, D, A_1, \dots and F, G, F_1, G_1, \dots will be metavariables ranging over preformulas of $\theta(\rightarrow)$ and formulas of θ , respectively.

2.4. *Axioms* of $\theta(\rightarrow)$ are defined by the following:

- (i) $F \rightarrow F, F \rightarrow \top, \&AB \rightarrow \&BA$.
- (ii) If F is an axiom (schemes) of θ , then $\top \rightarrow F$ is an axiom (scheme) of $\theta(\rightarrow)$.
- (iii) If $\frac{G_1, \dots, G_{k-1}, G_k}{G}$ is a rule (scheme) of inference of θ , then the formula $\&G_1 \& \dots \&G_{k-1} G_k \rightarrow G$ is an axiom (scheme) of $\theta(\rightarrow)$.
- (iv) Axioms are only those formulas obtained by (i), (ii) and (iii),

2.5. The *rule schemes* of $\theta(\rightarrow)$ are

$$\text{(IKS)} \frac{A \rightarrow B \quad A \rightarrow C}{A \rightarrow \&BC}, \quad \text{(IKA)} \frac{A \rightarrow B}{\&AC \rightarrow B}, \quad \text{(TR)} \frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}.$$

2.6. LEMMA. (1) $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow A}$;

- (2) $\frac{}{\vdash_{\theta(\rightarrow)} \&\&ABC \rightarrow \&A\&BC}$; $\frac{}{\vdash_{\theta(\rightarrow)} \&A\&BC \rightarrow \&\&ABC}$;
- (3) $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow \top}$; $\frac{}{\vdash_{\theta(\rightarrow)} \&A\top \rightarrow A}$;
- (4) if $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow B}$ and $\frac{}{\vdash_{\theta(\rightarrow)} C \rightarrow D}$ then $\frac{}{\vdash_{\theta(\rightarrow)} \&AC \rightarrow \&BD}$;
- (5) $\frac{}{\vdash_{\theta(\rightarrow)} \&AB \rightarrow C}$ iff $\frac{}{\vdash_{\theta(\rightarrow)} \&BA \rightarrow C}$;
- (6) $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow \&BC}$ iff $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow \&CB}$;
- (7) $\frac{}{\vdash_{\theta(\rightarrow)} \&\&ABC \rightarrow D}$ iff $\frac{}{\vdash_{\theta(\rightarrow)} \&A\&BC \rightarrow D}$;
- (8) $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow \&\&BCD}$ iff $\frac{}{\vdash_{\theta(\rightarrow)} A \rightarrow \&B\&CD}$;
- (9) if $A \rightarrow \&BC$ is k -provable in $\theta(\rightarrow)$, then $A \rightarrow B$ is k_1 -provable and $A \rightarrow C$ is k_2 -provable in $\theta(\rightarrow)$, where $k_1, k_2 < k^1$.

Proof. (1)–(8) Directly—using the rule schemes of $\theta(\rightarrow)$. (9) By induction on k .

According to 2.6. Lemma (5)–(8), we see that the preformulas $\&AB$ and $\&\&ABC$ can be replaced by $\&BA$ and $\&A\&BC$, respectively, in the framework of $\theta(\rightarrow)$, and conversely².

2.7. LEMMA. Iff $\frac{}{\vdash_{\theta} F}$, then $\frac{}{\vdash_{\theta(\rightarrow)} \top \rightarrow F}$.

Proof. By induction on the length of the proof for F in θ .

2.8. COROLLARY. If $\frac{}{\vdash_{\theta} F_1}$ and \dots and $\frac{}{\vdash_{\theta} F_n}$, then $\frac{}{\vdash_{\theta(\rightarrow)} \top \rightarrow F_1 \& \dots \& F_n}$ ($n \geq 1$).

2.9. THEOREM. $F_1 \dots, F_n \frac{}{\vdash_{\theta} F}$ iff $\frac{}{\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow F}$ ($n \geq 1$).

¹ F is k -provable in θ iff the length of the shortest proof of F in θ is k .

²So, we can write $A\&B$ instead of $\&AB$.

Proof. The “only if” part. By induction on the length m of the proof of $F_1, \dots, F_n \vdash_{\theta} F$.

Case $m = 1$: If F is the axiom of θ , then $\top \rightarrow F$ is the axiom of $\theta(\rightarrow)$. By 2.6. Lemma (3) $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow \top$, and by (TR) we have $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow F$. If F is F_1 for some $1 \leq i \leq n$, say that F is F_1 , then $F_1 \rightarrow F$ is the axiom of $\theta(\rightarrow)$. Hence, using the rule scheme (IKA) we have $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow F$.

Case $m > 1$: The following subcases are possible:

(i) F is either the axiom of θ or F_1 (for some $1 \leq i \leq n$);

(ii) F is the consequence of some preceding formulas by the rule (scheme) $\frac{G_1, \dots, G_k}{F}$. The subcase (i) is the same as the case $m = 1$. Subcase (ii): by induction hypothesis: $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_1$,

$$\dots \dots \dots \vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_k;$$

using rule scheme (IKS) (K-1) times, we have

$$\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_k.$$

$G_1 \& \dots \& G_k \rightarrow F$ is the axiom scheme of $\theta(\rightarrow)$. Therefrom, by (TR) we derive $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow F$.

The “if” part. By induction on the number m of uses of the rule schemes (IKS), (IKA) and (TR).

Case $m = 0$: $F_1 \& \dots \& F_n \rightarrow F$ is the axiom of $\theta(\rightarrow)$. Then either $\frac{F_1, \dots, F_n}{F}$ is the rule of θ or $n = 1$ and F_1 is F or F_1 is \top or F is \top . In any case we have $F_1, \dots, F_n \vdash_{\theta} F$.

Case $m > 0$: The following subcases are possible:

(i) in the last step of the proof we used the rule scheme (IKA) on the formula $F_1 \& \dots \& F_i \rightarrow F$ (for some $1 \leq i \leq n$);

(ii) in the last step of the proof we used the rule scheme (TR) on the formulas $F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_k$ and $G_1 \& \dots \& G_k \rightarrow F$. Subcase (i): by induction hypothesis

$$F_1, \dots, F_i \vdash_{\theta} F$$

and so $F_1, \dots, F_i, F_{i+1}, \dots, F_n \vdash_{\theta} F$. Subcase (ii): by 2.6 Lemma (9) we have

$$\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_1,$$

...

$$\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_k, \text{ and by induction hypothesis:}$$

$$F_1, \dots, F_n \vdash_{\theta} G_1,$$

...

$$F_1, \dots, F_n \vdash_{\theta} G_k,$$

therefrom $F_1, \dots, F_n \vdash_{\theta} F$ which was to be demonstrated.

In accordance with 2.7. Lemma, and 2.9. Theorem, we see that the binary predicate \rightarrow of $\theta(\rightarrow)$ is, in fact, a formalization of the deduction relation $\frac{}{\theta}$, the binary operation $\&$ is related to the metatheoretic “and”, while the constant \top characterizes the set $\text{Th}(\theta)$ of all theorems of θ in the sense that $\text{Th}(\theta) = \{F \mid \top \rightarrow F \text{ is provable in } \theta(\rightarrow)\}$.

Assording to (TR) and 2.6. Lemma (1), we will call $\theta(\rightarrow)$ an *inequational* description of θ .

2.10. THEOREM. $\text{For}(\theta) = \text{Th}(\theta)$ iff $\text{For}(\theta(\rightarrow)) = \text{Th}(\theta(\rightarrow))$.

Proof. Let $F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_m$ be any formula of $\theta(\rightarrow)$. If $\text{For}(\theta) = \text{Th}(\theta)$, then, in accordance with 2.7. Lemma, $\frac{}{\theta(\rightarrow)} \top \rightarrow G_1, \dots, \frac{}{\theta(\rightarrow)} \top \rightarrow G_m$; so using the rule scheme (IKS) $\frac{}{\theta(\rightarrow)} \top \rightarrow G_1 \& \dots \& G_m$. According to 2.6. Lemma (3), $\frac{}{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow \top$, and so, by (TR) we have $\frac{}{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_m$. On the other hand, if $\text{For}(\theta(\rightarrow)) = \text{Th}(\theta(\rightarrow))$, then all formulas of the form $\top \rightarrow F$ are theorems of $\theta(\rightarrow)$, and according to 2.9. Theorem $\text{Th}(\theta) = \text{For}(\theta)$.

Consequently, (syntactical) consistency is preserved in passing from θ to $\theta(\rightarrow)$.

3. Let $A \leftrightarrow B$ iff $A \rightarrow B$ and $B \rightarrow A$ in $\theta(\rightarrow)$.

3.1. COROLLARY. $\frac{}{\theta} F$ iff $\frac{}{\theta(\rightarrow)} F \leftrightarrow \top$.

3.2. LEMMA. (1) $\frac{}{\theta(\rightarrow)} A \leftrightarrow A$; (2) $\frac{}{\theta(\rightarrow)} A \& \top \leftrightarrow A$; $\frac{}{\theta(\rightarrow)} A \& A \leftrightarrow A$; (3) if $A \leftrightarrow B$, then $B \leftrightarrow A$; (4) if $A \leftrightarrow B$ and $B \leftrightarrow C$, then $A \leftrightarrow C$; (5) if $A \leftrightarrow B$ and $C \leftrightarrow D$, then $A \& C \leftrightarrow B \& D$; (6) if $A \rightarrow B$, then $A \leftrightarrow A \& B$.

Proof. For instance (6). Of course, $A \rightarrow B$ and $A \rightarrow A$, by rule scheme (IKS) we derive $A \rightarrow A \& B$. On the other hand, from $A \rightarrow A$, by rule scheme (IKA) we derive $A \& B \rightarrow A$.

3.3. Let $\theta(\sim)$ be the equational formal theory described in [1] of S. B. Prešić. The following statement is the consequence of the preceding Lemma.

3.4 LEMMA. If $\frac{}{\theta(\sim)} A \sim B$, then $\frac{}{\theta(\rightarrow)} A \leftrightarrow B$.

3.5. THEOREM. If $\frac{}{\theta(\rightarrow)} A \leftrightarrow B$, then $\frac{}{\theta(\sim)} A \sim B$.

Proof. Let $A \leq B$ in $\theta(\sim)$ iff $A \& B \sim A$ in $\theta(\sim)$. First we will prove that if $A \rightarrow B$ in $\theta(\rightarrow)$, then $A \leq B$ in $\theta(\sim)$. From $F \& F \sim F$ we have $F \leq F$; if F is an axiom of θ , then $\top \& F \sim \top$ in $\theta(\sim)$, and $\top \leq F$; if $A \& \top \sim A$ in $\theta(\sim)$, then $A \leq \top$ in $\theta(\sim)$; for a rule of inference $\frac{G_1, \dots, G_k}{G}$ of θ , $G_1 \& \dots \& G_k \& G \sim G_1 \& \dots \& G_k$ is an axiom of $\theta(\sim)$, therefrom $G_1 \& \dots \& G_k \leq G$; if $A \leq B$ and $A \leq C$, then $A \leq B \& C$; if $A \leq B$, then $A \& C \leq B$; if $A \leq B$ and $B \leq C$, i.e. $A \& B \sim A$ and $B \& C \sim B$, then we derive immediately $A \& B \& C \sim A \& C$ and $A \& B \& C \sim A \& B$, and so $A \& C \sim A$, i.e. $A \leq C$. Also, $A \leq B$ and $B \leq A$ iff $A \sim B$, Therefore, if $A \rightarrow B$ and $B \rightarrow A$ in $\theta(\rightarrow)$, then $A \leq B$ and $B \leq A$ in $\theta(\sim)$, i.e. $A \sim B$ in $\theta(\sim)$.

4. THEOREM. *If θ is a formal theory in a language containing the binary operations \wedge, \Rightarrow such that (1) $F, G \vdash F \wedge G$, $F \wedge G \vdash F$ and $F \wedge G \vdash G$; and (2) $F \vdash G$ iff $\vdash F \Rightarrow G^3$, then $\vdash_{\theta(\rightarrow)} F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_m$ iff $\vdash_{\theta} F_1 \wedge \dots \wedge F_n \Rightarrow G_1 \wedge \dots \wedge G_m$ ($m, n \geq 1$).*

Proof. This can be proved almost in the same way as Theorem 3 in [1].

In this case, when conditions (1) and (2) of the preceding theorem are satisfied, by mapping g : $\text{For}(\theta(\rightarrow)) \rightarrow \text{For}(\theta)$, defined by equality $g(F_1 \& \dots \& F_n \rightarrow G_1 \& \dots \& G_m) = F_1 \wedge \dots \wedge F_n \Rightarrow G_1 \wedge \dots \wedge G_m^4$, the theorems of $\theta(\rightarrow)$ will be mapped into theorems of θ . We will call the formal theory $\theta(\rightarrow)$ an *inequational reformulation* of θ .

4.1. It can be proved (cf. [1] and [2]) that the “corresponding” formal theories $\theta(\sim)$, and consequently $\theta(\rightarrow)$, are in the cases of the intuitionistic propositional calculus, classical propositional calculus and classical first-order predicate calculus (these are the cases of the equational (inequational) reformulations) just corresponding algebras: pseudo-Boolean, Boolean and cylindric. Furthermore, there is an isomorphism between $\theta(\sim)$ (or $\theta(\rightarrow)$) and the Lindenbaum algebra of θ .

REFERENCES

- [1] S. B. Prešić, *Equational reformulation of formal theories*, Publ. Inst. Math. (N. S.) (Beograd) **19(33)** (1975), 131–138. (Math. Rev. 1977: 4921)
- [2] B. R. Boričić, *Equational reformulations of intuitionistic propositional calculus and classical first-order predicate calculus*, Publ. Inst. Math. (Beograd) (N. S.) **29(43)**, (1981), 23–28.

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(Received 14 12 1982)

³Cf. Cond. 1. and Cond. 2. in [1].

⁴If $F_i = \top$ or $G_j = \top$ in $\theta(\rightarrow)$ (for some $1 \leq i \leq n$ or $1 \leq j \leq m$), then F_i or G_j in θ must be replaced by some axiom of θ .