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A NOTE ON EXTENSIONS OF BEAR AND P. P.-RINGS

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Bear rings are rings in which the left (right) annihilator of each subset is generated by an idempotent [**2**]. Closely related to Bear rings are left P. P. -rings; these are the rings in which each principal left ideal is projective, or equivalently, ring in which the left annihilator of each element is generated by an idempotent. In [**1**] Armendariz showed that if R is a ring which has no nonzero nilpotent elements then $R[X]$ is a Bear or P.P.-ring if and only if R is a Bear or P.P.-ring. In this note we generalize this result. A semigroup G is called au u.p. semigroup if, when A and B are nonempty finite subsets of G, then there always exists at least one $x \in G$ which has an unique representation in the form $x = ab$ with $a \in A$ and $b \in B$. We prove that if R is a reduced ring and G a u. p. semigroup then the semigroup ring RG is a Bear or P.P.-ring if and only if R is a Bear or P.P.-ring.

We will assume throughout that rings have a unit. In a reduced ring left and right annihilators coicide for any subset U of R, hence we let $\mathrm{ann}_R(U) = l(U_=r(U) = \{a \in R : aU = 0\}$.

The key lemma is the following characterization of zero divisors in RG when R is a reduced ring.

LEMMA 1. [3, COROLLARY 3.2] *Let G be an u. p. semigroup and let R be a reduced ring. Let G be an u.p. semigroup and let $p, q \in RG$ such that $pq = 0$. Then for any $g, h \in G$ we have $p_q q_h = 0$.*

COROLLARY 1. *If R is a reduced ring anf $f \in RG$, G an u.p. semigroup, such that $f^2 = f$ then $f \in R$.*

Proof. Let $f = \sum_{i=1}^{n} a_i g_i$. It is easy to show that $g_i = e$ for at least one i. Hence we may, without any loss in generality, put $f = a_1 e + a_2 g_2 + \ldots + a_n g_n$. Now $f(f-1) = 0$. From Lemma 1 we have $a_1(a_1 - 1) = 0$ and $a_i = 0$ for $i \geq 2$. Herce $f = a_1 = a_1^2 \in R$.

If $f \in RG$ and $f = \sum_{i=1}^{n} a_i g_i$ let $S_f = \{a_1, a_2, \ldots, a_n\}$.

COROLLARY 2. *Let R be a reduced ring and $U \subseteq RG$. If $T = \bigcup_{f \in U} S_f$ then* $\mathrm{ann}_{RG} U = \mathrm{ann}_R(T)G$.

Proof. This follows easily from Lemma 1.

THEOREM 1. *Let R be a reduced ring and G an u.p. semigroup. Then RG is a P.P.-ring if and only if R is a P.P.-ring.*

Proof. If RG is a P.P.-ring and $a \in R$ then $\mathrm{ann}_R(a) = R \cap \mathrm{ann}_{RG}(a) = R \cap (RG)e$ with $e^2 = e$. By Corollary 1, $e \in R$ and thus $R \cap RGe = Re$.

Now assume R is a P.P.-ring. Let $a, b \in R$ with $\mathrm{ann}_R(a) = Re_1$, $\mathrm{ann}_R(b) = Re_2$, where $e_1^2 = e_1$, $e_2^2 = e_2$. Put $e = e_1 e_2$. Because the idempotents of R are central we have $e^2 = e$. We show that $\mathrm{ann}_R\{a, b\} = Re$. If $xa = xb = 0$ then $x = xe_1 = xe_2$ and $xe = xe_1 e_2 = x$. Hence $\mathrm{ann}_R\{a, b\} \subseteq Re$. Further, let $t \in Re$, say $t = re_1 e_2$. Now $ta = re_1 e_2 a = re_2 e_1 a = 0$ and $tb = re_1 e_2 b = 0$. Hence $Re \subseteq \mathrm{ann}_R\{a, b\}$. Therefore, $Re = \mathrm{ann}_R\{a, b\}$. Thus for any finite subset $U \subseteq R$, $\mathrm{ann}_R(U) = Re$ for some idempotent $e \in R$. I $f \in RG$ then by Corollary 2, $\mathrm{ann}_{RG}(f) = \mathrm{ann}_R(S_f)G = (Re)G = (RG)e$ with $e^2 = e$, as Sf is finite. Thus RG is a P.P.-ring.

Similarly we can establish

THEOREM 2. *Let R be a reduced ring and G an u. p. semigroup. Then RG is a Bear ring if and only if R is a Bear ring.*

COROLLARY 3 [1, THEOREM A] *Let R be a reduced ring. Then $R[x]$ is a P.P.-ring if and only if R is a P.P.-ring.*

Proof. It follows from the fact that the infinite cyclic semigroup $\langle X \rangle$ is an u.p. semigroup.

COROLLARY 4 [1, THEOREM B]. *Let R be a reduced ring. Then $R[x]$ is a Bear ring if and only if R is a Bear ring.*

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