

ABELIAN TYPE THEOREMS FOR SOME INTEGRAL OPERATORS IN \mathbf{R}^n

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Abstract. For integral transforms of functions defined on cones in the n -dimensional Euclidean space we prove two theorems of the following “Abelian” type: The transform of a regularly varying function is regularly varying.

1. Introduction. Let, as usual, \mathbf{R}^n denote the real n -dimensional Euclidean space. If $x = (\xi_1, \dots, \xi_n)$ and $y = (\eta_1, \dots, \eta_n)$ are elements of \mathbf{R}^n , their inner product is denoted by $x \cdot y = \sum_{j=1}^n \xi_j \eta_j$ and the norm of x by $|x| = (x \cdot x)^{1/2}$.

The set $\Gamma \subset \mathbf{R}^n$ is a *cone* if $x \in \Gamma$ implies $\lambda x \in \Gamma$, for every $\lambda > 0$. An example of cone is $\mathbf{R}_+^n = \{x \in \mathbf{R}^n : \xi_1 > 0, \dots, \xi_n > 0\}$, the positive octant. We shall always assume that the cone Γ is closed, convex, that it has nonempty interior and that it is acute; see [3] or [4].

When examining the asymptotic relations the notion of regular variation is very useful. Regularly varying functions were first defined by Karamata – in the one-dimensional case (see [2]). This definition was generalized to the n -dimensional case by Yakymiv in the following way.

Definition [5]. Let Γ be a cone in \mathbf{R}^n . A measurable function $R : \Gamma \rightarrow \mathbf{R}_+$ is *regularly varying* (at infinity) if a function $\varphi : \Gamma \rightarrow \mathbf{R}_+$ exists such that

$$(1) \quad \lim_{\lambda \rightarrow \infty} \sup_{x \in B} \left| \frac{R(\lambda x)}{R(\lambda e)} - \varphi(x) \right| = 0$$

for a fixed $e \in \Gamma$ ($e \neq 0$) and for every compact set $B \subset \Gamma \setminus \{0\}$.

Recall that in the definition of regularly varying functions in one variable only the existence of the pointwise limit of $R(\lambda x)/R(\lambda)$ is required and that (the analogue of) (1) follows by the Uniform Convergence Theorem [2]. It was proved by Yakymiv that for $n \geq 2$ the existence of the pointwise limit would not imply the existence of the uniform limit and this motivates the Definition.

It is easily seen that the class of functions satisfying the Definition does not depend on the choice of e ; we can assume that $\varphi(e) = 1$.

The function φ in (1) is homogeneous, i.e. there is an $\rho \in \mathbf{R}$, called the order of φ , such that $\varphi(\lambda x) = \lambda^\rho \varphi(x)$, for every $\lambda > 0$ and $x \in \Gamma$; moreover, φ is continuous and bounded away from zero. The order of φ is also called the index of the regularly varying function R . We shall consistently use the following notation: R will stand for a regularly varying function, φ for its ‘‘index’’ function, r for the function in one variable $R(\lambda e)$ (which is obviously regularly varying in \mathbf{R}_+).

A regularly varying function whose index function is 1 is called a *slowly varying function*. Every regularly varying function R can be represented in the following way $R = \varphi L$, where φ is the index function and L is a slowly varying function.

Finally, let us define regular variation at zero. A measurable function $R : \Gamma \rightarrow \mathbf{R}_+$ is *regularly varying at zero* (with index ρ) if the limit

$$\lim_{\lambda \rightarrow \infty} \frac{R(x/\lambda)}{r(\lambda)} = \varphi(x)$$

exists uniformly in $x \in B$, for every compact set $B \subset \Gamma \setminus \{0\}$.

The main objective of the present paper is the investigation of some integral operators of the form

$$(2) \quad \mathcal{K}F(x) = \int_{\Gamma} F(u)k(u, x)du, \quad x \in G,$$

which transform functions defined on a cone Γ into functions defined on a cone G . We shall consider two kinds of such operators (the first applicable to arbitrary regularly varying functions – Section 2, and the second only to the monotone ones – Section 3) and we shall prove that for these operators the regular variation of F implies the regular variation of $\mathcal{K}F$. In Section 4 we deduce analogous results for the regular variation at zero. These statements are n -dimensional counterparts of the one-dimensional results from [1].

Let us first state some facts about regularly varying functions.

- (i) A regularly varying function on Γ is locally bounded on Γ (= bounded on every compact set $B \subset \Gamma \setminus \{0\}$) [5].
- (ii) For every slowly varying function L there is an asymptotically equivalent radial slowly varying function L_r [5].

This is easily seen by putting $L_r(x) = L(|x|e)$, for some $e \in \Gamma$; let l be defined by $l(|x|) = L_r(x)$.

Let L be a slowly varying function on Γ and let F be a locally integrable function on Γ . Let $|u| < \delta$ denote the set $\{u \in \Gamma : |u| < \delta\}$ and similarly for

$|u| > \Delta$. Then for some $\eta > 0$ and some positive constants C_1 and C_2 we have

$$(iii) \quad \int_{|u| < \delta} L(\lambda u) F(u) du \leq C_1 \delta^\eta l(\lambda \delta) \int_{|u| < \delta} |u|^{-\eta} F(u) du$$

$$(iv) \quad \int_{|u| < \delta} L(\lambda u) F(u) du \leq C_2 \Delta^{-\eta} l(\lambda \Delta) \int_{|u| > \Delta} |u|^\eta F(u) du$$

for λ large enough.

The proof of (iii) and (iv) is similar to the proof of the corresponding one-dimensional statements [2]. For (iii) we use the following property of slowly varying functions $\sup_{|u| < |x|} |u|^\eta L(u) \sim |x|^\eta l(|x|)$, for $\eta > 0$, $|x| \rightarrow \infty$ (see [5]), and similarly for (iv).

2. Operators with absolutely integrable kernels. Let Γ and G be two cones in \mathbf{R}^n . A measurable function $k : \Gamma \times G \rightarrow \mathbf{R}$ will be called a *kernel* (on $\Gamma \times G$). In this section we shall consider kernels which satisfy the following condition. For a given $\rho \in \mathbf{R}$

$$(A) \quad \int_{\Gamma} \max(|u|^{-\eta+\rho}, |u|^{\eta+\rho}) |k(u, x)| du \leq C(x), \quad x \in G,$$

for some $\eta > 0$ and a positive function $C(x)$.

The number ρ is called the *index* of the kernel k .

Now consider an operator defined as in (2). Assume moreover that the function k is homogeneous of order α , for some $\alpha \in \mathbf{R}$ (i.e. $k(\lambda u, \lambda x) = \lambda^\alpha k(u, x)$). Then we have

$$(3) \quad \mathcal{K}F(\lambda x) = \lambda^{\alpha+n} \int_{\Gamma} F(\lambda u) k(u, x) du.$$

Indeed, by making a change of variables we have

$$\begin{aligned} \mathcal{K}F(\lambda x) &= \int_{\Gamma} F(u) k(u, \lambda x) du = \lambda^n \int_{\Gamma} F(\lambda v) k(\lambda v, \lambda x) dv \\ &= \lambda^{\alpha+n} \int_{\Gamma} F(\lambda v) k(v, x) dv \end{aligned}$$

which proves (3),

PROPOSITION 1. *Let Γ and G be two cones in \mathbf{R}^n . Let k be a kernel on $\Gamma \times G$ satisfying (A) with index 0, homogeneous of order α , and let L be a slowly varying function on Γ . Then*

$$\left| \frac{\mathcal{K}L(\lambda x)}{\lambda^{\alpha+n} l(\lambda)} - \mathcal{K}1(x) \right| \leq \varepsilon(\lambda) C(x), \quad x \in G,$$

where $\varepsilon(\lambda) \rightarrow 0$, as $\lambda \rightarrow \infty$.

Proof. We have by (3)

$$(4) \quad \begin{aligned} \frac{\mathcal{K}L(\lambda x)}{\lambda^{\alpha+n}l(\lambda)} - \mathcal{K}1(x) &= \int_{\Gamma} \left(\frac{L(\lambda u)}{l(\lambda)} - 1 \right) k(u, x) du = \\ &= \int_{|u| < \delta} + \int_{\delta \leq |u| \leq \Delta} + \int_{\delta \leq |u|} = I_1 + I_2 + I_3 \end{aligned}$$

where δ and Δ will be chosen later.

For the integral I_1 we have

$$|I_1| \leq \frac{1}{l(\lambda)} \int_{|u| < \delta} L(\lambda u) |k(u, x)| du + \int_{|u| < \delta} |k(u, x)| du.$$

If we apply property (iii) of slowly varying functions and then condition (A) we obtain

$$(5) \quad \begin{aligned} |I_1| &\leq \frac{1}{l(\lambda)} C_1 \delta^\eta l(\lambda \delta) \int_{|u| < \delta} |u|^{-\eta} |k(u, x)| du + \int_{|u| < \delta} |k(u, x)| du \leq \\ &\leq C_1 (l(\lambda \delta)/l(\lambda) + 1) \delta^\eta C(x) \leq C_3 \delta^\eta C(x) \end{aligned}$$

for λ large enough.

Analogously, using (iv) instead of (iii) we have for I_3

$$(6) \quad |I_3| \leq C_4 \Delta^{-\eta} C(x)$$

for λ large enough.

Now for a given $\varepsilon > 0$ we choose δ and Δ such that $C_3 \delta^\eta < \varepsilon$ and $C_4 \Delta^{-\eta} < \varepsilon$. Then from (5) and (6) we have

$$(7) \quad |I_1| + |I_3| < \varepsilon C(x)$$

Next for δ and Δ chosen as above consider I_2 . Since the set $\delta \leq |u| \leq \Delta$ is compact it follows from the definition of slowly varying functions that

$$(8) \quad |I_2| < \varepsilon C(x)$$

for λ large enough. Now the proof of the proposition follows by substituting (7) and (8) into (4).

THEOREM 1. *Let Γ and G be two cones in \mathbf{R}^n . Let k be a kernel satisfying (A) with index ρ , homogeneous of order α , and let R be a regularly varying function of index p in Γ . Then*

$$(9) \quad \left| \frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x) \right| \leq C(x)\varepsilon(\lambda), \quad x \in G$$

where $\varepsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. The function φ is homogeneous of order ρ and continuous and bounded away from zero; therefore

$$C_1|u|^\rho \leq \varphi(u) = |u|^\rho \varphi(u') \leq C_2|u|^\rho$$

and from this it follows that (A) is equivalent to

$$\int_{\Gamma} \max(|u|^{-\eta}, |u|^\eta) \varphi(u) k(u, x) du \leq C_3 C(x)$$

which means that the function $k_1(u, x) = \varphi(u)k(u, x)$ is a kernel satisfying (A) with index 0. It is moreover homogeneous of order $\rho + \alpha$, and thus it satisfies the conditions of Proposition 1 (with α replaced by $\rho + \alpha$). We shall apply this proposition to the operator \mathcal{K}_1 with kernel k_1 and to the slowly varying function $L = R/\varphi$. First observe that

$$\frac{R(\lambda v)}{r(\lambda)} = \frac{L(\lambda v)\varphi(\lambda v)}{l(\lambda)\varphi(\lambda e)} = \frac{L(\lambda v)\lambda^\rho \varphi(v)}{l(\lambda)\lambda^\rho}.$$

From this, by an application of (3), it follows that

$$\begin{aligned} \frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x) &= \int_{\Gamma} \left(\frac{R(\lambda v)}{r(\lambda)} - \varphi(v) \right) k(v, x) dv = \\ &= \int_{\Gamma} \left(\frac{L(\lambda v)}{l(\lambda)} - 1 \right) \varphi(v) k(v, x) dv = \frac{\mathcal{K}_1 L(\lambda x)}{\lambda^{\alpha+n+\rho}l(\lambda)} - \mathcal{K}_1 1(x). \end{aligned}$$

Now an application of Proposition 1 completes the proof of the theorem.

Remark. We have not proved yet that $\mathcal{K}R$ is regularly varying, for (9) gives only the pointwise convergence of $\mathcal{K}R(\lambda x)/\lambda^{\alpha+n}r(\lambda)$ (and not the uniform). However, it is immediate from (9) that we shall indeed have regular variation if $C(x)$ is assumed to be locally bounded (for instance, $C(x) = C$). We shall formulate this remark as a corollary and observe that similar considerations are in order whenever we have an equation similar to (9) (see (19) in Theorem 2 below).

COROLLARY 1. *Let k and R satisfy the conditions of Theorem 1 and let the function $C(x)$ in (A) be bounded on compact sets in $G_1 \setminus \{0\}$, where G_1 is a cone contained in, or equal to G . Then $|\mathcal{K}R|$ is regularly varying in G_1 with index function $|\mathcal{K}\varphi|$ and index $\alpha + n + \rho$.*

In the next corollary we shall consider several examples of operators to which Theorem 1 applies.

Let us write $\langle a, b \rangle$ for the “interval”, defined by $\langle a, b \rangle = \{x \in \Gamma : a \leq_{\Gamma} x \leq_{\Gamma} b\} = (a + \Gamma) \cap (b - \Gamma)$. We shall write $\rangle a, b \rangle$ for the difference of two intervals $\rangle a, b \rangle = \langle 0, b \rangle \setminus \langle 0, a \rangle$, and also by abusing the notation $\langle a, \infty \rangle = a + \Gamma$ and $\rangle a, \infty \rangle = \Gamma \setminus \langle 0, a \rangle$.

Consider the following three operators:

$$(10) \quad \mathcal{I}_\Gamma f(x) = \int_{\langle 0, x \rangle} f(u) du, \quad x \in \Gamma$$

$$(11) \quad \mathcal{J}_\Gamma f(x) = \int_{\langle x, \infty \rangle} f(u) du, \quad x \in \Gamma$$

$$(12) \quad \mathcal{J}_\Gamma^1 f(x) = \int_{\rangle x, \infty \rangle} f(u) du, \quad x \in \Gamma.$$

COROLLARY 2. *Let R be a regularly varying function on a cone Γ , with index ρ and index function φ .*

a) *Let $-n < \rho < 0$. Then $\mathcal{I}_\Gamma R$ is regularly varying in Γ with index $\rho + n$ and index function $\mathcal{I}_\Gamma \varphi$.*

b) *Let $\rho > -n$. Then $\mathcal{J}_\Gamma R$ and $\mathcal{J}_\Gamma^1 R$ are regularly varying in Γ with index $\rho + n$ and index functions $\mathcal{J}_\Gamma \varphi$ and $\mathcal{J}_\Gamma^1 \varphi$ respectively.*

Proof. We shall prove a) only, the proof of b) being similar. Obviously (10) is an operator of the form (2) with kernel $k(u, x) = \theta_{\langle 0, x \rangle}(u)$ (where θ_A is the characteristic function of the set A). We have to prove that this kernel satisfies the conditions of Corollary 1. Since $k(\lambda u, \lambda x) = k(u, x)$, we see that k is homogeneous of order 0. Next we prove that (A) holds. We have

$$\int_{\Gamma} \max(|u|^{\rho-\eta}, |u|^{\rho+\eta}) |k(u, x)| du = \int_{|u| \leq 1} + \int_{|u| > 1} = I_1 + I_2.$$

For the first integral we have

$$(13) \quad I_1 = \int_{|u| \leq 1} |u|^{\rho-\eta} |k(u, x)| du \leq \int_{|u| \leq 1} |u|^{\rho-\eta} du \leq C$$

if $0 < \eta < \rho + n$; and for the second, if $0 < \eta < -\rho$

$$(14) \quad I_2 = \int_{|u| > 1} |u|^{\rho+\eta} |k(u, x)| du \leq \int_{|u| > 1} k(u, x) du \leq \int_{\langle 0, x \rangle} du \equiv V(x).$$

From (13) and (14) it follows that if we choose $\eta < \min(-\rho, \rho + n)$, then $I_1 + I_2 \leq C + V(x)$, and this proves (A) with $C(x) = C + V(x)$.

If B is a compact set in $\Gamma \setminus \{0\}$, then $\sup_{x \in B} V(x)$ is bounded, since $V(x)$ is the volume of a bounded set. Now Corollary 1 can be applied to complete the proof.

The functions defined in (10), (11) and (12) may be called primitive functions for f . They are monotone in the following sense.

A cone Γ defines a partial order in \mathbf{R}^n . We say that $x \leq_{\Gamma} y$ if $y - x \in \Gamma$. A real function F is said to be monotone increasing (decreasing) on Γ if $x \leq_{\Gamma} y$ implies $F(x) \leq F(y)$ ($F(x) \geq F(y)$).

Now, it is obvious that for positive f the function $\mathcal{I}_{\Gamma} f$ is monotone increasing in Γ , and $\mathcal{J}_{\Gamma} f$ and $\mathcal{J}_{\Gamma}^1 f$ are monotone decreasing.

Remark. For monotone functions it is possible to obtain the converse of Corollary 2 (the Tauberian theorem for operators (10), (11) and (12)), i.e., that for monotone R the regular variation of $\mathcal{I}_{\Gamma} R$ (or $\mathcal{J}_{\Gamma} R$ or $\mathcal{J}_{\Gamma}^1 R$) implies the regular variation of R . This was proved (implicitly) by Yakymiv [5] (see the proof of Theorem 9.1).

3. Operators with nonabsolutely integrable kernels. In this section we shall consider operators \mathcal{K} with a different kind of kernel (satisfying condition (B) below). These operators will be applied to monotone functions and the integrals defining them will not be absolutely convergent.

Let F be a locally integrable function on Γ . The integral $\int_{\Gamma} F(t) dt$ is said to converge in the sense of principal value if the limit $\lim_{|b| \rightarrow \infty} \int_{\langle 0, b \rangle} F(t) dt$ exists for $b \in \Gamma_1$, where Γ_1 is a cone such that $\Gamma_1 \subset \int \Gamma$.

Let Γ and G be two cones in \mathbf{R}^n . In this section we shall consider kernels on $\Gamma \times G$ which satisfy the following condition

$$(B) \quad \left| \int_{\langle a, b \rangle} k(u, x) du \right| \leq C(x), \quad x \in G$$

for some positive function $C(x)$ and all intervals $\langle a, b \rangle \subset \Gamma$.

Let us call, for short, the function F *monotone primitive* if there is a positive integrable function f on Γ such that $F(x) = \int_{x+\Gamma} f(t) dt = \mathcal{J}_{\Gamma} f(x)$.

In the following proposition we prove the existence of the integral $\mathcal{K}F(x)$. This proposition is the analogue of Dirichlet's criterion for nonabsolutely convergent integrals.

PROPOSITION 2. *Let Γ and G be two cones in \mathbf{R}^n . Let k be a kernel on $\Gamma \times G$ satisfying (B) and let F be a monotone primitive function on Γ . Then the integral*

$$(13) \quad \mathcal{K}F(x) = \int_{\Gamma} F(u) k(u, x) du, \quad x \in G$$

exists in the sense of principal value.

The proof of the proposition is based on the following lemma.

LEMMA 1. Let F and k satisfy the conditions of Proposition 2. Let $F(u) = \int_{u+\Gamma} f(t)dt$. Then

$$(14) \quad |I_{a,b}| \equiv \left| \int_{\rangle a,b\rangle} F(u)k(u,x)du \right| \leq C(x) \int_{\rangle a,\infty\rangle} f(t)dt.$$

With this lemma, Proposition 2 follows at once. Indeed, to show that (13) converges in the sense of principal value we have to show that the “remainder” $I_{a,b}$ of this integral tends to 0, as $|a|, |b| \rightarrow \infty$. Now, by (14) we have $|I_{a,b}| \leq C(x) \int_{\rangle a,\infty\rangle} f(t)dt$, and the last integral tends to 0 as $|a| \rightarrow \infty$, since f is integrable. This proves Proposition 2.

Proof of Lemma 1. First we shall prove that for k satisfying (B) we have

$$(15) \quad \left| \int_{\rangle a,b\rangle} k(u,x)du \right| \leq 2C(x)$$

Indeed, since $\rangle a,b\rangle = \langle 0,b\rangle \setminus \langle 0,a\rangle$ we have $\int_{\rangle a,b\rangle} = \int_{\langle 0,b\rangle} - \int_{\langle 0,a\rangle}$. Now an application of (B) yields (15).

To prove (14) we shall apply Fubini’s Theorem to the integral

$$(16) \quad I_{a,b} = \int_{\rangle a,b\rangle} F(u)k(u,x)du = \int_{\rangle a,b\rangle} k(u,x) \int_{u+\Gamma} f(t)dtdu.$$

The domain of integration in the double integral is defined by $u \in \rangle a,b\rangle$, which means (i) $u \in \rangle a,\infty\rangle$ and (ii) $u <_{\Gamma} b$, and by $t \in u + \Gamma$, which means (iii) $t >_{\Gamma} u$.

Now, it is easily seen that by (i) and (iii) we have $t \in \rangle a,\infty\rangle$, and that for u we have, by (ii) and (iii), $u <_{\Gamma} b$ and $u <_{\Gamma} t$, which, if we put $c = \min_{\Gamma}(b,t)$, is equivalent with $u <_{\Gamma} c$; and this together with (i) gives $u \in \rangle a,c\rangle$.

Thus if we reverse the order of integration in (16) we have

$$I_{a,b} = \int_{\rangle a,b\rangle} k(u,x) \int_{u+\Gamma} f(t)dtdu = \int_{\rangle a,\infty\rangle} f(t) \int_{\rangle a,c\rangle} k(u,x)dudt.$$

And now we apply (15) to the last integral:

$$|I_{a,b}| = \int_{\rangle a,\infty\rangle} f(t) \left| \int_{\rangle a,c\rangle} k(u,x)du \right| dt \leq C(x) \int_{\rangle a,\infty\rangle} f(t) dt$$

which completes the proof of the lemma.

Now we shall consider the regular variation of monotone primitive functions. First we prove a lemma.

LEMMA 2. Let R be monotone primitive on a cone Γ , $R(x) = \int_{x+\Gamma} f(t)dt$ such that f is also monotone, and let R be regularly varying with index ρ , $-n < \rho < 0$. Then

a) the function f is regularly varying with index function ψ such that $\varphi(x) = \int_{x+\Gamma} \psi(u)du$.

b) Let k satisfy condition (B), then

$$(17) \quad \frac{1}{r(\lambda)} \left| \int_{\langle a, \infty \rangle} R(\lambda u)k(u, x)du \right| \leq C(x) \int_{\langle a, \infty \rangle} \psi(u)du.$$

Proof. Part a) follows by the remark at the end of Section 2.

b) By Lemma 1 we have

$$(18) \quad \frac{1}{r(\lambda)} \left| \int_{\langle a, b \rangle} R(\lambda u)k(u, x)du \right| \leq \frac{C(x)}{r(\lambda)} \int_{\langle \lambda a, \infty \rangle} f(u)du.$$

But since f is regularly varying, an application of Corollary 2 b) yields that $\mathcal{I}_\Gamma^1 f(a) = \int_{\langle a, \infty \rangle} f(u)du$ is regularly varying and has the index function $\mathcal{I}_\Gamma^1 \psi(a)$.

Thus letting $\lambda \rightarrow \infty$ in (18) we shall have $1/r(\lambda) \int_{\langle \lambda a, \infty \rangle} f(u)du \rightarrow \int_{\langle a, \infty \rangle} \psi(u)du$ and this proves (17).

THEOREM 2. Let Γ and G be two cones in \mathbf{R}^n . Let $-n < \rho < 0$. Let R be regularly varying on Γ with index ρ , and let R be monotone primitive, $R(x) = \int_{x+\Gamma} f(t)dt$, such that f is also monotone. Let k be a bounded kernel on $\Gamma \times G$ satisfying (B), homogeneous of order α . Then

$$(19) \quad \left| \frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x) \right| \leq \varepsilon(\lambda)C(x), \quad x \in G$$

where $\varepsilon(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. We have, as in the proof of Theorem 1 (see (3)),

$$(20) \quad \begin{aligned} \frac{\mathcal{K}R(\lambda x)}{\lambda^{\alpha+n}r(\lambda)} - \mathcal{K}\varphi(x) &= \int_{\Gamma} \left(\frac{R(\lambda v)}{r(\lambda)} - \varphi(v) \right) k(v, x)dv = \\ &= \int_{\langle 0, a \rangle} + \int_{\langle a, A \rangle} + \int_{\langle A, \infty \rangle} = I_1 + I_2 + I_3, \end{aligned}$$

say. Consider I_1 . Since R is regularly varying with index ρ , $-n < \rho < 0$, from Corollary 2 a) it follows that $\mathcal{I}_\Gamma R(x) = \int_{\langle 0, x \rangle} R(u)du$ is regularly varying with index

function $\mathcal{I}_\Gamma \varphi(x) = \int_{\langle 0, x \rangle} \varphi(u) du$ (and with index $\rho + n$). From this we shall have, since k is bounded

$$|I_1| \leq \frac{1}{r(\lambda)} \int_{\langle 0, a \rangle} R(\lambda v) dv + \int_{\langle 0, a \rangle} \varphi(v) dv = \frac{\mathcal{I}_\Gamma R(\lambda a)}{\lambda^n r(\lambda)} + \mathcal{I}_\Gamma \varphi(a) < 2\mathcal{I}_\Gamma \varphi(a)$$

for λ large enough. Now given ε (depending on λ) we can choose a small enough in Γ such that

$$(21) \quad |I_1| < \varepsilon.$$

For the integral I_3 we have

$$|I_3| \leq \frac{1}{r(\lambda)} \left| \int_{\rangle A, \infty \rangle} R(\lambda v) k(v, x) dv \right| + \left| \int_{\rangle A, \infty \rangle} \varphi(v) k(v, x) dv \right|$$

and since R satisfies the conditions of Lemma 2, we have by (17)

$$|I_3| \leq 2C(x) \int_{\rangle A, \infty \rangle} \psi(u) du.$$

Now if we choose A large enough in Γ we shall have $\int_{\rangle A, \infty \rangle} \psi(u) du < \varepsilon$. Thus we have proved

$$(22) \quad |I_3| < \varepsilon C(x).$$

For a and A chosen as above we consider I_2

$$(23) \quad |I_2| \leq \int_{\rangle A, \infty \rangle} \left| \frac{R(\lambda v)}{r(\lambda)} - \varphi(v) \right| dv \rightarrow 0$$

as $\lambda \rightarrow \infty$, by the definition of regularly varying functions, since $\rangle a, A \rangle$ is compact in $\Gamma \setminus \{0\}$. By substituting (21), (22) and (23) into (20) we complete the proof of the theorem.

Remark. Obviously, we can obtain as a corollary of this theorem a statement similar to Corollary 1 (see also the Remark after Theorem 1); namely, assuming the local boundedness of $C(x)$ the regular variation of R implies the regular variation of $|\mathcal{K}R|$.

4. Asymptotic behavior at zero. In the present section we prove two corollaries of Theorems 1 and 2.

COROLLARY 4. *Let R and k satisfy the assumptions of Theorem 1, or the assumptions of Theorem 2. Then*

$$\lim_{\lambda \rightarrow \infty} \frac{R(u/\lambda)}{r(\lambda)} = \varphi(u), \quad u \in \Gamma,$$

uniformly in compact sets in $\Gamma \setminus \{0\}$, implies

$$(23) \quad \lim_{\lambda \rightarrow \infty} \frac{\mathcal{K}R(x/\lambda)}{\lambda^{-\alpha-n}r(\lambda)} = \mathcal{K}\varphi(x), \quad x \in G,$$

and the convergence in (23) is uniform in compact sets in $G \setminus \{0\}$, under the additional assumption that $C(x)$ is locally bounded.

Proof. We have

$$\begin{aligned} \mathcal{K}R(x/\lambda) &= \int_{\Gamma} R(u)k(u, x/\lambda)du = \int_{\Gamma} R(v/\lambda)k(v/\lambda, x/\lambda)\lambda^{-n}dv = \\ &= \lambda^{-n-\alpha} \int_{\Gamma} R(v/\lambda)k(v, x)dv \end{aligned}$$

and if we use this equation instead of (3), the proof of the corollary follows along the same lines as the proof of Theorem 1 or the proof of Theorem 2.

Next we consider kernels which instead of being homogeneous of order α satisfy the following condition

$$(24) \quad k(\lambda u, x/\lambda) = k(u, x).$$

COROLLARY 5. *Let R and k satisfy the assumptions of Theorem 1 or the assumptions of Theorem 2, – only let k satisfy (24) instead of being homogeneous of order α . Then*

$$\lim_{\lambda \rightarrow \infty} \frac{R(\lambda u)}{r(\lambda)} = \varphi(u), \quad u \in \Gamma,$$

uniformly in compact sets in $\Gamma \setminus \{0\}$, implies

$$(25) \quad \lim_{\lambda \rightarrow \infty} \frac{\mathcal{K}R(x/\lambda)}{\lambda^n r(\lambda)} = \mathcal{K}\varphi(x), \quad x \in G.$$

Here also we have uniform convergence in (25) if the function $C(x)$ is locally bounded and the proof is very similar to the proof of Corollary 4.

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