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A HEREDITARY PROPERTY OF HM-SPACES

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Abstract. We have shown that a class of HM-spaces is invariant under projective topology, in particular, arbitrary product, subspace and separated quotient. We have shown also that $(E, \sigma(E, E'))$ is an HM-space for every locally convex space (E, t) (see [**2**, Theorem 513]).

The nonstandard theory of topological vector spaces and in particular the construction of the nonstandard hull of an arbitrary topological vector space has been studied by Henson and Moore in [**2**] and [**3**]. We recount the principal ideas and definitions.

Let (E, t) be a locally convex space and let ${}^*\mathcal{M}$ be a nonstandard extension of a superstructure $\mathcal{M}$ which contains (E, t). An element $p \in^* E$ is called t-finite if for every t-neighborhood U of zero there exists an integer n such that $x \in n^*U$ and the set of t-finite elements of ${}^*E$ is denoted by $\operatorname{fin}_t({}^*E)$. The monad of O is defined by $\mu_t(O) = \bigcap_{U \in \mathcal{U}} {}^*U$, where $\mathcal{U}$ is a base of balanced, covex neighborhood of zero. Both $\operatorname{fin}_t({}^*E)$ and $\mu_t(O)$ are vector spaces over the same field as E. We denote the quotient vector space $fin_t({}^*E)/\mathcal{M}_t(O)$ by $\hat{E}$, the canonical quotient mapping of $\operatorname{fin}_t({}^*E)$ onto $\hat{E}$ by π and the quotient topology defined on $\hat{E}$ by $\hat{t}$. The nonstandard hull of (E, t) with respect to ${}^*\mathcal{M}$ is the separated quotient space $(\hat{E}, \hat{t})$. Clearly, the map taking x to $\pi({}^*x)$ is a topological vector space isomorphism of (E, t) into $(\hat{E}, \hat{t})$. An element $p \in^* E$ is called t-pre-near-standard if for each t-neighborhood U there exists $y \in E$ such that $x \in^* y +^* U$ and the set of t-pre-near-standard elements of ${}^*E$ is denoted by $\operatorname{pns}_t({}^*E)$.

The nonstandard hull $(\hat{E}, \hat{t})$ of a locally convex space (E, t) always contains $\pi(\operatorname{pns}_t({}^*E))$ = the completion of (E, t) and the nonstandard hulls determined by different nonstandard extensions ${}^*\mathcal{M}$ need not even be isomorphic to each other. When every nonstandard hull $(\hat{E}, \hat{t})$ is equal to $\pi(\operatorname{pns}_t({}^*E))$, we say that the nonstandard hulls of (E, t) are invariant, i.e. we say that a locally convex space (E, t) is an HM-space (see [**4**]).

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Throughout this paper (E, t) will denote a separated locally convex space over K, where K denotes the real or complex numbers.

Henson and Moore have shown the following [**3**]:

THEOREM 1 (Henson and Moore). *The following conditions on a locally convex space (E, t) are equivalent:*

(a) *The nonstandard hulls of (E, t) are invariant;*

(b) *The nonstandard hull of (E, t) is isomorphic to the completion of (E, t) for every choice of the enlargement ${}^*\mathcal{M}$;*

(c) $\mathrm{fin}_t({}^*E) = \mathrm{pns}_t({}^*E)$ *for every choice of ${}^*\mathcal{M}$;*

(d) *The nonstandard hull of (E, t) is isomorphic to the completion of (E, t) for some choice of the enlargement ${}^*\mathcal{M}$;*

(e) $\mathrm{fin}_t({}^*E) = \mathrm{pns}_t({}^*E)$ *for some choice of ${}^*\mathcal{M}$;*

(f) *If $\mathcal{F}$ is an ultrafilter on E and for each t-neighborhood U of zero there is an integer n such that $nU \in \mathcal{F}$, then $\mathcal{F}$ is a Cauchy filter.*

For proofs of our theorems ([2], [3], [4] and [5]) we use a conditions (e) or (f) of preceding theorem.

THEOREM 2. *Every separated quotient $(E/H, t)$ of an HM-space (E, t) is an HM-space.*

Proof. Let ${}^*\mathcal{M}$ be a nonstandard extension of a superstructure $\mathcal{M}$ which contains a space (E, t). According to the condition (e) of the preceding theorem and [**2**, Theorem 1.2] it is sufficient to prove that $\mathrm{fin}_t{}^*(E/H) \subset \mathrm{pns}_t{}^*(E/H)$. Let $x \in \mathrm{fin}_t{}^*(E/H)$. Then, by [**1**, Definition 1.1] for each t-neighborhood U of zero there exists an integer n such that $x \in n^*(U + {}^*H) = n^*U + {}^*H$, i.e. $x \in \mathrm{fin}_t({}^*E) + {}^*H$. By assumption (E, t) is an HM-space and then $x \in \mathrm{pns}_t({}^*E) + {}^*H$. From this and [**2**, Theorem 1.2. iv] it follows that for every t-neighborhood U of zero there is $y \in E$ such that $x \in {}^*y + {}^*U + {}^*H = {}^*(y + H) + {}^*(U + H)$, i.e. $x \in \mathrm{pns}_t{}^*(E/H)$. Hence, the separated quotient space $(E/H, t)$ of a HM-space (E, t) is an HM-space.

THEOREM 3. *A locally convex,space (E, t) is an HM-space, if and only if all its subspaces are HM-spaces.*

Proof. The sufficiency follows immediately from [**1**, Theorem 4.6]. To prove the necessity we utilize condition (f) of Theorem 1. Let $\mathcal{F}$ be an ultrafilter on its subspace (H, t_H) with the property that for each t_H-neighborhood U of zero, there exists an integer n such that $nU \in \mathcal{F}$ (t_H is the relative topology in H). By [**1**, chapter I (6), propositon 10] $\mathcal{F}$ is the base of an ultrafilter on the space (E, t) such that for every t-neighborhood V of zero, there exists an integer n such that $nV \in \mathcal{F}$ ($V \supset V \cap H$ and for some: $n(V \cap H) \in \mathcal{F}$ i.e. $nv \in \mathcal{F}$). By assumption (E, t) is an HM-space and then $\mathcal{F}$ is a base of Cauchy filter according to Theorem 1 (f). We want to show that $\mathcal{F}$ is a Cauchy filter on the space (H, t_H). If V

is a t_H-neighborhood of zero, there exists a t-neighborhood U of zero such that $V \supset U \cap H$. For t-neighborhood U of zero there exists $A \in \mathcal{F}$ such that $A - A \subset U$, i.e. $A \cap H - A \cap H \subset A - A \subset U$ and then $A \cap H - A \cap H \subset U \cap H \subset V$. Hence, for each t_H-neighborhood V of zero there exists $B \in \mathcal{F}$ ($B = A \cap H$) such that $B - B \subset V$, i.e. $\mathcal{F}$ is a Cauchy filter on the space (H, t_H), so the condition is necessary.

THEOREM 4. *The topological product of a family (E_i, t_i) $i \in I$ of HM-spaces is an HM-space if and only if every space (E_i, t_i) is an HM-space.*

Proof. The necessity follows from the preceding theorem. To prove the sufficiency we use condition (e) of Theorem 1. Clearly, it is sufficient to prove that $\operatorname{fin}_{t^*}(\prod_{i\in I} E_i) \subset \operatorname{pns}_{t^*}(\prod_{i\in I} E_i)$, where t is a product topology. Let $x \in \operatorname{fin}_{t^*}(\prod_{i\in I} E_i)$ and let $U = \prod_{i=1}^{n} V_1 \times \prod_{i\neq 1,2,\ldots,n} E_i$ be a neighborhood of zero of the space $\left(\prod_{i\in I} E_i, t_i\right)$. By [**2**, Definition 1.1] there exists an integer n such that $x \in n^*U = n^*\left(\prod_{i=1}^{n} V_1 \times \prod_{i\neq 1,2,\ldots,n} E_i\right)$. According to transfer principle it follows that $x(i) \in n^*V_i$ for $i \in \{1, 2, \ldots, n\}$ and $x(i) \in {}^*E_i$ for $i \neq 1, 2, \ldots, n$. Therefore, $x(i) \in \operatorname{fin}_{t_i}({}^*E_i) = \operatorname{pns}_{t_i}({}^*E_i)$ for $i \in \{1, 2, \ldots, n\}$, i.e. for neighborhoods V_i there exists $y(i) \in E_i$ such that $x(i) \in {}^*y(i) + {}^*V_i$ and $y(i) = O \in E_i$ for $i \neq 1, 2, \ldots, n$. Hence, for each t-neighborhood U of zero of the space $\left(\prod_{i\in I} E_i, t\right)$ there exists $y \in \prod_{i\in I} E_i$ such that $x \in {}^*y + {}^*U$, i.e. $x \in \operatorname{pns}_{t^*}\left(\prod_{i\in I} E_i\right)$.

THEOREM 5. *Let (E_i, t_i), $i \in I$, be a family of HM-spaces. Then the linear space E equipped with a locally convex separated topology t is an HM-space, if t is the projective topology for t_i, $i \in I$.*

Proof. Let $\mathcal{F}$ be an ultrafilter on E with the property that for each t-neighborhood V of zero, there exists an integer n such that $nV \in \mathcal{F}$. By [**1**, Chapter I(6), Proposition 10] every $f_i(\mathcal{F})$ is a base of ultrafilter for which the condition (f) of Teorem 1 holds. But, locally convex spaces (E_i, t_i) are HM-spaces and therefore for every t_i-neighborhood V_i of zero there exists $A \in \mathcal{F}$ such that $A - A \subset f_i^{-1}(f_i(A - A)) \subset f_i^{-1}(V_i)$, so $A - A \subset \bigcap_{i=1}^{n} f_i^{-1}(V_i)$. Hence, $\mathcal{F}$ is a Cauchy filter on the space (E, t) and according to Theorem 1 (f), the theorem is proved.

COROLLARY 1. *The projective limit of any family of HM-spares is an HM-space.*

Proof. The proof follows Theorem 5 and [**5**, chapter II.5].

COROLLARY 2. *If (E, t_i), $i \in I$ is any family of a HM-spaces, then $(E, \sup t_i)_{i \in I}$ is an HM-space.*

Proof. A topology $\sup t_i$, $i \in I$ is projective for the systems (E, t_i) and then the proof follows by the preceding theorem.

COROLLARY 3. *For every locally convex space (E, t), the asssociated space $(E, \sigma(E, E'))$ is an HM-space.*

Proof. The weak topology is projective for the systems $(K_i, t_i)_{i \in E'}$, where K is the field of the real or complex numbers and t_i is the usual euclidean topology for every $i \in E'$ (E' is the vector space of continuous linear functionals on a locally convex space (E, t)). Hence, by the preceding theorem the space $(E, \sigma(E, E'))$ is an HM-space.

COROLLARY 4 [**5**, Chapter IV, (3.3)]. *The space (E, τ_m) is an HM-space, if τ_m, is the minimal locally convex topology on E.*

Proof. According to [**5**] $\tau_m = \sigma((E')^{\#}, E')$ and by Corollary 3 (E, τ_m) is an HM-space.

Remark 1. By [**3**, Theorem 2] and [**5**, Chapter II. 6, Example 1] it follows that the class of HM-spaces is not invariant under direct sums and inductive limit.

Remark 2. According to Corollary 3, it is easy to see that the barrelled (quasi-barrelled, bornological, ultra-bornological) space associated to an HM-space is not an HM-space, in general. (About associated spaces see [**6**]).

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