

## ON EDGE-COLORABILITY OF PRODUCTS OF GRAPHS

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**Abstract.** Let  $\chi'(G)$  denote the edge-chromatic number and  $\Delta(G)$  the maximum vertex degree of a graph  $G$ . A graph  $G$  is said to be *of class 1* if  $\chi'(G) = \Delta(G)$  and *of class 2* otherwise. Some sufficient conditions for various graph products (the Cartesian, lexicographic, tensor and strong product) to be of class 1 are given.

**1. Introduction and definitions.** This note extends results of Himmelwright and Williamson [3], Kotzig [4], Mahmoodian [5] and Mohar, Pisanski and Shawe-Taylor [6, 7] concerning the edge-colorability of various graph products. Up till now we considered only the products of regular graphs, being inspired by the work of Kotzig [4]. The paper [6] summarizes our work in the regular case. However, most of the results can be extended to the products of non-regular graphs with almost no extra effort. In some cases, though, the condition of regularity is essential.

We will leave the basic definitions of graph theory to a standard reference book [2], and will limit ourselves to defining only less known terms and those which may cause confusion.

Let  $v$  and  $u$  be vertices of a graph  $G$ . We write  $v \sim u$  to denote that  $v$  and  $u$  are adjacent. Let  $\chi'(G)$  denote the edge-chromatic number and  $\Delta(G)$  the maximum vertex degree of a graph  $G$ . By the well-known Theorem of Vizing [9] on edge-colorability of graphs we can classify graphs into two classes. A graph  $G$  is said to be *of class 1* if  $\chi'(G) = \Delta(G)$ , and *of class 2* if  $\chi'(G) = \Delta(G) + 1$ .

The *Cartesian product*  $G \times H$  of graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$  and edge set

$$E(G \times H) = \{(u, v)(u', v'); \text{ either } (u = u' \text{ and } v \sim v') \text{ or } (u \sim u' \text{ and } v = v')\}.$$

The *lexicographic product*  $G \circ H$  of graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$  and edges  $E(G \circ H) = \{(u, v)(u', v'); \text{ either } u \sim u' \text{ or } (u = u' \text{ and } v \sim v')\}$ . Note that  $G \circ H$  is often written as  $G[H]$ .

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*AMS Subject Classification* (1980): 05 C 15, 05 C 70

*Key words and phrases:* Edge-coloring, Graph products.

This work was supported in part by the Boris Kidrič Fund, Ljubljana, Yugoslavia.

The *tensor product* of graphs  $G$  and  $H$  is defined as the graph  $G \otimes H$  with vertex set  $V(G) \times V(H)$  and the edge set

$$E(G \times H) = \{(u, v)(u', v'); u \sim u' \text{ and } v \sim v'\}.$$

If  $G$  and  $H$  have the same vertex set  $V = V(G) = V(H)$ , and disjoint edge sets,  $E(G) \cap E(H) = \emptyset$ , then their *sum*  $G \oplus H$  is defined as the graph having the edge set  $E(G \oplus H) = E(G) \cup E(H)$ . We also say that  $G$  and  $H$  are *factors* of  $G \oplus H$ .

The *strong product*  $G * H$  is defined as  $G * H = (G \circ H) \oplus (G \times H)$ .

All defined products are associative thus the products of more than two graphs can be defined without confusion.

**2. Results.** First of all we state a simple lemma, whose trivial proof we omit.

**2.1. LEMMA.** *Let  $F_1$  and  $F_2$  be two graphs which are of class 1 and let  $F = F_1 \oplus F_2$ . If*

$$\Delta(F) = \Delta(F_1) + \Delta(F_2) \quad (1)$$

*then  $F$  is of class 1.*

Now we present the generalization of Kotzig's result [4] for the cartesian product of graphs.

**2.2. THEOREM.** *Let  $H$  be the cartesian product of graphs  $G_1, G_2, \dots, G_n$  and let one of the following two conditions be satisfied*

- (i) *at least one of the graphs  $G_i$  is nontrivial and of class 1,*
- (ii) *there exist at least two distinct indices  $i$  and  $j$ , such that  $G_i$  and  $G_j$  both contain a 1-factor.*

*Then  $H$  is of class 1.*

It is worth noting that the sufficient condition (i) of Theorem 2.2 was obtained before by several authors. It was first given as an exercise in Bondy and Murty's book [1, exercise 6.2.6], then obtained by Himmelwright and Williamson [3] for the case of regular graphs and later extended by Mahmoodian [5] to the general case. At last this condition appeared in Kotzig's paper [4]. Condition (ii) can also be found there, though it is given only for the case of regular graphs. A proof similar to that in [4] can be applied to the nonregular case: Let  $G_i = F_i \oplus H_i$  and  $G_j = F_j \oplus H_j$ , where  $F_i$  and  $F_j$  are 1-factors. Then  $G_i \times G_j = (F_i \oplus H_i) \times (F_j \oplus H_j) = (F_i \times H_j) \oplus (H_i \times F_j)$ . By (i), both graphs  $F_i \times H_j$  and  $H_i \times F_j$  are of class 1 as  $F_i$  and  $F_j$  are of class 1. It is easy to see that the factorization  $(F_i \times H_j) \oplus (H_i \times F_j)$  satisfies condition (1) of Lemma 2.1. We conclude that  $G_i \times G_j$  is of class 1. Now, part (i) of the theorem applies to the general case.

Let us now consider the tensor and strong products.

**2.3 THEOREM.** *Let  $K$  be the tensor and  $H$  the strong product of graphs  $G_1, G_2, \dots, G_n$  and let at least one of the graphs  $G_i$  be nontrivial and of class 1. Then  $K$  and  $H$  are of class 1.*

We will omit the proof of Theorem 2.3, as it is the same as the one given in [7] for the regular case. Again, condition (1) of Lemma 2.1 is easily verified for all the factorizations used in the proof.

**2.4. THEOREM.** *The lexicographic product  $K = G_1 \circ G_2 \dots \circ G_n$  is of class 1 if at least one of the following conditions is satisfied:*

- (i)  *$G_1$  is a nontrivial graph which is of class 1,*
- (ii) *for some  $i$  ( $2 \leq i \leq n$ ) the graph  $G_i$  is nontrivial and of class 1 and for some  $j \geq i$  the graph  $G_j$  is of even order or*
- (iii) *there exists two distinct indices  $i$  and  $j$ , such that the graphs  $G_i$  and  $G_j$  both contain a 1-factor.*

Before proving the theorem we need a lemma:

**2.5. LEMMA.** *Let  $H$  be a lexicographic product of graphs  $F_1, F_2, \dots, F_n$  and assume that for some  $i$  ( $1 \leq i \leq n$ ) the graph  $F_i$  has a 1-factor. Then  $H$  has a 1-factor.*

*Proof.* By the law of associativity of the lexicographic product

$$H = (F_1 \circ \dots \circ F_{i-1}) \circ F_i \circ (F_{i+1} \circ \dots \circ F_n)$$

It follows that it suffices to prove the lemma only for the case  $n = 2$ . But since

$$F_1 \circ F_2 = (F_1 \times F_2) \oplus (F_1 \oplus K_m), \quad m = |V(F_2)|$$

and  $F_1 \times F_2$  contains a 1-factor if either  $F_1$  or  $F_2$  does, the lemma follows.

**PROOF OF THEOREM 2.4.** From

$$K = G_1 \circ (G_2 \circ \dots \circ G_n) = (G_1 \circ \dots \circ G_{i-1}) \circ (G_i \circ (G_{i+1} \circ \dots \circ G_n))$$

we can conclude that it suffices to give a proof of (i) and (ii) only for the case  $n = 2$ . Since

$$G_i \circ \dots \circ G_j = G_i \circ (G_{i+1} \circ \dots \circ G_j)$$

and since by Lemma 2.5  $G_{i+1} \circ \dots \circ G_j$  has a 1-factor if  $G_j$  has a 1-factor, the same is true for the case (iii).

Let  $K = G_1 \circ G_2$ . The lexicographic product can be factored in an obvious way,

$$K = G_1 \circ G_2 = (G_1 \times G_2) \oplus (G_1 \oplus K_m), \quad m = |V(G_2)|. \quad (2)$$

By Theorem 2.2, the cartesian product  $G_1 \times G_2$  is of class 1 if at least one of the conditions (i)–(iii) is satisfied. The maximum degrees of graphs in the factorization (2) are:

$$\Delta(G_1 \circ G_2) = m\Delta(G_1) + \Delta(G_2)$$

$$\Delta(G_1 \times G_2) = \Delta(G_1) + \Delta(G_2)$$

$$\Delta(G_1 \oplus K_m) = \Delta(G_1) \cdot \Delta(K_m) = \Delta(G_1).$$

Therefore Lemma 2.1 can be applied and we have to prove only that the graph  $G_1 \oplus K_m$  is of class 1. But this is obvious: in case (i) the graph  $G_i$  is of class 1, whereas in cases (ii) and (iii)  $m$  is even and thus  $K_m$  is of class 1 (existence of a 1-factor implies that  $G_2$  has an even number of vertices). By Theorem 2.3 the tensor product  $G_1 \oplus K_m$  is of class 1. This completes the proof.

The requirement in Theorem 2.4 (ii) that the graph  $G_i \circ G_{i+1} \circ \dots \circ G_n$  has an even number of vertices is essential. For example the graphs  $C_3 \circ (K_1 \cup K_2)$  and  $C_{2n+1} \circ P_{2k+1}$  ( $n \leq k$ ) are of class 2, though  $K_1 \cup K_2$  and  $P_{2k+1}$  are of class 1. In fact there is a general construction of such graphs: Let  $G$  be any regular graph of odd order. Then for all  $k$ 's large enough,  $G \circ P_{2k+1}$  is of class 2. To see this one observes that the given examples have only vertices of degree  $d$  and  $d - 1$ . If the graph has an odd number of vertices, and, moreover, if the number of vertices of degree  $d - 1$  is less than  $d$  one can verify that such a graph must be of class 2.

**3. Concluding remarks.** It is interesting that Theorem 2.2 can be put in a more general setting. In particular, the results for the cartesian product can be seen as a special case of a similar result for graph bundles, obtained by Pisanski, Shawe-Taylor and Vrabec [8].

It is worth noting that the sufficient conditions of Theorems 2.2, 2.3 and 2.4 are not necessary, as it is shown in [4, 7] in the case of regular graphs.

Finally we mention some open problems concerning the classification of various graph products.

PROBLEM 1. *Let  $G$  and  $H$  both have 1-factors. Does it follow that  $G \oplus H$  is of class 1?*

The answer to problem 1 is not known even for regular graphs.

PROBLEM 2. *Let  $H$  be a nontrivial graph of class 1 and of odd order and let  $G$  be of even order. Is the lexicographic product  $G \circ H$  of class 1?*

PROBLEM 3. *Are the conditions (i) or (ii) of Theorem 2.2 sufficient for some more general graph products to be of class 1?*

Finally I would like to thank Dr. Tomo Pisanski and John Shawe-Taylor for their assistance and helpful comments on this paper.

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(Received 21 07 1983)