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NONEXISTENCE OF NONMOLECULAR GENERIC SETS

Donald D. Steiner and **Alexander Abian**

Abstract. Generic subsets of partially ordered sets play an important role in giving significant examples of Zermelo-Fraenkel set-theoretical models. The significance of these models lies in the fact that a generic subset G of a partially ordered set P, in general, does not exist in a model M in which P exists. Thus, by adjoining G to M an interesting extended model may ensue which has properties not shared by M. Thus, in considering generic extensions of set-theoretical models it is quite relevant to know whether or not a generic subset of a partially ordered set P exists in the same model in which P exists. In this paper, we give a necessary and sufficient condition for P to have a generic subset in the same model.

Let $(P, \leq)$ be a partially ordered set. As usual, when no confusion is likely to arise, we represent $(P, \leq)$ simply by P. If P has a *minimum* (i.e., the *smallest*) element, we represent it by 0 and we call it the *zero* element of P.

Definition 1. *A* subset D of a partially ordered set $(P, \leq)$ is called a dense (or, a coinitial) subset of P if and only if for every nonzero element x of P there exists a nonzero element y of D such that $y \leq x$.

It is an interesting fact that, as shown in [1], a partially ordered set P has either finitely many or else continuum many dense subsets. Clearly, every nonzero minimal element of P is an element of every dense subsets. Clearly, every nonzero minimal element of P is an element of every dense subset of P. On the other hand, if 0 of P exists, 0 need not be an element of every dense subset of P. However, if D is a dense subset of P then $D - \{0\}$ as well as $D \cup \{0\}$ is a dense subset of P. Indeed, every superset of a dense subset of P is a dense subset of P.

Based on the notion of a dense subset of a partially ordered set, we introduce the notion of a generic subset of a partially ordered set as follows:

Definition 2. Let $(P, \leq)$ be a partially ordered set. A subset G of P is called a generic subset of P (or, simply generic) if and only if: (1) $0 \notin G$ and G is nonempty; (2) For every element x and y of P if $x \in G$ and $x \leq y$ then $y \in G$; (3) Every two elements of G have a lower bound in G; (4) G has a nonempty intersection with every dense subset of P.

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We observe that by (1) and (3) every two elements of G have a nonzero lower bound in G. Also, we observe that (1), (2) (3) imply that G is a filter. Therefore, a generic subset G of a partially ordered set P is a filter of P such that G has a nonempty intersection with every dense subset of P.

As shown below, not every partially ordered set has a generic subset. In fact, according to Theorem 1 below, a partially ordered set has a generic subset if and only if it has a molecule. In the literature, a modified notion of a generic subset is usually considered where (4) is replaced by:

(4a) G has a nonempty intersection with every dense subset of P belonging to a preassigned collection M of dense subsets of P.

If in Definition 1, condition (4) is replaced by (4a), then G is called *generic over* M and is denoted by G/M. It can be readily verified [3], that if M is a denumerable collection $\{D_0, D_1, D_2, D_3, \dots\}$ of dense subsets D_i of $(P, \leq)$, then P always has a *generic over* M subset G with $p \in G$ for any nonzero element p of P. Indeed, since D_i's are dense in P there exists a nonincreasing sequence:

$$\dots \leq d_3 \leq d_2 \leq d_1 \leq d_0 \leq p$$

where $d_i \in D_i$. But then clearly $G = \{x \mid x \in P \text{ and } d_i \leq x \text{ for some } i\}$ is a generic over M subset of P.

Definition 3. A nonzero element m of a poset P is called a molecule of P if and only if every two nonzero elements of P which are less than or equal to m have a nonzero lower bound.

Clearly, every nonzero element of a simply ordered set S is a molecule of S.

A typical partially ordered set without a molecule is provided with the following example. Let H be the set of all finite sequences of the natural numbers $0, 1, 2, 3, \dots$. Let us consider the partial order $(H, \leq)$ where $p \leq q$ if and only if p is an extension of q. Thus, $(0,3,5,2,1) \leq (0,3,5)$ and $(0,3,5,7) \leq (0,3,5)$. It can be readily verified that $(H, \leq)$ has no molecule.

Definition 4. Let m be a molecule of a partially ordered set $(P, \leq)$. Then a subset $G(m)$ of P is called molecular, or generated by the molecule m if and only if:

$$G(m) = \{x \mid (x \in P) \wedge (\exists y)((y \neq 0) \wedge (y \in P) \wedge (y \leq m) \wedge (y \leq x))\} \tag{5}$$

Accordingly, $G(m)$ consists of those elements of P each of which is greater than or equal to some element of P which is less than or equal to m.

From (5) it obviously follows that

$$m \in G(m) \tag{6}$$

Next, we prove [cf. 4, p. 26].

Theorem 1. *A subset G of a partially ordered set $(P, \leq)$ is generic if and only if G is generated by a molecule of P.*

Proof. Let G be a generic subset of P. First, we show that there exists $m \in G$ such that

$$\{y \mid (y \in P) \wedge (0 \neq y) \wedge (y \leq m)\} \subseteq G \text{ and } m \text{ is a molecule of } P. \tag{7}$$

Assume on the contrary that for every $m \in G$ there exists a nonempty subset $N(m)$ of P such that $z \in N(m)$ if and only if $z \leq m$ and $z \in (P - G)$. But then clearly, $P - G$ is a dense subset of P which has an empty intersection with G, contradicting (4). Thus, the first part of (7) is established. Now, from this and (3) and the fact that G is generic, it follows that m is nonzero and that every two nonzero elements of P which are lees than or equal to m have a nonzero lower bound in G (and a fortiori in P). Thus, m is a molecule of P, according to Definition 3 and the proof of (7) is complete. Next, we show that $G = G(m)$. Let $x \in G$. By (3), we see that x and m must have a lower bound, say, y in G. Thus, $y \leq x$ and $y \leq m$. But then, from (5) it follows that $x \in G(m)$. But then, from (5) and (7), it follows that $x \geq y$ for some $y \in G$ which by (2) implies that $x \in G$. Hence, indeed $G = G(m)$.

To complete the proof of the theorem, it remains to show that $G(m)$ as given by (5) is a generic subset of P.

We observe that (1) follows directly from (5). To establish (2), it is enough to observe that if $z \in G(m)$ then by (5) we see that $z \geq y$ for some nonzero $y \leq m$. Therefore, if $z \leq x$ then $x \leq y$ with $y \leq m$, which implies $x \in G(m)$. To establish (3), it is enough to observe that if $x_1 \in G(m)$ and $x_2 \in G(m)$, then by (5) we see that $y_1 \leq x_1$ and $y_2 \leq x_2$ for some nonzero $y_1 \leq m$ and nonzero $y_2 \leq m$. But since m is a molecule, y_1 and y_2 have a nonzero lower bound, which by (5) is an element of $G(m)$. Hence, x_1 and x_2 have a lower bound in $G(m)$. To establish (4), let D be a dense subset of P. But then D has a nonzero element d such that $d \leq m$. From (5) it follows that $d \in G(m)$ and therefore $G(m)$ has a nonempty intersection with every dense subset of P.

COROLLARY 1. *If k is a nonzero minimal element of a partially ordered set $(P, \leq)$ then k is a molecule of P and $\{x \mid x \in P$ and $k \leq x\}$ is a generic subset of P.*

Proof. Since k has no nonzero predecessors, k is trivially a molecule of P. But then the conclusion of the corollary follows from (5).

COROLLARY 2. *A partially ordered set has a generic subset if and only if it has a molecule.*

This is an immediate consequence of Theorem 1 and (6). Accordingly, the partial order $(H, \leq)$ mentioned above, has no generic subset.

Let us recall the following:

DEFINITION 5. A Boolean algebra $(A, \leq)$ is a complemented distributive lattice with a minimum 0 and a maximum 1. Moreover, a nonzero element a of A is called an atom of A if and only if for every $x \in A$ it is the case that $x < a$ implies $x = 0$.

Furthermore, a subset U of A is called an ultrafilter of A if and only if:

(8) $0 \notin U$; (9) For every element x and y of A if $x \in U$ and $x \leq y$ then $y \in U$; (10) Every two elements of U have a lower bound in U; (11) For every element x of A either $x \in U$ or else $x' \in U$ where x' is the complement of x.

The reader is advised to compare (8), (9), (10), (11) with (1), (2), (3), (4). Clearly, $1 \in U$ by (11) so that U is nonempty.

LEMMA 1. *Let $(A, \leq)$ be a Boolean algebra. An element a of A is an atom of A if and only if a is a molecule of A.*

Proof. Let a be an atom of A. Clearly, a is a nonzero minimal element of A and therefore by Corollary 1, we see that a is a molecule of A. Conversely, let a be a molecule of A. To prove that a is an atom of A it is enough to show that $x < a$ for no nonzero element x of A. Assume on the contrary that $x < a$ for some nonzero element x of A. But then, since A is a Boolean algebra $a - x$ exists, is nonzero and $(a - x) < a$. However, from Definition 3 it follows that x and $a - x$ must have a nonzero lower bound. But this leads to a contradiction since the only lower bound of x and $a - x$ in A is 0. Thus, our assumption is false and a is an atom of A.

Definitions 2 and 5 indicate that an ultrafilter of a Boolean algebra somewhat resembles a generic subset of it. However, in view of the discrepancies between (4) and (11), we must not expect that every ultrafilter of a Boolean algebra is also a generic subset of it. Indeed, in view of Theorem 1 and Lemma 1, we have:

THEOREM 2. *A subset G of a Boolean algebra A is generic if and only if G is generated by an atom of A (i.e., if and only if G is a principal ultrafilter of A).*

Proof. By Theorem 1 and Lemma 1, we see that G must be generated by an atom a of A which in view of (5) implies:

$$G = G(a) = \{x \mid x \in A \;\; and \;\; a \leq x\}$$

But then it is a routine matter to verify that the above equality implies that $G(a)$ is an ultrafilter of A (as defined by (8), (9), (10), (11)) generated by the atom a of A.

The following lemmas show some significant properties of dense subsets of a Boolean algebra.

LEMMA 2. *Let $A, \leq)$ be a Boolean algebra and D a dense subset of A. Then* $\operatorname{lub} D = 1$.

Proof. Clearly, 1 is an upper bound of D. To prove the lemma we show that every upper bound u of D is equal to 1. Assume on the contrary that u is an upper bound of D and $u < 1$. But then $1 - u$ (i.e., the complement u' of u) is a nonzero element of A. However, no element of D is less than or equal to $1 - u$, contradicting the denseness of D. Hence $u = 1$, as desired.

LEMMA 3. *Let $(A, \leq)$ be a Boolean algebra and H a subset of A such that* $\operatorname{lub} H = 1$. *Then the subset D of A given by*

$$D = \{x \mid x \in A \;\; and \;\; x \leq h \;\; for\ some \;\; h \in H\} \tag{12}$$

is a dense subset of A.

Proof. We must show that for every nonzero element p of A there exists a nonzero element d of D such that $d \leq p$. Since $\operatorname{lub} H = 1$ and since $(A, \leq)$ is a Boolean algebra, we see that

$$p = p \wedge (\operatorname{lub} H) = \operatorname*{lub}_{h \in H} (p \wedge h) \tag{13}$$

Since $p \neq 0$, from (13) it follows that $(p \wedge h) \neq 0$ for some $h \in H$. Let $d = p \wedge h$. Thus, d is nonzero and since $d \leq h$, we see by (12) that $d \in D$. Clearly, $d \leq p$ as desired.

As mentioned earlier, we called a nonempty subset G of a partially ordered set a filter if and only if G satisfies (1), (2), (3). Very often, in the literature [2], condition (3) is replaced by "*the greatest lower bound of every two elements of G exists and is an element of G*". However, this point is immaterial for our purposes.

THEOREM. *Let A be a Boolean algebra. Then a filter G of A is a generic subset of G if and only if for every family $(a_i)_{i \in E}$ of A it is the case that*

$$\operatorname*{lub}_{i \in E} a_i = 1 \ \textit{ implies } \ a_i \in G \ \textit{ for some } \ i \in E \tag{14}$$

Proof. Let G be a generic subset of A. But then by Theorem 2 wee see that A has an atom a and $a \in G$. Now, let $\operatorname{lub}_{i \in E} a_i = 1$. Since A is a Boolean algebra, we have:

$$a = a \wedge (\operatorname{lub} a_i) = \operatorname{lub}(a \wedge a_i) \ \textit{ with } \ i \in E \tag{15}$$

However, since a is an atom, $a \wedge a_i = 0$ or $a \wedge a_i = a$. But then from (15) it follows that $a \wedge a_i = a$ for some $i \in E$. Hence, $a \leq a_i$ and since $a \in G$ and G is a filter, $a_i \in G$ by (2). Thus, (14) is established.

Conversely, let G be a filter of A satisfying (14). We show that G is a generic subset of A. To this end, in view of (4) it is enough to prove that G has a nonempty intersection with every dense subset of A. Now, let D be a dense subset of A. By Lemma 2 we see that $\operatorname{lub} D = 1$ and by (14) we derive that $d \in G$ for some $d \in D$. Thus, indeed G has a nonempty intersection with every dense subset of G, as desired. In view of Theorem 2, clearly G in also a principal ultrafilter of A.

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MCC, 9430 Research Blvd., Austin, Texas 78759, U.S.A.
(Received 14 11 1983)
Department of Mathematics, Iowa State University,
Ames, Iowa 50011, U.S.A.
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