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ON THE ABSOLUTE SUMMABILITY OF LACUNARY FOURIER SERIES

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Abstract. Let $f \in L[-\pi,\pi]$ and let its Foirer Series $\sigma(f)$ be lacynary. The absolute convergence of $\sigma(f)$ when f satisfies Lipschitz condition of order α, $0<\alpha<1$, only at a point and when $\{n_k\}$ satisfies the gap condition $n_{k+1}-n_k \geq An_K^{\beta}k^{\gamma}$ $(0<\beta<1,\ \gamma\geq 0)$ is obtained by Patadian and Shah when $\alpha\beta+\alpha\gamma>(1-\beta)/2$. Here we study the absolute summability of $\sigma(f)$ when $\alpha\beta+\alpha\gamma\leq(1-\beta)/2$.

1. Let

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{nk}\sin n_k x) \tag{1.1}$$

be the Fourier series of a 2π-periodic function $f\in L[-\pi,\pi]$ with an infinity of gaps (n_k, n_{k+1}), where $\{n_k\}$ $(k\in N)$ is a strictly increasing sequence of natural numbers. Noble [**7**], Kennedy [**4, 5, 6**], and several other mathemaicians , have studied the absolute convergence of the Fourier series (1.1), as well as the order of magnitude of Fourier coefficients, by considering various properties of f either on an arbitrary subinterval or on an arbitrary subset of $[-\pi,\pi]$ of positive measure. This way they obtained a number of results under different lacunarity conditions. Izumi and Izumi [**3**], Chao [**1**], and Patadia and Shah [**8**], have studied this problem for the Fourier series (1.1) with some lacunae when the function satisfies Lipschitz condition only at a point. Chao [**1**] proved the following theorems:

THEOREM A. [**1**; Theorem 1]. *If*

$$\text{(i)}\quad f\in \operatorname{Lip}\alpha\,(\alpha>0)\ \textit{at a point}\ x_0\in(-\pi,\pi), \tag{1.2}$$

$$\text{(ii)}\quad n_{k+1}-n_k\geq A\,F(n_k) \tag{1.3}$$

where $F(n_k)\uparrow\infty$ *as* $k\to\infty$, $F(n_k)\leq n_k$ *for all* k *and* A *is a positive constant, then*

$$a_{n_k}, b_{n_k} = O(F(n_k)^{-\alpha}),\ k=1,2,\dots \tag{1.4}$$

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THEOREM B. [**1**; Theorem 2]. *If f satisfies* (1.2) *and if*

$$n_{k+1} - n_k \geq A n_k^{\beta} k^{\gamma} \ (0 < \beta < 1, \ \gamma \geq 0) \tag{1.5}$$

where A is a positive constant, then the Fourier series (1.1) *of f converges absolutely when $\alpha\beta + \alpha\gamma + \beta > 1$.*

Furthermore, Patadia and Shah [**8**] considered the same gap condition (1.5) and proved the following theorem:

THEOREM C. *If f satisfies* (1.2), *and if $\{n_k\}$ satisfies* (1.5), *then*

$$\sum_{k=1}^{\infty} (\mid a_{n_k} \mid^r + \mid b_{n_k} \mid^r) < \infty \ \ 0 < r \leq 1 \tag{1.6}$$

when $\alpha\beta r + \alpha r\gamma > (1 - r/2)(1 - \beta)$.

We observe that the particular case of theorem C when $r = 1$ provides us with a generalization of Theorem B, ensuring the absolute convergence o the Fourier series (1.1) when $\alpha\beta + \alpha\gamma > (1 - \beta)/2$. It may be noted here that when $\alpha\beta + \alpha\gamma = (1 - \beta)/2$, the absolute convergence of (1.1) is obtained by Patadie and Shah [**9**] by taking at apoint a little stronger condition than $\mathrm{Lip}\,\alpha$ on f.

Now, it is quite natural to inequire into the behaviour of the Fourier series (1.1) of a function f in $\mathrm{Lip}\,\alpha$ at a point, when $\alpha\beta + \alpha\gamma \leq (1 - \beta)/2$. In this regard, we propose to study the absolute summability (c, θ) of the series (1.1). We prove the following theorem:

THEOREM. *If $f \in Lip\,\alpha (0 < \alpha < 1)$ at a point $x_0 \in (-\pi, \pi)$, and if $\{n_k\}$ satisfies* (1.5) *with some suitable constant A, then the Fourier series* (1.1) *of f is absolutely summable (c, θ) for $0 < \theta \leq 1$ when*

$$\alpha > \max\left\{ \frac{1 - \beta - \theta - \gamma\theta}{\beta + \gamma}, \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma} \right\}.$$

Remark 1. Theorems 1 and 2 due to Patel [**10**] are particular cases of this theorem when $\theta = 1, \ \gamma = 0$, and $\theta = 1/2, \ \gamma = 0$ respectively.

Remark 2. It is interesting to observe that when $\gamma = 1$, the theorem gives the absolute summability $(c, 1)$ of the Fourier series (1.1) for every $\alpha > 0$; and that, when $\gamma = 3/2$, we get the absolute summability $(c, 1/2)$ of (1.1) for every $\alpha > 0$

2. We need the following lemma due to Patadia and Shah [**9**].

LEMMA. *If $\{n_k\}$ satisfies* (1.5) *with $A > 2^M - 1$, M being a positive integer reater than, δ, where $\delta = (1 + \gamma)/(1 - \beta)$, then*

$$n_k \geq k^{\delta} \ \ \textit{for all} \ \ k \in N \tag{2.1}$$

Proof of the Theorem. For a real number s, which is not a negative integer, put $E_n^s = \binom{n+s}{n}$ where $n \in N$ and $E_0^s = 1$. Denoting the n-th Cesaro mean of order $\theta > 0$ by $\sigma_n^\theta(x)$, and replacing the absent terms in (1.1) by zeros, we have [**2**]:

$$| \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) |= \frac{1}{n_k \cdot E_{n_k}^\theta} \Big| \sum_{p=1}^{k} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p} \cos n_p x + b_{n_p} \sin n_p x) \Big|$$
$$\leq \frac{1}{n_k \cdot E_{n_k}^\theta} \Big\{ | n_k(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) | +$$
$$+ \Big| \sum_{p=1}^{k-1} E_{n_k-n_p}^{\theta-1} \cdot n_p \cdot (a_{n_p}) \cos n_p x + b_{n_p} \sin n_p x) \Big| \Big\}. \tag{2.2}$$

Let $0 < \theta \leq 1$. Now.

$$\text{(i)} \quad E_n^\theta \simeq \frac{n^\theta}{\Gamma(\theta+1)}, \quad \text{(ii)} \ a_{n_k},\ b_{n_k} = O\Big(\frac{1}{n_k^{\alpha\beta} \cdot k^{\gamma\alpha}}\Big), \ \ k = 1, 2, 3, \dots,$$

by taking $F(n_k) = n_k^\beta k^\gamma$ in Theorem A, and

$$\text{(iii)} \quad | n_k - n_p | \geq | n_k - n_{k-1} | \quad \text{for } p = 1, 2, 3, \dots, k-1$$
$$\geq A n_k^\beta k^\gamma, \ \text{ by (1.5)}.$$

Hence, from (2.1) and (2.2), we obtain

$$| \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) |$$
$$=0(1) \frac{1}{n_k n_k^\theta} \Big\{ n_k n_k^{-\alpha\beta} k^{-\gamma\alpha} + \sum_{p=1}^{k-1} \frac{1}{(n_k - n_p)^{1-\theta}} \cdot n_p \cdot n_p^{-\alpha\beta} p^{-\gamma\alpha} \Big\}$$
$$=0(1) \frac{1}{n_k^{1+\theta}} \Big\{ n_k^{1-\alpha\beta} k^{-\gamma\alpha} + \Big(\frac{1}{n_k^\beta k^\gamma}\Big)^{1-\theta} \sum_{p=1}^{k-1} \frac{n_p^{1-\alpha\beta}}{p^{\gamma\alpha}} \Big\}$$
$$=0(1) \frac{1}{n_k^{1+\theta}} \Big\{ n_k^{1-\alpha\beta} k^{-\gamma\alpha} + \Big(\frac{1}{n_k^\beta k^\gamma}\Big)^{1-\theta} \cdot k \cdot n_k^{1-\alpha\beta} \Big\},$$

as $p^{-\gamma\alpha} \leq 1$ and $n_p^{1-\alpha\beta} \leq n_k^{1-\alpha\beta}$, $0 < \alpha, \beta < 1$. Therefore

$$| \sigma_{n_k}^\theta(x) - \sigma_{n_k-1}^\theta(x) | = 0(1) \Big\{ \frac{1}{n_k^{\theta+\alpha\beta} k^{\gamma\alpha}} + \frac{1}{n_k^{\theta+\beta-\beta\theta+\alpha\beta} k^{\gamma-\gamma\theta-1}} \Big\}$$
$$= 0(1) \Big\{ \frac{1}{k^{\delta(\theta+\alpha\beta)+\gamma\alpha}} + \frac{1}{k^{\delta(\theta+\beta-\beta\theta+\alpha\beta)+\gamma-\gamma\theta-1}} \Big\}$$
$$= 0(1) \Big\{ \exp_k \Big(\frac{\theta + \alpha\beta + \gamma\theta + \alpha\gamma}{1-\beta} \Big) +$$
$$+ \exp_k \Big(\frac{\theta + 2\beta - \beta\theta + \alpha\beta\gamma\alpha\beta + \gamma - 1}{1-\beta} \Big) \Big\}, \tag{2.3}$$

as $\delta = (1+\gamma)/(1-\beta)$, ($\exp_k A$ denotes k^{-A}). Finally, since $\alpha > (1-\beta-\theta\gamma\theta)/(\beta+\gamma)$, it follows that $(\theta + \alpha\beta + \gamma\theta + \alpha\gamma)/(1-\beta) > 1$; and since

$$\alpha > \frac{2 - 3\beta - \gamma + \beta\theta - \theta}{\beta + \beta\gamma}$$

we have

$$\frac{\theta + 2\beta - \beta\theta + \alpha\beta + \alpha\beta\gamma + \gamma - 1}{1-\beta} > 1.$$

Hence, from (2.3) we have

$$\sum_{k=1}^{\infty} \mid \sigma_{n_k}^{\theta}(x) - \sigma_{n_k-1}^{\theta}(x) \mid < \infty,$$

which implies the absolute summability (c, θ) of (1.1). This completes the proof of the theorem.

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