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ON FINITE-ELEMENT SIMPLE EXTENSIONS OF A COUNTABLE COLLECTION OF COUNTABLE GROUPOIDS

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Abstract. Belkin and Gorbunov [**2**] showed that any two finite groupoids can be imbedded into a finite simple groupoid. We prove here a stronger result: Any countable collection $\{A_i\}_{i\in I}$ of countable grupoids can be embedded into a simple groupoid $K(\bigcup_{i\in I} A_i)$ such that $K(\bigcup_{i\in I} A_i) - \bigcup_{i\in I} A_i$ contains only a single element which generates the whole groupoid.

A universal algebra $\mathfrak{U} = \langle A; F\rangle$ is said to be simple if its lattice of congruences $\operatorname{Con}\mathfrak{U}$ is isomorphic to the two element chain. A variety (= equational class) of universal algebras of fixed type is said to have the simple extension property if any algebra in this variety can be imhedded into a simple algebra in this variety.

A groupoid $\langle G; 0\rangle$ is a universal algebra of type (2). The vairety $\mathfrak{G}(2)$ of groupoids has the simple extension property. This result was discovered independently by several authors ([**3, 4, 5**]).

V. P. Belkin and V. A. Gorbunov [**1**] showed that in the lattice $L(\mathfrak{G}(2))$ of quasivarieties of groupoids every nonunit filter of $L((\mathfrak{G}(2))$ has cardinality $2^{\aleph}$. In order to establish this result they show that any two finite groupoids A, B are embedded into a strongly simple partial groupoid $P(A,B)$. We recall that a partial groupoid is strongly simple if any full extension groupoid of this partial groupoid is simple.

Their construction of $P(A,B)$ is given as follows:

Assume $A = \{2,3,\ldots,s\}$, $B = \{s+1,\ldots,t\}$. Let p be a prime number greater than t and set $P(A,B) = \{0,1,2,\ldots,p-1\}$. The multiplication on $P(A,B)$ is defined as follows:

$$\begin{aligned}
&0\circ 0 = 1\\
&r\circ 0 = 0\circ r,\ \text{ if } r\neq 0\\
&1\circ r = r\circ 1 = r+1 \quad (\mod p)\\
&r\circ r = 0 \quad \text{if } r\notin\{0,1,\ldots,t\}
\end{aligned}$$

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multiplication on the subsets A and B coincides with the multiplication in the groupoids A and B, respectively. Note that in the original and the translation versions of the above article there are two obvious errors in the definition of the multiplication.

Belkin and Gorbunov showed that the partial groupoid $P(A, B)$ has the following feature: $P(A, B)$ is generated by any element in $P(A, B) - (A \cup B)$ and it is strongly simple. Because of the irregularity of the distribution of the prime numbers, this construction may be practically useless if one has to add billions of new elements to $A \cup B$ in order to obtain the simple groupoid $P(A, B)$. This leads us to the following concept:

Definition 1. For each integer $n \geq 1$, a groupoid P is called the n-element simple extension of a collection of groupoids $\{A_i\}_{i \in I}$ if

(1) each A_i is a subgroupoid of P,

(2) P is a simple groupoid,

(3) $P - (\bigcup_{i \in I} A_i)$ has only n elements.

We have the following result:

THEOREM 1. *Any finite collection of finite groupoids has a* 2*-element simple extension.*

Proof. Suppose we have $A_i = \{a_{i,1}, \ldots, a_{i,t(i)}\}$ $i = 1, \ldots, n$ a finite groupoid. We call $a_{i,1}$ and $a_{i,t(i)}$ the initial and terminal element of A_i respectively. Let $S(A_1, \ldots, A_n) = \{0, 1\} \bigcup_{a=1}^{n} A_i$ where 0,1 are two elements not in $\bigcup_{i=1}^{n} A_i$.

We define the partial multiplication $\circ$ on $S(A_1, \ldots, A_n)$ as follows:

$$0 \circ 0 = 1 \qquad 1 \circ 1 = a_{1,1}$$

$$0 \circ x = x \circ 0 = 0 \text{ for all } x \neq 0 \text{ in } S(A_1, \ldots, A_n),$$

$$1 \circ a_{i,j} = a_{i,j} \circ 1 = \begin{cases} a_{i,j+1} & \text{if } j \neq t(i) \\ a_{i+1,1} & \text{if } i \neq n, j = t(i) \\ 0 & \text{if } i = n, j = t(n), \end{cases}$$

multiplication on the subsets A_i coincides with the multiplication in the original groupoids.

To see that the partial groupoid $\langle S(A_1, \ldots, A_n); \circ \rangle$ is strongly simple, we let θ be a non-identity congruence relation on a full extension of $S(A_1, \ldots, A_n)$ and x, y are distinct elements such that $x\theta y$.

We distinguish the following cases:

Case 1. $x = 0$ and $y = 1$. Left multiplying both sides of the congruence $0\theta 1$ by $a_{i,j} \in A_i$ we have $0\theta a_{i,j+1}$ or $0\theta a_{i+1,1}$ depending on whether $j \neq t(i)$ or $j = t(i)$ and $i \neq n$. Multiplying both sides of the congruence $0\theta 1$ by 1 we obatain $0\theta a_{1,1}$. Therefore θ is the universal congruence.

Case 2. $x = 0$ and $y = a_{i,j}$ for some $a_{i,j}$ in A_i. Multiplying both sides of the congruence by 0 we obtain $1\theta 0$ which reduces to Case 1.

Case 3. $x = a_{i,j}$ and $y = a_{k,s}$ $i \leq k$. Multiplying the both sides of the congruence by 1 successively several times, we obtain $a_{p,q}\theta 0$ which reduces to Case 2.

All these cases show that θ is the universal congruence. Hence the groupoid $S(A_1, \dots, A_n)$ is simple.

Remark. For $n = 2$, the reader may notice that our partial groupoid $S(A_1, A_2)$ is similar to the partial groupoid $P(A_1, A_2)$ of Belkin and Gorbunov. However, we threw away the superfluous elements: $t+1, \dots, p-1$ which reduced the size of our groupoid.

Modifying the above construction we obtain the following result:

THEOREM 2. *Any finite collection of finite groupoids has a* 1*-element simple extension.*

Let $\{A_1\}_{i=1,\dots,n}$ be a finite collection of finite groupoids. We adjoin to $\bigcup_{i=1}^{n} A_i$ a new element, say 0. We let $OS(A_1, \dots, A_n) = \{0\} \cup \bigcup_{i=1}^{n} A_i$ and define a binary partial operation as follows:

$$0 \circ 0 = a_{1,1}, a_{1,1} \circ 0 = a_{1,1} \text{ and } x \circ 0 = 0 \text{ for } x \in OS(A_1, \dots, A_n) - \{0, a_{1,1}\}$$

$$0 \circ a_{i,j} = \begin{cases} a_{i,j+1} & \text{if } j \neq t(i) \\ a_{i+1,1} & \text{if } i \neq n, j = t(i) \\ 0 & \text{if } i = n, j = t(n), \end{cases}$$

multiplication on the subset A_i coincides with the multiplication in the original groupoid and the product of other elements are arbitrary. We can show that the partial groupoid $OS(A_1, \dots, A_n)$ is strongly simple.

The proof can be ground out in the similar way as the proof of Theorem 1. We omit the details of the proof.

Now a question arises: can we have a simple one-element extension of a finite collection of groupoids A_i each of which is of cardinality of at most countable? We can ask further whether one can have a simple one-element extension of a countable collection of groupoids A_i such that each of which is of cardinality at most countable.

We can show that:

THEOREM 3. *Any countable collection* $\{A_i\}_{i \in I}$ *of countable groupoids has a simple one-element extension* $K(\bigcup_{i \in I} A_i)$ *such that it is monogeneric.*

Evans showed in [**2**] that any countable groupoid is a subgroupoid of a monogeneric groupoid, i.e. it is generated by a single element. Our result shows that the groupoid can be chosen to be simple.

Suppose $A_i = \{a_{i,1}, \dots\}$. If A_i is finite we denote its last element by $a_{i,t(i)}$.

Let $K(\bigcup_{i \in I}) A_i) = \{0\} \cup \bigcup_{i \in I} A_i$ where 0 is an element not in $\bigcup_{i \in I} A_i$. We define a binary operation on $K(\bigcup_{i\in I} A_i)$ as follows:

$$0\circ 0 = a_{1,1}, \tag{1}$$

$$0\circ a_{i,j} = \begin{cases} a_{i+1,1} & \text{if } j=1, \text{ and } i<\aleph_0<|I|,\\ a_{1,1} & \text{if } j=1, i=|I|<\aleph_0,\\ a_{i,j+1} & \text{if } j\neq 1,\\ a_{i,1} & \text{if } j=|A_i|<\aleph_0, \end{cases} \tag{2}$$

$$a_{i,j}\circ 0 = \begin{cases} a_{i,1} & \text{if } j=1, \text{ and } |A_i|=1,\\ a_{i,2} & \text{if } j=1, |A_i|\geq 2,\\ 0 & \text{if } j=2,\\ a_{i,j-1} & \text{otherwise}, \end{cases} \tag{3}$$

$$a_{i,j}\circ 0 = \begin{cases} a_{1,1} & \text{if } i<j,\\ a_{i,p}a_{i,q} & \text{if } i=j,\\ 0 & \text{if } i>j, \end{cases} \tag{4}$$

The proof of the simplicity of $K(\bigcup_{i\in I} A_i)$, is tedious but not difficult. The reader can complete it by imitating the proof of Theorem 1.

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