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AN APPLICATION OF THE RUSCHEWEYH DERIVATIVES II

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Abstract. We introduce the class $R(\alpha)$ of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0)$$

satisfying the condition

$$\Re\{D^{\alpha+1} f(z)/D^{\alpha} f(z)\} > \alpha/(\alpha+1)$$

for some $(\alpha \geq 0)$ and for all $z \in U = \{z : |z| < 1\}$, where $D^{\alpha} f(z)$ denotes the Hadamard product of $z/(1-z)^{\alpha+1}$ and $f(z)$. The object of the present paper is to prove some distortion and closure theorems for functions $f(z)$ in $R(\alpha)$, and to give the result for the modified Hadamard product of functions $f(z)$ belonging to the class $R(\alpha)$. Furthermore, we determine the radii of starlikeness and convexity of functions $f(z)$ in the class $R(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = \sum{}^{*} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ in A, and by S^* and K the subclasses of S whose members are starlike with respect to the origin and convex in the unit disk U, respectively. A function $f(z)$ belonging to the class A is said to be starlike of order β $(0 \leq \beta < 1)$ in the unit disk U if and only if

$$\Re\{zf'(z)/f(z)\} > \beta \qquad (z \in U) \tag{1.2}$$

for some β $(0 \leq \beta < 1)$. Further, a function $f(z)$ belonging to the class A is said to he convex of order β $(0 \leq \beta < 1)$ in the unit disk U if and only if

$$\Re\{1 + zf''(z)/f'(z)\} > \beta \qquad (z \in U) \tag{1.3}$$

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$^*\sum$ stands for $\sum_{k=2}^{\infty}$ unless stated otherwise.

for some $\beta\,(0 \leq \beta < 1)$. We denote by $S^*(\beta)$ and $K(\beta)$ the subclasses of A whose members satisfy (1.2) and (1.3), respectively. Then, it is well-known that $S^*(\beta) \subset S^*$, $K(\beta) \subset K$ for $O < beta < 1$, and that $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\beta = 0$.

Let $f * g(z)$ denote the Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum b_k z^k,$$

then

$$f * S(z) = z + \sum a_k b_k z^k.$$

By using the Hadamard product, Ruscheweyh [**15**] defined

$$D^\alpha f(z) = (z/(1-z)^{\alpha+1}) * f(z) \qquad (\alpha \geq 1)$$

which implies that

$$D^n f(z) = \{z(z^{n-1} f(z))^{(n)}/n! \tag{1.4}$$

for $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \ldots\}$. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The symbol $D^n f(z)$ was named by Al-Amiri [**1**] the n-th order Ruscheweyh derivative of $f(z)$. With the notation (1.4), Ruscheweyh [**15**] introduced the classes K_n of functions $f(z)$ in A satisfying the following condition

$$\Re\{D^{n+1} f(z)/D^n(z)\} > 1/2 \qquad (z \in U) \tag{1.5}$$

for $n \in N \cup \{0\}$, and he showed the basic property

$$K_{n+1} \subset K_n \tag{1.6}$$

for each $n \in N \cup \{0\}$, where $K_0 \equiv S^*(1/2)$ and $K_1 \equiv K$. Further, in the notation (1.5) a class K_{-1} can be defined as the class of functions $f(z)$ in A satisfying

$$\Re\{f(z)/z\} > 1/2 \qquad (z \in U).$$

Since $K_0 \equiv S^*(1/2) \subset S^*$, Ruscheweyhs result (1.6) implies that $K_n \subset S^* \subset S$ for each $n \in N \cup \{0\}$.

Recently, by using the n-th order Ruscheweyh derivative of $f(z)$, Singh and Singh [**18**] introduced the subclass R_n of A whose members are characterized by the following condition

$$\Re\{D^{n+1} f(z)/D^n f(z)\} > n/(n+1) \qquad (z \in U)$$

for $n \in N \cup \{0\}$. We can immediately see that $R_0 \equiv S^*$ and $R_n \subset K_n$ for each $n \in N$. Hence R_n is a subclass of $S^* \subset S$ for each $n \in N \cup \{0\}$. Further we can observe that $R_{n+1} \subset R_n$ for every $n \in N \cup \{0\}$.

In recent years, many classes defined by using the n-th order Ruscheweyh derivative of $f(z)$ were studied by Al-Amiri ([**2**], [**3**]), Bulboaca [**4**], Goel and Sohi ([**6**], [**7**]), and Owa ([**10**], [**11**]).

In this paper, we introduce the following classes $R(\alpha)$ by using the symbol $D^{\alpha}f(z)$.

Definition 1. We say that $f(z)$ is in the class $R(\alpha)$ $(\alpha > 0)$, if $f(z)$ defined by

$$f(z) = z - \sum a_k z^k \qquad (a_k \geq 0) \tag{1.7}$$

satisfies the condition

$$\Re\{D^{\alpha+1}f(z)/D^{\alpha}f(z)\} > x/(\alpha+1) \qquad (z \in U) \tag{1.8}$$

for some α $(\alpha \geq 0)$.

Since $D^1 f(z) = zf'(z)$ and $D^0 f(z) = f(z)$, we can see that $R(0) \equiv T^*$ for $\alpha = 0$, which is the subclass of S^* consisting of functions $f(z)$ of the form (1.7) and was studied by Silverman [**17**]. $R(1)$ is the subclass of $K_1 \equiv K$ consisting of functions $f(z)$ of the form (1.7). Further, $R(n)$ is the subclass of R_n consisting of functions $f(z)$ of the form (1.7).

2. Fractional calculus

Many essentially equivalent definitions of the fractional calculus that is, fractional derivatives and fractional integrals, have been given in the literature (cf. e.g., [**5**, Chapter 13), [**8**], [**9**], [**14**], [**16**], and [**19**, P. 28 et seq.]). We find it to be convenient to recall here the following definitions which were used recently by Owa [**12**].

Definition 2. The fractional integral of order a is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3. The fractional derivative of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha}},$$

where $0 \leq \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 4. Under the hypotheses of Definition 3, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = d^n D_z^{\alpha} f(z)/dz^n,$$

where $0 \leq \alpha < 1$ and $n \in N \cup \{0\}$.

With these definitions, recently, Owa [**13**] showed the following lemma.

LEMMA. *Let the function* $f(z)$ *be defined by* (1.7). *Then*

$$D^{n+\alpha}f(z)=z\left\{D_z^{n+\alpha}(z^{n+\alpha-1}f(z))\right\}/\Gamma(n+\alpha+1)$$

for $0\le\alpha<1$, $n\in N\cup\{0\}$ *and* $z\in U$.

3. Distortion theorems

By using the lemma, we state and prove

THEOREM 1. *Let* $n\in N\cup\{0\}$, $0\le\alpha<1$ *and the function* $f(z)$ *be defined by* (1.7). *Then* $f(z)$ *is in the class* $R(n+\alpha)$ *if and only if*

$$\sum\frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!}a_k\le\Gamma(n+\alpha+2). \tag{3.1}$$

Equality holds for functions $f(z)$ *given by*

$$f(z)=z-\frac{\Gamma(n+\alpha+2)(k-1)!}{(n+\alpha k+k)\Gamma(n+\alpha+k)}z^k \qquad (k\ge 2) \tag{3.2}$$

Proof. Assume that the inequality (3.1) holds true and let $|z|=1$. Then, by virtue of the lemma, we obtain that

$$\left|\frac{D^{n+\alpha+1}f(z)}{D^{n+\alpha}f(z)}-1\right|=\left|\frac{\sum\frac{(1-k)\Gamma(n+\alpha+k)}{(n+\alpha+1)(k-1)!}a_kz^{k-1}}{\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k)}{(k-1)!}a_kz^{k-1}}\right|$$

$$\le\frac{\sum\frac{\Gamma(n+\alpha+k)}{(n+\alpha+1)(k-2)!}a_k}{\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k)}{(k-1)!}a_k}\le 1/(\alpha+1).$$

This shows that the values of $D^{n+\alpha+1}f(z)/D^{n+\alpha}f(z)$ lie in a circle centered at $w=1$ whose radius is $1/(\alpha+1)$. Consequently, we can see that the function $f(z)$ satisfies the condition (1.8), hence, $f(z)\in R(n+\alpha)$.

For the converse, assume that the function $f(z)$ belongs to the class $R(n+\alpha)$ for $n\in N\cup\{0\}$ and $0\le\alpha<1$. Then we get

$$\Re\left\{\frac{D^{n+\alpha+1}f(z)}{D^{n+\alpha}f(z)}\right\}= \tag{3.3}$$

$$\Re\left\{\frac{\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k+1)}{(n+\alpha+1)(k-1)!}a_kz^{-1}}{\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k)}{(k-1)!}a_kz^{k-1}}\right\}>\alpha/(\alpha+1)$$

for $z\in U$. Choose values of z on the real axis so that $D^{n+\alpha+1}f(z)/D^{n+\alpha}f(z)$ is real. Upon clearing the denominator in (3.3) and letting $z\to 1^-$ through real values, we can observe that

$$\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k+1)}{(n+\alpha+1)(k-1)!}a_k$$

$$\ge\frac{\alpha}{\alpha+1}\left\{\Gamma(n+\alpha+1)-\sum\frac{\Gamma(n+\alpha+k)}{(k-1)!})a_k\right\}$$

which implies (3.1).

Finally we can show that the function $f(z)$ given by (3.2) is an extremal function for the theorem. This completes the proof of Theorem 1.

COROLLARY 1. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \le \alpha < 1$. *Then*

$$a_k \le \frac{(k-1)!\Gamma(n+\alpha+2)}{(n+\alpha k+k)\Gamma(n+\alpha+k)}$$

for $k \ge 2$. *The eguality holds for the function* $f(z)$ *of the form* (3.2).

Applying Theorem 1, we prove

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \le \alpha < 1$. *Then*

$$|z| - (n+2\alpha+2)^{-1}|z|^2 \le f(z) \le |z| + (n+2\alpha+2)^{-1}|z|^2$$

for $z \in U$. *The result is sharp.*

Proof. Since $f(z)$ belongs to the class $R(n+\alpha)$, by using Theorem 1, we have

$$(n+2\alpha+2)\Gamma(n+a+2)\sum a_k \le \sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!}a_k \le \Gamma(n+\alpha+2)$$

which gives that $\sum a_k \le (n+2\alpha+2)^{-1}$. Hence we can see that

$$|f(z)| \ge |z| - |z|^2 \sum a_k \ge |z| - (n+2\alpha+2)^{-1}|z|^2,$$
$$|f(z)| \le |z| + |z|^2 \sum a_k \le |z| + (n+2\alpha+2)^{-1}|z|^2$$

for $z \in U$.

Further, by taking the function

$$f(z) = z - (n+2\alpha+2)^{-1}z^2$$

we can prove that the result of the theorem is sharp.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \le \alpha < 1$. *Then* $f(z)$ *is included in a disk with its center at the origin and radius* R *given by* $R = (n+2\alpha+3)(n+2\alpha+2)^{-1}$.

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \le \alpha < 1$. *Then, for* $z \in U$,

$$1 - (2+\alpha)|z|/(n+2\alpha+2) \le |f'(z)| \le 1 + (2+\alpha)|z|/(n+2\alpha+2).$$

Proof. In view of Theorem 1, we have

$$(n+2\alpha+2)\Gamma(n+\alpha+2)(2+\alpha)^{-1}\sum ka_k$$
$$\le \sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!}a_k \le \Gamma(n+\alpha+2)$$

which implies that

$$\sum ka_k \le (2+\alpha)(n+2\alpha+2)^{-1}. \tag{3.4}$$

Consequently, by using (3.4), we have

$$|f'(z)| \ge 1 - |z|\sum ka_k \ge 1 - (2+\alpha)|z|/(n+2\alpha+2),$$
$$|f'(z)| \le 1 + |z|\sum ka_k \le 1 + (2+\alpha)|z|/(n+2\alpha+2)$$

for $z \in U$.

Remark. We have not been able to obtain a sharp estimate $|f'(z)|$ in Theorem 3.

COROLLARY 3. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \le \alpha < 1$. Then $f'(z)$ is included in a disk with its center at the origin and radius R' given by $R' = (n+3\alpha+4)(n+2\alpha+2)^{-1}$.*

The following sharp estimation for $|f'(z)|$ is due to Professor M. Obradovič.

THEOREM 3′. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \le \alpha < 1$. Then, for $z \in U$,*

$$1 - 2|z|/(n+2\alpha+2) \le |f'(z)| \le 1 + 2|z|/(n+2\alpha+2).$$

The result is sharp.

Proof. By using Theorem 1, we have

$$\Gamma(n+\alpha+2) \le \sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!}a_k$$
$$= \sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!}ka_k. \tag{*}$$

Consider the expression

$$\frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!} = \frac{(n+\alpha k+k)(n+\alpha k+k-1)\dots(n+\alpha+2)\Gamma(n+\alpha+2)}{k!}$$

for $n = 0$:

$$(\alpha k+k)\Gamma(\alpha+k)/k! \ge (\alpha+1)\Gamma(\alpha+2)$$

for $n \in N$:

$$(n+\alpha k+k)\Gamma(n+\alpha+k)/k! \ge (n+2\alpha+2)\Gamma(n+\alpha+2)/2.$$

So, we can conclude that, for every $n \in N \cup \{0\}$, we have

$$(n+\alpha k+k)\Gamma(n+\alpha+k)/k! \ge (n+2\alpha+2)\Gamma(n+\alpha+2)/2,$$

and then from (*) we obtain

$$\sum ka_k \geq 2/(n+2\alpha+2).$$

The rest of the proof is as for Theorem 3.

Furthermore, that this estimation is the best possible is shown by the function

$$f(z) = z - z^2/(n+2\alpha+2).$$

We derive some distortion inequalities for the fractional calculus of functions belonging to the class $R(n+\alpha)$.

THEOREM 4. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then*

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{2|z|}{(2+\lambda)(n+2\alpha+2)}\right\} \tag{3.5}$$

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{2|z|}{(2+\lambda)(n+2\alpha+2)}\right\} \tag{3.6}$$

for $\lambda > 0$ and $z \in U$. The results in (3.5) and (3.6) are sharp.

Proof. Let

$$F(z) = \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z) = z - \sum \frac{k!\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} a_k z^k \tag{3.7}$$

for $\lambda > 0$. Then we can observe that

$$0 < k!\Gamma(2+\lambda)/\Gamma(k+1+\lambda) < 2/(2+\lambda) \tag{3.8}$$

for $\lambda > 0$ and $k \leq 2$. Hence, with the help of (3.8) and Theorem 1, we have

$$|F(z)| \geq |z| - \frac{2}{2+\lambda}|z|^2 \sum a_k \geq |z| - \frac{2|z|^2}{(2+\lambda)(n+2\alpha+2)}$$

$$|F(z)| \leq |z| + \frac{2}{2+\lambda}|z|^2 \sum a_k \geq |z| + \frac{2|z|^2}{(2+\lambda)(n+2\alpha+2)}$$

which give (3.5) and (3.6), respectively. Further, we can see that the results in (3.5) and (3.6) are sharp for the function $f(z)$ defined by

$$f(z) = z - \frac{z^2}{n+2\alpha+2}.$$

COROLLARY 4. *Let the function $f(z)$ defined by (1.7) be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R^{-\lambda}$ given by*

$$R^{-\lambda} = \frac{1}{\Gamma(2+\lambda)} \left\{1 + \frac{2}{(2+\lambda)(n+2\alpha+2)}\right\}.$$

where $\lambda > 0$.

Applying Theorem 3′ to the function $F(z)$, we have

THEOREM 5.*Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \leq \alpha < 1$. *Then*

$$\left|D_z^{-\lambda} f(z)\right| \geq \operatorname{Max}\left\{0, \frac{|z|}{\Gamma(2+\lambda)}\left(1-\lambda\frac{(2+\lambda)|z|}{(n+2\alpha+2)}\right)\right\}, \tag{3.9}$$

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{\lambda}}{\Gamma(2+\lambda)}\left(1+\lambda+\frac{(2+\lambda)|z|}{(n+2\alpha+2)}\right) \tag{3.10}$$

for $\lambda > 0$ *and* $z \in U$.

Proof. It is easy to see that the function $F(z)$ defined by (3.7) is also the class $R(n+\alpha)$. Hence, by means of Theorem 3′ we get

$$1 - 2|z|/(n+2\alpha+2) \leq F'(z) \leq 1 + 2|z|/(n+2\alpha+2)$$

for $z \in U$ which gives (3.9) and (3.10).

COROLLARY 5. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $R(n+\alpha)$ $n \in N \cup \{0\}$ *and* $0 \leq \alpha < 1$. *Then* $D_z^{1-\lambda} f(z)$ *is included in a disk with its center at the origin and radius* $R^{1-\lambda}$ *given by*

$$R^{1-\lambda} = \frac{1}{\Gamma(2+\lambda)}\left(1+\lambda+\frac{2+\lambda}{n+2\alpha+2}\right), \quad \text{where } \lambda > 0.$$

4. Closure theorems

THEOREM 6. *Let the functions*

$$f_i(z) = z \sum a_{k,i} z^k \qquad (a_{k,i} \geq 0) \tag{4.1}$$

be in the class $R(n+\alpha)$ *for* $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ *and every* $i = 1, 2, 3, \ldots, m$. *Then the function* $h(z)$ *defined by*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geq 0)$$

is also in the class $R(n+\alpha)$, *where* $\sum_{i=1}^{m} c_i = 1$.

Proof. By means of the definition of $h(z)$, we can write

$$h(z) = z - \sum\left(\sum_{i=1}^{m} c_i a_{k,i}\right) z_k.$$

Further, by virtue of Theorem 1, we have

$$\sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!} a_{k,i} \leq \Gamma(n+\alpha+2)$$

for every $i = 1, 2, 3, \dots, m$. Hence we can see that

$$\begin{aligned}&\sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!}\left(\sum_{i=1}^{m} c_i a_{k,i}\right)\\ &\sum_{i=1}^{m} c_i \left\{\sum \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!} a_{k,i}\right\}\\ &\leq \left(\sum_{i=1}^{m} c_i\right)\Gamma(n+\alpha+2) = \Gamma(n+\alpha+2)\end{aligned}$$

which implies that $h(z)$ is in the class $R(n+\alpha)$ with the aid of Theorem 1.

THEOREM 7. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{(k-1)!\Gamma(n+\alpha+k)}{(n+\alpha k+k)\Gamma(n+\alpha+k)} z_k \qquad (k \in N - \{1\}). \tag{4.2}$$

Then $f(z)$ *is in the class* $R(n+\alpha)$ *for* $n \in N \cup \{0\}$ *and* $0 \leq \alpha < 1$ *if and only if it can be expressed in the form*

$$f_k(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \tag{4.3}$$

where $\lambda_k \geq 0$ *for* $k \in N$ *and* $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum \frac{(k-1)!\Gamma(n+\alpha+2)}{(n+\alpha k+k)\Gamma(n+\alpha+k)} \lambda_k z^k.$$

Then we obtain that

$$\sum \left\{\frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!} \frac{(k-1)!\Gamma(n+\alpha+2)}{(n+\alpha k+k)\Gamma(n+\alpha+k)} \lambda_k\right\} \leq \Gamma(n+\alpha+2).$$

This shows that $f(z)$ belongs to the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $O \leq \alpha < 1$. Again, by using Theorem 1.

For the converse, suppose that $f(z)$ belongs to the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Again, by using Theorem 1, we can observe that

$$a_k \leq \frac{(k-1)!\Gamma(n+\alpha+2)}{(n+\alpha k+k)\Gamma(n+\alpha+k)} \qquad (k \in N - \{1\}).$$

Now, setting

$$\lambda_k = \frac{(n+\alpha+k)\Gamma(n+\alpha+k)}{(k-1)!\Gamma(n+\alpha+2)} a_k \qquad (k \in N - \{1\})$$

and $\lambda_1 = 1 - \sum \lambda_k$, we have the representation (4.3). This completes the proof of Theorem 7.

5. Modified Hadamard product

Let $f(z)$ be defined by (1.7) and $g(z)$ be defined by

$$g(z) = z - \sum b_k z^k \qquad (b_k \geq 0).$$

Further, let $f * g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is, $f * g(z) = z - \sum a_k b_k z^k$.

THEOREM 8. *Let the functions $f_i(z)$ defined by (4.1) be in the classes $R(n_i + \alpha_i)$ for $n_i \in N \cup \{0\}$, $0 \leq \alpha < 1$ and each $i = 1, 2, 3, \dots, m$, respectively. Then the modified Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $R(n + \alpha)$, where $n + \alpha = underset 1 \leq i \leq m \to \mathrm{Min}\{n_i + \alpha_i\}$.*

Proof. Since $f_i(z) \in R(n + \alpha_i)$ for each $i = 1, 2, 3, \dots, m$, in view of Theorem 1, we get

$$a_{k,i} \leq 1/(n_i + 2\alpha + 2) \qquad (i = 1, 2, 3, \dots, m). \tag{5.1}$$

Therefore we can show that

$$\sum \frac{(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k-1)!} \left(\prod_{i=1}^{m} a_{k,i} \right)$$
$$\leq \frac{(n + 2\alpha + k)\Gamma(n + \alpha + k)}{\prod_{i=1}^{m}(n_i + 2\alpha_i + 2)} \leq \Gamma(n + \alpha + 2)$$

with the help of (5.1) and Theorem 1. Thus we have Theorem 8.

COROLLARY 6. *Let the functions $f_i(z)$ defined by (4.1) be in the same class $R(n+\alpha)$ for $n \in N \cup \{0\}$, $0 \leq \alpha < 1$ and every $i = 1, 2, 3, \dots, m$. Then the modified Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ also belongs to the class $R(n + \alpha)$.*

6. Radii of starlikeness and convexity

We determine the radii of starlikeness and convexity of functions $f(z)$ belonging to the class $R(n + \alpha)$.

THEOREM 9. *Let the function $f(z)$ defined by (1.7) be in the class $R(n + \alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is starlike of order β $(0 \leq \beta < 1)$ in the disk $|z| < r_0$, where*

$$r_0 = \inf_{k \in N - \{1\}} \left\{ \frac{(1-\beta)(n + \alpha k + k)\Gamma(n + \alpha + k)}{(k - \beta)(k-1)!\Gamma(n + \alpha + 2)} \right\}^{1/(k-1)}$$

with equality for the function $f(z)$ given by (4.2).

Proof. It suffices to shaw that

$$|zf'(z)/f(z) - 1| < 1 - \beta$$

for $|z| < r_0$. Now, we can observe that

$$|zf'(z)/f(z) - 1| \leq \frac{\sum(k-1)a_k|z|^{k-1}}{1 - \sum a_k|z|^{k-1}} \leq 1 - \beta$$

if and only if

$$\sum((k-\beta)/(1-\beta))a_k|z|^{k-1} \leq 1.$$

Hence, by virtue of Theorem 1, we need only find values of $|z|$ for which

$$\left(\frac{k-\beta}{1-\beta}\right)|z|^{k-1} \leq \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{(k-1)!\Gamma(n+\alpha+2)} \qquad (k \geq 2),$$

which will be true when $|z| \leq r_0$. This completes the proof of the Theorem.

Finally, we have

THEOREM 10. *Let the function $f(z)$ defined by* (1.7) *be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is convex of order β $(0 \leq \beta < 1)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_{k \in N - \{1\}} \left\{ \frac{(1-\beta)(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!\Gamma(n+\alpha+2)} \right\}^{1/(k-1)}.$$

Proof. Since $f(z)$ defined by (1.7) is convex of order β if and only if $zf'(z)$ is starlike of order β, we can show that the proof of the Theorem follows the proof of Theorem 9, with a_k replaced by ka_k.

COROLLARY 7. *Let the function $f(z)$ defined by* (1.7) *be in the class $R(n+\alpha)$ for $n \in N \cup \{0\}$ and $0 \leq \alpha < 1$. Then $f(z)$ is univalent and starlike for $|z| < r_2$, where*

$$r_2 = \inf_{k \in N - \{1\}} \left\{ \frac{(n+\alpha k+k)\Gamma(n+\alpha+k)}{k!\Gamma(n+\alpha+2)} \right\}^{1/(k-1)}.$$

Proof. By taking $\beta = 0$ in Theorem 9, we have the Corollary.

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REFERENCES

[1] H.S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38**(1980), 87–94.

[2] H.S. Al-Amiri, *Certain analogy of the a-convex functions*, Rev. Roumaine Math. Pures Appl. **23**(1978), l449–1454.

[3] H.S. Al-Amiri, *On the Ruscheweyh-Mocanu alpha convex functions of order n*, Mathematica **22**(1980), 207–?13.

[4] T. Bulboaca, *Asupra unor noi clase de functii analitice*, Studia Univ. Babes-Bolyai Math. **26**(1981), 42–46.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms, Vol. II*, McGraw-Hill, New York, 1954.

[6] R.M. Goel and N.S. Sohi, *Subclasses of univalent functions*, Tamkang J. Math. **11**(1980), 77–81.

[7] R.M. Goel and N.S. Sohi, *A new criterion for univalence and its applications*, Glas. Mat. **16**(1981), 39–49.

[8] K. Nishimoto, *Fractional derivative and integral. Part I*, J. College Engrg. Nihon Univ. Ser. B **17**(1976), 11–19.

[9] T.J. Osler, *Leibniz rule for fractional derivative generalized and an application to infinite series*, SIAM J. Appl. Math. **18**(1970), 658–674.

[10] S. Owa, ıOn the Ruscheweyh's new criteria for univalent functions, Math. Japon. **27**(1982), 77–96.

[11] S. Owa, *On new criteria for analytic functions*, Tamkang J. Math. **13**(1982), 201–213.

[12] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J. **18**(1978), 53–59.

[13] S. Owa, *An application of the Ruscheweyh derivatives*, Math. Japon. **30**(1985), 927–946.

[14] B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus*, in Fractional Calculus and its Applications (B. Ross, ed.), Springer, Berlin, 1975, 1–36.

[15] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[16] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. College General Ed. Kyushu Univ. **11**(1978), 135–143.

[17] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975), 109–116.

[18] R. Singh and S. Singh, *Integrals of certain univalent functions*, Proc. Amer. Math. Soc. **77**(1979), 336–340.

[19] H.M. Srivastava and R.G. Buschman, *Convolution Integral Equations with Special Function Kernels*, John Wiley and Sons, New York, 1977.

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