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HYPERSURFACES OF C2-LIKE FINSLER SPACES

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Abstract. The notion of $C2$-like Finesler spaces has been introduced by Matsumoto and Numata [**1**]. The purpose of the present paper is to study the properties of hypersurfaces immersed in $C2$-like Finsler spaces. We prove that each non-Riemannian hypersurface of a $C2$-like Finsler space is $C2$-like. The condition under which a hypersurface of a $C2$-like Landsberg space is Landsberg is obtained. Finally after using the so called T-conditions [**6**] we explore the situation in which a hypersurface of a $C2$-like Finsler space F_n satisfying the T-conditions also satisfies the T-condition.

1. Introduction Let F_n be a Finsler space of dimension n with the fundamental function $F(x,y)$, $(y^i = \dot{x}^i)$. The following are the two well known properties of Finsler spaces:

$(P1)$ The Berwald connection parameter G^i_{jk} [**3**] is not, in general, independent on the direction element y^i.

(P_2) $g_{ij(k)} = -2C_{ijk|0} \neq 0$ in general, where (k) stands for Berwald's process of covariant derivation, $C_{ijk} = 1/2 \cdot dg_{ij}(x,y)$ and suffix 0 stands for transvection with respect to y^i.

A Finsler space in which G^i_{jk} is independent on y is called a Berwald space. This space is characterized by the condition $C_{ijk|0} = 0$ [**3**].

A Finsler space in which $g_{ij(k)} = 0$ is called a Landsberg space. This space is characterized by $C_{ijk|0} = 0$.

It is obvious that each Berwald space is a Landsberg space. Further, the relation $\Gamma^{*i}_{jk} = G^i_{jk} - C^i_{jk|0}$ [**3**] involving Cartan's connection parameter Γ^{*i}_{jk} proves,

LEMMA 1. *In a Landsberg space, Cartan's and Berwald's connection parameters are identical and in Bervald's space the Cartan's connection parameter is independent on y.*

Definition 1. Finsler space F_n $(n \geq 2)$ with $C^2 = C^i C_i \neq 0$ is called $C2$-like [**1**], if the (h) hv-torsion tensor C_{ijk} can be written in the form

$$C_{ijk} = C_i C_j C_k / C^2 \quad \text{where} \quad C_i = g^{jk} C_{ijk}. \tag{1.1}$$

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The following lemma can be easily deduced with the help of the equation (1.1) and definition of Berwald and Landsberg spaces.

LEMMA 2. *The necessary and sufficient condition that a $C2$-like Finsler space be a Bervvald space (or Landsberg space) is that $C_{i|h} = 0$ (or $C_{i|0} = 0$).*

2. Hypersurfaces of a C2-like Finsler space. Consider a non-Riemannian hypersurface F_{n-1} of F_n ($n \geq 3$), characterized by the equation $x^i = x^I(u^\alpha)$, where we assume that all the Latin indices $i, j, \dots$ take values $1, 2, \dots n$, while all the Greek indices $\alpha, \beta, \dots$ take values $1, 2, \dots n-1$. The fundamental tensor of F_{n-1} is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y) B^i_\alpha B^i_\beta, \quad \text{where } B^i_\alpha = \partial x^i / \partial u^\alpha. \tag{2.1}$$

We shall use the notation, $B^{ij\dots k}_{\alpha\beta\dots\gamma} = B^i_\alpha B^j_\beta \dots B^k_\gamma$.

Now in a hypersurface F_{n-1} of F_n we have

$$C_{\alpha\beta\gamma} = C_{ijk} B^{ijk}_{\alpha\beta\gamma}. \tag{2.2}$$

If F_n is $C2$-like, then by means of the equation (1.1), the equation (2.2) reduces to

$$C_{\alpha\beta\gamma} = \overline{C}_\alpha \overline{C}_\beta \overline{C}_g amma / C^2, \tag{2.3}$$

where

$$\overline{C}_\alpha = C_i B^i_\alpha = C^2 / \overline{C}^2 \cdot C_\alpha, \tag{2.4}$$

where we have put $\overline{C}^2 = \overline{C}_\alpha \overline{C}^\alpha \neq 0$ and $C_\alpha = g^{\beta\gamma} C_{\alpha\beta\gamma}$.

The equations (2.3) and (2.4) give the following

$$C_{\alpha\beta\gamma} = C^4 / \overline{C}^6 \cdot C_\alpha C_\beta C_\gamma \tag{2.5}$$

A direct calculation will give

$$C^4 / \overline{C}^6 = 1 / \overline{C}^2 \tag{2.6}$$

where $\overline{\overline{C}}$ stands for $C_\alpha C^\alpha$ and this must be non-zero, for if it is zero then $C_\alpha = 0$. Therefore by Diecke's theorem the hypersurface is Riemannian, which is a contradiction to our assumption. Thus (2.5) reduces to $C_{\alpha\beta\gamma} = C_\alpha C_\beta C_\gamma / \overline{\overline{C}}^2$ which proves the following

THEOREM 2.1. *The hypersurface F_{n-1} of a $C2$-like Finsler space F_n is $C2$-like.*

Throughout the paper it with be assumed that $\overline{\overline{C}}^2 \neq 0$.

The differences between the intrinsic and induced connection parameters $\hat{\Gamma}^\alpha_{\beta\gamma}$ and $\Gamma^{*\alpha}_{\beta\gamma}$ of a hypersurface has been obtained by Rund [**2**]. If the space F_n is $C2$-like then this difference tensor $\Lambda_{\alpha\beta\gamma} = \hat{\Gamma}_{\alpha\beta\gamma} - \Gamma^*_{\alpha\beta\gamma}$ reduces to the form

$$\Lambda_{\alpha\beta\gamma} = \rho C^2 / \overline{C}^4 \cdot [C_\beta C_\gamma \Omega_{\alpha 0} + C_\alpha C_\beta \Omega_{\gamma 0} - C_\gamma C_\alpha \Omega_{\beta 0} - C_\alpha C_\beta C_\gamma \Omega_{00}] \tag{2.7}$$

where $\rho = N^i C_i$, $\Omega_{\alpha\beta}$ are the components of the second fundamental tensor of F_{n-1}, and N^i are the components of the unit vector normal to F_{n-1}. If we suppose that intrinsic and induced connection parameters of F_{n-1} are identical, then (2.7) gives either $\rho = 0$, or $\Omega_{\alpha 0} = 0$, or $C_\alpha = 0$. But $C_\alpha = 0$ gives that the hypersurface is Riemannian, which is a contradiction to our assumption. This proves the following:

THEOREM 2.2. *The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a $C2$-like Finsler space be equal is that either $\Omega_{\alpha 0} = 0$ or the vector C_i is tangential to the hypersurface.*

In order to derive a condition under which a hypersurface of a $C2$-like Landsberg space is a Landsberg space we note that the induced covariant differentiation of the relation $C_\alpha = \overline{C}^2/C^2 \cdot C_i B^i_\alpha$ yields

$$C_{\alpha\|\beta} = \overline{C}^2/C^2(C_{i|h}B^{ih}_{\alpha\beta} + \partial C_1/\partial \dot{u}^\alpha \cdot \Omega_{\beta 0}N^1 + \rho\Omega_{\alpha\beta}) + \overline{C}_\alpha(\overline{C}^2/C^2)_{\|\beta} \tag{2.8}$$

where we have used the fact that $\partial C_i/\partial y^i$ is symmetric in the indices $i, 1$. (The double vertical bar stands for induced covariant derivative). The transvection of the relation (2.8) with respect to $\dot{u}^\beta$ gives

$$C_{\alpha\|0} = \overline{C}^2/C^2(C_{i|0}B^i_\alpha + \partial C_1 \partial \dot{u}^\alpha \cdot \Omega_{\beta 0}N^1 \rho\Omega_{\alpha 0}) + \overline{C}_\alpha(\overline{C}^2/C^2)_{\|0} \tag{2.9}$$

If we take $\rho = 0$ then equation (1.1) shows that the tensor defined by, $M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k$ vanishes. The properties of the hypersurfaces in this case have been discussed by Brown [**4**]. He has shown that in this case

$$\partial N^1 . \partial \dot{u}^\alpha = -M_\alpha N^1, \text{ where } M_\alpha = C_{ijk}B^i_\alpha N^j N^k.$$

This relation and the condition $\rho = C_1 N^1 = 0$ give

$$\frac{\partial C_1}{\partial \dot{u}^\alpha} N^1 = -C_1 \frac{\partial N_1}{\partial \dot{u}^\alpha} = C_1 N^1 M_\alpha = 0.$$

A direct calculation will give $\overline{C}^2 = C^2 - \rho^2$.

This shows that the condition $\rho = 0$ will reduce the equation (2.9) to $C_{\alpha\|0} = C_{i|0}B^i_\alpha$ Again, Brown [**4**] has shown that for $M_{\alpha\beta} = 0$, the intrinsic and induced connection parameters are identical. Hence and from Lemma 2 we obtain the following

THEOREM 2.3. *A hypersurface of a $C2$-like Landsberg space will be a Landsberg space if the vector C_i is tangential to the hypersuface.*

Now we want to find the condition under which the induced connection parameter $\Gamma^{*\alpha}_{\beta\gamma}$ of a hypersurface of a $C2$-like Berwald space is independent of $\dot{u}^\alpha$. Rund [**3**] has given the following relation for induced connection parameter of F_{n-1},

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i \left(\frac{\partial^2 x^i}{\partial u^\beta \partial u^\beta} + \Gamma^{i*}_{jk} B^{jk}_{\beta\gamma} \right), \text{ where } B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta. \tag{2.10}$$

If the Finsler space F_n is Berwald, then equation (2.10) by means of Lemma 1 gives that $\Gamma^{*\alpha}_{\beta\gamma}$ is independent of $\dot{u}^\alpha$ if and only if B^α_i is independent of $\dot{u}^\alpha$. Rund [**3**] has given the following relation

$$\partial B^\alpha_i/\partial\dot{u}^\lambda = 2g^{\alpha\delta}B^{h\delta}_{\lambda\delta}C_{jkh}N^jN_i$$

which in view of (1.1) and (2.4) reduces to

$$\partial B^\alpha_i/\partial\dot{u}^\lambda = 2\rho C^2/\overline{C}^4\cdot C^\alpha C_\lambda N_i. \tag{2.11}$$

Hence we have the following

THEOREM 2.4. *The necessary and sufficient condition that the induced conneetion parameter of a hypersurface of a $C2$-like Berwald space be independent on the direction element is that the vector C_i is tangential to the hypersurface.*

Theorems 2.2. and 2.4 give the following:

THEOREM 2.5. *If the included connection parameter of a hypersurface of a $C2$-like Berwald space is independent on the direction element then the induced and intrinsic connection parameters are equal.*

The two normal curvature vectors denoted by $I^{\alpha\beta}_i$ and $\overset{\circ}{H}{}^{\alpha\beta}_i$ are given by Rund [**3**] and Davies [**5**]. These vectors are related by [**3**] as follows.

$$\overset{\circ}{H}{}^i_{\alpha\beta} = I^{\alpha\beta}_i + N^iN_jC^j_{hk}B - \beta^h\overset{\circ}{H}{}^k_{\alpha\lambda}\dot{u}^\lambda \tag{2.12}$$

The relation (2.12) after transvection with respect to $\dot{u}^\beta$ gives

$$\overset{\circ}{H}{}^i_{\alpha\beta}\dot{u}^\beta - I^i_{\alpha\beta} = \Omega_{\alpha 0}N^i. \tag{2.13}$$

The equations (1.1), (2.4), (2.12) and (2.13) give

$$\overset{\circ}{H}{}^i_{\alpha\beta} = I^i_{\alpha\beta} + (\rho^2/\overline{C})\Omega_{\alpha 0}C_\beta N^i$$

which proves the following:

THEOREM 2.6. *The necessary and sufficient condition that Rund's and Davies's normal curvature vectors of the hypersurface of a $C2$-like Finsler space are identical is that either $\Omega_{\alpha 0} = 0$, or the vector C_i is tangential to the hypersurface.*

The following theorems is a consequence of theorems 2.2 and 2.6.

THEOREM 2.7. *A necessary und sufficient condition that Rund's and Davics's normal curvature vectors of the hypersurface of a $C2$-like Finsler space are identical is that their induced and intrinsic connection parameters, are identical.*

Theorems 2.4 and 2.6 yield the following:

THEOREM 2.8, *If the induced connection parameter of a hypersuface of a $C2$-like Berwald. space is independent an the direction element then Rund's and Davies's normal curvature vectors of hypersurface are identical.*

3. T-Conditions. We now consider the T-tensor (Matsumoto [**6**] and Kawaguchi [**7**]) given by

$$T_{hijk} = FC_{hijk|k} + C_{hij}l_k + C_{hik}l_j + C_{hjk}l_i + C_{ijk}l_h \tag{3.1}$$

where $C_{hij|k}$ stands for the v-covariant derivative of C_n with respect to y^k. The corresponding expression for the T-tensor $T_{\alpha\beta\gamma\delta}$ in F_{n-1} can be written as

$$T_{\alpha\beta\gamma\delta} = FC_{\alpha\beta\gamma|\delta} + l_\alpha C_{\beta\gamma\delta} + l_\beta C_{\alpha\gamma\delta} + l_\gamma C_{\alpha\beta\delta} + l_\delta C_{\alpha\beta\gamma}. \tag{3.2}$$

The relation (2.2) yields

$$C_{\alpha\beta\gamma|\delta} = C_{ijk|\delta} B^{ijk}_{\alpha\beta\gamma} + C_{ijk} B^{jk}_{\alpha\gamma} Z^i_{\alpha\delta} + C_{ijk} B^{ik}_{\alpha\gamma} Z^j_{\beta\delta} + C_{ijk} B^{ij}_{\alpha\beta} Z \tag{3.3}$$

where $Z^i_{\alpha\delta} = B^i_{\alpha|\delta} = N^i M_{\alpha\delta}$. A direct calculation will give

$$C_{ijk|\delta} = C_{ijk|h} B^h_\delta \tag{3.4}$$

$$Z^i_{\alpha\delta} = \rho/C^2 \cdot N^i \overline{C}_\alpha \overline{C}_\delta \tag{3.5}$$

By virtue of the equations (1.1), (2.4), (3.4) and (3.5) the relation (3.3) reduces to the form

$$C_{\alpha\beta\gamma|\delta} = C_{ijk|h} B^{ijkh}_{\alpha\beta\gamma\delta} + 3\rho^2 C^4/\overline{C}^8 \cdot C_\alpha C_\beta C_\delta \tag{3.6}$$

The equations (2.2), (3.1), (3.2), (3.6) and the well known relation $l_\alpha = i_i B^i_\alpha$ give

$$T_{\alpha\beta\gamma\delta} = C_{hijk} B^{ijkh}_{\alpha\beta\gamma\delta} + 3\rho^2 C^4/\overline{C}^8 \cdot C_\alpha C_\beta C_\gamma C_\delta$$

The space F_n is said to satisfy the T-condition if and only if $T_{hijh} = 0$. Therefore we have the following theorem.

THEOREM 3.2. *If a C2-like Finsler space F_n satisfies the T-condition then the necessary and sufficient condition for its hypersurface F_{n-1} to satisfy the T-condition is that the vector field C_i is tangential to the space F_{n-1}.*

The theorems 2.2, 2.3, 2.4, 2.6, and 3.1 yield the following:

THEOREM 3.7. *If a $C2$-like Bervald space F_n and its hypersurface F_{n-1} satisfy the T-condition then the induced connection parameter of F_{n-1}, is independent on the direction element, its intrinsic and induced connection parameters are identical, its Rund's and Davies's normal curvature vectors are identical and the hyperssurfuce is a Landsber'g space.*

It can be easily shown that in a hypersurface of a $C2$-like space the v-curvature tensor $S_{\alpha\beta\gamma\delta} = 0$, which proves the following

THEOREM 3.3. *The hypersurface of a C2-like Finsler space is a flat space.*

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