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ISOTROPIC SECTIONS AND CURVATURE PROPERTIES OF HYPERBOLIC KAEHLERIAN MANIFOLDS

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Abstract. In [**4**,2] curvature properties of pseudo-Riemannian manifolds were investigated with respect to isotropic vectors and isotropic sections. Further, analogous properties have been treated in [**1**] for Kaehlerian manifolds with an indefinite metric. In this paper we consider hyperbolic Kaehlerian manifolds, and study how the curvature properties of one- and two-dimensional isotropic tangential spaces determine the curvature properties of the manifold.

1. Preliminaries

Let M be a $2n$-dimensional hyperbolic Kaehlerian manifold, i.e. M is a Riemannian manifold with an indefinite metric g and an almost product structure satisfying the conditions:

$$p^2 = id, \qquad g(PX, PY) = -g(X, Y) \tag{1}$$

for arbitrary vector fields X, Y and $\nabla P = 0$. The metric g is of signature (n, n) and P trace $= 0$.

R, ρ and T will stand for the curvature tensor, the Ricci tensor and the scalar curvature respectively. The curvature tensor R satisfies the condition

$$R(X, Y, Z, U) = -R(X, Y, PZ, PU) \tag{2}$$

for arbitrary vectors in the tangential space T, M, p in M. The Ricci tensor ρ has the property

$$\rho(X, Y) = -\rho(PX, PY); \quad X, Y \text{ in } T_pM. \tag{3}$$

Further, we consider the tensors:

$$\begin{aligned} \varphi(Y, Z, U) &= g(Y, Z)\rho(X, U) - g(X, Z)\rho(Y, U) \\ &\quad + g(X, U)\rho(Y, Z) - g(Y, U)\rho(X, Z); \\ \psi(X, Y, Z, U) &= -g(Y, PZ)\rho(X, PU) + g(X, PZ)\rho(Y, PU) \\ &\quad - g(X, PU)\rho(Y, PZ) + g(Y, PU)\rho(X, PZ) \\ &\quad + 2g(X, PY)\rho(Z, PU) + 2g(Z, PU)\rho(X, PY); \end{aligned}$$

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$$\pi_1(X,Y,Z,U) = g(Y,z)g(X,U) - g(X,Z)g(Y,U);$$
$$\pi_2(X,Y,Z,U) = -g(Y,PZ)g(X,PU) + g(X,PZ)g(Y,PU) + 2g(X,PY)g(Z,PU).$$

Let α be a section (2-plane) in T_pM. The section a is said to be nondegenerate, weakly isotropic, strongly isotropic, if the rank of the restriction of the metric g on α is 2, 1, 0 respectively. With respect to the structure P a section α is said to be holomorphic (totally real) if $P\alpha = \alpha (P\alpha \neq \alpha, P\alpha \perp \alpha)$.

We shall use two kinds of special bases of T_pM:

1) An adapted basis $\{a_1 \dots, a_n; x_1, \dots, x_n\}$ is characterized with the property that the matrices g and P with respect to such a basis are

$$g = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the unit matrix.

2) A separate basis $\{\eta_1, \dots, \eta_n; \xi_1, \dots, \xi_n\}$ consists of eigen vectors of P, so that $\{\xi_1, \dots, \xi_n\}$ form a basis of the eigen space V^+, corresponding to the eigen value $+1$ of P. The vectors $\{\eta_1, \dots, \eta_n\}$ form a basis of the eigen space V^-. With respect to a separate basis the matrices g and P are

$$P = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

The following equation is fulfilled $T_pM = V^+ \oplus V^-$ (nonorthogonal). The second condition of (1) implies that every eigen vector ξ of P is isotropic, i.e. $g(\xi, \xi) = 0$. Given an adapted basis, one obtains a separate basis by the formulae:

$$\xi_i = (a_i + x_i)/\sqrt{2}, \quad \eta_i = (a_i - x_i)/\sqrt{2}; \qquad i = 1, \dots, n.$$

These formulae give also an inverse transition.

In what follows, x, y, z will denote unit space-like vectors, i.e. $g(x,x) = 1$; a, b, c will denote unit time-like vectors, i.e. $g(a,a) = -1$; u, r, v will denote isotropic vectors which are not eigen vectors, i.e. $g(u,u) = 0$, $Pu \neq \pm u$; ξ, η, ζ will denote eigenvectors of P, i.e. $P\xi = \pm\xi$.

Taking into account both structures, we find the following types of holomorphic and totally real sections in T_pM:

A. *Holomorphic sections.*

A1. *Nondegenerate holomorphic sections.* These sections have an orthonormal basis of type $\{x, Px\}$ or $\{a, Pa\}$ and a basis of type $\{\xi, \eta\}$, $P\xi = \xi$, $P\eta = -\eta$, $g(\xi, \eta) \neq 0$.

A2. *Strongly isotropic holomorphic sections of hybrid type.* These sections exist by $n \geq 2$, and have a basis of type $\{u, Pu\}$. Another kind of useful bases for such sections are $\{\xi, \eta\}$, $P\xi = \xi$, $P\eta = -\eta$, $g(\xi, \eta) \neq 0$.

A3. *Strongly isotropic holomorphic sections of pure type.* By $n \geq 2$ these sections are the sections in V^+ and in V^-.

B. *Totally real sections.*

B1. *Nondegenerate totally real sections of pure type.* These sections exist by $n \geq 2$ and have an orthonormal basis of type $\{x, y\}$, $g(x, Py) = 0$ or $\{a, b\}$, $g(a, Pb) = 0$.

B2. *Nondegenerate totally real sections of hybrid type.* These sections exist by $n \geq 2$ and have an orthonormal basis of type $\{x, a\}$, $g(x, Pa) = 0$.

B3. *Weakly isotropic totally real sections of the I type.* These sections exist by $n \geq 2$ and have a basis of type $\{x, \xi\}$, $g(x, \xi) = 0$; $\{a, \xi\}$; $g(a, \xi) = 0$.

B4. *Weakly isotropic totally real sections of the II type.* These sections exist by $n \geq 3$ and have a basis of type $\{x, u\}$, $g(x, u) = g(x, Pu) = 0$; $\{a, u\}$, $g(a, u) = g(a, Pu) = 0$.

B5. *Strongly isotropic totally real sections of the I type.* These sections exist by $n \geq 3$ and have a basis of type $\{\xi, u\}$, $g(\xi, u) = 0$.

B6. *Strongly isotropic totally real sections of the II type.* These sections exist by $n \geq 4$ and have a basis of type $\{u, v\}$, $g(u, v) = g(u, Pv) = 0$.

2. Holomorphic curvatures

If α is a nondegenerate section in T_pM with a basis $\{X, Y\}$, its curvature is given by

$$K(\alpha, p) = K(X, Y) = R(X, Y, Y, X)/\pi_1(X, Y, Y, X).$$

For an isotropic section such a curvature cannot be defined. If $\{X, Y\}$ forms a basis of an isotropic section α and

$$R(X, Y, Y, X) = 0, \tag{4}$$

this is a geometric property of the section α.

Now, let α be a nondegenerate holomorphic section. Curvatures of such sections will be called holomorphic sectional curvatures. As for Kaehlerian manifolds, we have.

Lemma 1. *Let T be a tensor of type $(0, 4)$ over T_pM with the properties:*

$$\begin{aligned} &1) \quad T(X, Y, Z, U) = -T(Y, X, Z, U); \\ &2) \quad T(X, Y, Z, U) = -T(X, Y, U, Z); \\ &3) \quad T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0; \\ &4) \quad T(X, Y, Z, U) = -T(X, Y, PZ, PU). \end{aligned} \tag{5}$$

If T has zero holomorphic sectional curvatures, then $T = 0$.

Proof. From the condition of the lemma it follows that

$$T(X, PX, PX, X) = 0 \tag{6}$$

for an arbitrary nonisotropic vector X in T_pM. Let Y be an arbitrary isotropic vector. Then $Y = \lambda(x + a)$, λ – real number, $g(x, x) = -g(a, a) = 1$, $g(x, a) = 0$. Substituting the vector $x + ta$, $|t| < 1$ in (6), we obtain a polynomial identity

$$f(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 = 0.$$

for $|t| < 1$. This implies $c_0 = \cdots = c_4 = 0$ and in particular $f(1) = 0$, i.e. $T(Y, PY, PY, Y) = 0$. Thus, (6) is fulfilled for an arbitrary vector. Now, as in the case of a Kaehlerian manifold [**5**], it follows that $T = 0$.

A hyperbolic Kaehlerian manifold is said to be of constant holomorphic sectional curvature μ if $K(\alpha, p) = \mu$, does not depend on the choice of the nondegenerate holomorphic section α in T_pM, p in M. The curvature identity characterizing these manifolds has been found in [**7**] with respect to local coordinates. We shall derive this identity from Lemma 1.

PROPOSITION. [**7**] *A hyperbolic Kaehlerian manifold is of constant holomorphic sectional curvature μ if and only if*

$$R = \mu(\pi_1 + \pi_2)/4, \qquad \mu = \tau/n(n+1). \tag{7}$$

Proof. The proposition follows by applying Lemma 1 to the tensor $T = R - (\mu/4)(\pi_1 + \pi_2)$.

The equality (7) implies $\rho = \mu((n+1)/2)g$, i.e. M is Einsteinian. Hence, if M is connected, μ is a constant on M.

Remark. In [**7**], hyperbolic Kaehlerian manifolds of constant holomorphic sectional curvature have been called manifolds of almost constant curvature.

Let $\mathcal{K}$ be the vector space of the tensors over T_pM having the properties (5). For T in $\mathcal{K}, \rho(T)$ and $\tau(T)$ will stand for the Ricci tensor and the scalar curvature with respect to T. The metric g induces in a natural way an inner product in $\mathcal{K}$. Using the same method as in [**6, 8**], we obtain the following decomposition theorem for $\mathcal{K}$.

THEOREM 1. *The following decomposition of $\mathcal{K}$ is orthogonal:*

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_w,$$

where

1) $\mathcal{K}_1 = \{T \in \mathcal{K} | \, T = \mu(\pi_1 + \pi_2)/4\}$;
2) $\mathcal{K}_w = \{T \in \mathcal{K} | \, \rho(T) = 0\}$;
3) $\mathcal{K}_2$ *is the orthogonal complement $\mathcal{K}_w$ in $\mathcal{K}_1^\perp$*;
4) $\mathcal{K}_1 \oplus \{T \in \mathcal{K} | \, \rho(T) = \tau(T)g/2n\}$;
5) $\mathcal{K}_2 \oplus \{T \in \mathcal{K} | \, \tau(T) = 0\}$.

The curvature tensor R of a hyperbolic Kaehlerian manifold has the properties (5). The component $B(R)$ of R in $\mathcal{K}_w$ (Weyl component) is said to be the Bochner curvature tensor. It is easy to check that this component is

$$B(R) = R - \frac{1}{2n(n+2)}(\varphi + \psi) + \frac{\tau}{4(n+1)(n+2)}(\pi_1 + \pi_2). \tag{8}$$

COROLLARY 1. *A hyperbolic Kaehlerian manifold $M(2n \geq 4)$ is of constant holomorphic sectional curvature if and only if M is Einsteinian and $B(R) = 0$.*

The Ricci curvature of a direction determined by a nonisotropic vector X is given by $\rho(X) = \rho(X,X)/g(X,X)$. Applying Lemma 1 we obtain

COROLLARY 2. *Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. M has a vanishing Bochner curvature tensor if and only if*

$$K(X,PX) - \frac{4}{n+2}\rho(X) + \frac{\tau}{(n+1)(n+2)} = 0 \tag{9}$$

for an arbitrary nonisotropic vector X in T_pM, p in M.

THEOREM 2. *Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent.*

1) *$R(u,Pu,Pu,u) = 0$ for arbitrary u in T_pM, i.e. the strongly isotropic holomorphic sections of hybrid type have the property* (4)*;*

2) $B(R) = 0$.

Proof. Let $\{a_1,\dots,a_n; x_1,\dots,x_n\}$ be an adapted basis for T_pM. From the condition 1) of the theorem we have $R(a_i + x_j, a_j + x_i, a_j + x_i, a_i + x_j) = 0$, $i \neq j$. These equalities imply

$$6K(a_i,x_j) + 2K(a_i,a_j) = K(a_i,x_i) + K(a_j,x_j); \qquad i \neq j. \tag{10}$$

Using $u = a_i + x_i + a_j - x_j$, $i \neq j$ and the condition 1) we obtain

$$K(a_i,a_j) = K(a_i,x_j), \qquad i \neq j. \tag{11}$$

The equalities (10) and (11) give

$$K(x_i,Px_i) - \frac{4}{n+2}\rho(x_i) + \frac{\tau}{(n+1)(n+2)} = 0,$$

which is equivalent to (9) and hence $B(R) = 0$. The inverse is a simple verification.

3. Totally real sections

The curvatures of nondegenerate totally real sections are said to be totally real sectional curvatures.

Lemma 2. *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(x,a,a,x) = 0$ whenever $a \perp x,\ Px$, i.e. the totally real sectional curvatures of hybrid type are zero;*

2) *$R(x,y,y,x) = 0$ whenever $x \perp y,\ Py$, i.e. the totally real sectional curvatures of pure type are zero;*

3) *$R = 0$.*

Proof. Let $\{x,y,a\}$ be an orthogonal triple spanning a 3-dimensional totally real space. For the pair $\{x, a' = (a+ty)/\sqrt{1-t^2}\}$, $|t| < 1$ we have $a' \perp x, Px$. Substituting this pair into the condition 1) of the lemma, we get $R(x, a+ty, a+ty, x) = 0$. The corresponding polinomial identity gives $R(x,y,y,x) = 0$, i.e. 1) implies 2). The inverse follows in a similar way.

Now, let $\{x,y,z\}$ be orthogonal and span a 3-dimensional totally real space. Applying 1) to the vectors $(x-y)/\sqrt{2}$, $(Px+Py)/\sqrt{2}$ and using 2) we find $K(x,Px) + K(y,Py) = 0$. Analogously, $K(y,Py) + K(z,Pz) = K(x,Px) + K(z,Pz) = 0$. Therefore $K(x,Px) = 0$ and Lemma 1) implies $R = 0$.

The following theorem has an easy proof using Lemma 2.

Theorem 3. *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *M is of constant totally real sectional curvature of hybrid type. i.e. $K(a,x) = v$, whenever $\perp x, Px$;*

2) *M is of constant totally real sectional curvature of pure type, i.e. $K(x,y) = v(K(a,b) = v)$, whenever $x \perp y, Py$ $(a \perp b, Pb)$;*

3) *M is of constant holomorphic sectional curvature $\mu = 4v$.*

Theorem 4. *Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(x,\xi,\xi,x) = 0$ whenever $\{x,\xi\}$ spans a weakly isotropic totally real section of I type;*

2) *$B(R) = 0$.*

Proof. Let the pair $\{x,y\}$ be orthogonal and span a totally real section. Applying the condition 1) of the theorem to the pair $\{x, \xi = y + Py\}$ we obtain

$$R(x,y,y,x) + R(x,Py,Py,x) = 0. \tag{12}$$

Now, we substitute the pair $\{x,y\}$ in (12) by $\{(x+y)/\sqrt{2}, (x-y)/\sqrt{2}\}$ and linearizing we find

$$8K(x,y) = K(x,Px) + K(y,Py). \tag{13}$$

Further, as in the proof of Theorem 2, (12) and (13) give $B(R) = 0$.

The inverse follows immediately by taking into account that $\rho(\xi, \xi) = 0$.

THEOREM 5. *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(x, u, u, x) = 0$, whenever $\{x, u\}$ spans a weakly isotropic totally real section of the II type;*

2) *M is of constant holomorphic sectional curvature.*

Proof. Let $\{a_1, \ldots, a_n; x_1, \ldots, x_n\}$ be an adapted basis for T_pM. Applying the condition 1) of the theorem to the pairs $\{x_i, x_j + a_k\}$ (i, j, k - different), we find $K(a_i, x_j) = \text{const}$; $i \neq j$. This is equivalent to the condition 1) of Theorem 3. Hence, M is of constant holomorphic sectional curvature.

The inverse is easy to check.

THEOREM 6. *Let $M(2n \geq 6)$ he a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(\xi, u, u, \xi) = 0$, whenever $\{\xi, u\}$ spans a strongly isotropic totally real section of the I type;*

2) *$B(R) = 0$.*

Proof. Let $\{\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n\}$ be a separate basis for T_pM. Applying the condition 1) to the pair $\{\xi_i, \eta_j + \lambda\xi_k\}$, $\lambda \neq 0$ (i, j, k - different) we obtain

$$0 = R(\xi_i, \eta_j, \eta_j, \xi_i); \qquad i \neq j. \tag{14}$$

The pairs $\{\xi_i, \eta_j\}$, $i \neq j$ span strongly isotropic holomorphic sections of hybrid type and (14) is equivalent to the condition 1) of Theorem 2. Hence, $B(R) = 0$.

THEOREM 7. *Let $M(2n \geq 8)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(u, v, v, u) = 0$, whenever $\{u, v\}$ spans a strongly isotropic totally real section of the II type;*

2) *$B(R) =)$.*

Proof. Let $\{\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n\}$ be a separate basis for T_pM. Substituting $\{u = \xi_i + \lambda\eta_j, v = \lambda\xi_k + \eta_l\}$, $\lambda \neq 0$ (i, j, k, l - different) in the condition 1), we get $R(\xi_i, \eta_l, \eta_l, \xi_i) = 0$, $i \neq 1$, which is (14) and therefore $B(R) = 0$.

THEOREM 8. *Let $M(2n \geq 8)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:*

1) *$R(x_i, x_j, x_k, x_l) = 0$, ($i, j, k, l$ - different), whenever $\{a_1, \ldots, a_n; x_i, \ldots, x_n\}$ is an adapted basis;*

2) *$K(x_i, x_j) + K(x_k, x_l) = K(x_j, x_k) + K(x_j, x_l)$, ($i, j, k, l$ - different) whenever $\{a_1, \ldots, a_n; x_i, \ldots, x_n\}$ is an adapted basis;*

3) *$B(R) = 0$.*

This theorem is analogous to a theorem in [**9**] for Kaehlerian manifolds and it can be checked in a similar way taking into account the properties of the structure P.

4. Pinching problems

A Ricci curvature cannot be defined for an isotropic direction. If X is an isotropic vector and $\rho(X,X)=0$, this is a geometric property of the isotropic direction, defined by X.

The following statement is a slight modification of a result in [**3**].

Lemma 3. *Let M be a hyperbolic Kaehlerian manifolod. The following conditions are equivalent:*

1) $\rho(u,u)=0$*, for arbitrary* u*;*

2) $\rho=(\tau/2n)g$*, i.e. M is Einsteinian.*

Theorem 9. *Let $M(2n\geq 4)$ be a hyperbolic Kaehlerian manifold. If the holomorphic sectional curvatures in every point are bounded, i.e. for an arbitrary nondegenerate holmorphic sertion α in T_pM*

$$|K(\alpha,p)|\leq c(p), \tag{15}$$

then M is of constant holomorphic sectional curvature.

Proof. Let $x=u+a$, $a\perp u, Pu$ and α be the holomorphic section spanned by $\{(x+ta/\sqrt{1-t^2},(Px+tPa)/\sqrt{1-t^2}$, $|t|<1$. From condition (15) we get

$$|R(x+ta,Px+tPa,Px+tPa,x+ta)\leq(1-t^2)^2c(p).$$

Hence, $R(u,Pu,Pu,u)=0$ and Theorem 2 implies $B(R)=0$, i.e.

$$\frac{4}{n+2}\rho(x)=K(x,Px)+\frac{\tau}{(n+1)(n+2)}.$$

This equality gives that the Ricci curvatures in every point are bounded

$$|\rho(x)|\leq c'(p). \tag{16}$$

Substituting x by $(x+ta)/\sqrt{1-t^2}$, $|t|<1$ in (16), we find $\rho(u)=0$ and Lemma 3 implies that M is Einsteinian. Now, the statement follows from Corollary 1.

Theorem 10. *Let $M(2n\geq 6)$ be a hyperbolic Keahlerian manifofd. If the totally real sectional curvatures of hybrid type are bounded in every point, i.e. if*

$$|K(x,a)|\leq c(p);\qquad a\perp x,Px, \tag{17}$$

then M is of constant holomorphic sectional curvature.

Proof. Let $u=x+a$ and $\{x,a,b\}$ span a totally real 3-dimensional space. Substituting the pair $\{x,a\}$ in (17) by $\{(x+ta)/\sqrt{1-t^2},b\}$, $|t|\leq 1$, we obtain

$$|R(x+ta,b,b,x+ta)|\leq(1-t^2)c(p).$$

Therefore, $R(u, b, b, u) = 0$, and Theorem 5 implies that M is of constant holomorphic sectional curvature.

Remark. The totally real curvatures of hybrid type in Theorem 10 can be replaced by totally real curvatures of pure type.

THEOREM 11. *Let $M(2n \geq 6)$ be a hyperbolic Kaelalerian manifold. If the totally real sectional curvatures are bounded from above, i.e. if*

$$\begin{aligned} K(x, a) &\leq c(p); \qquad a \perp x, Px, \\ K(x, y) &\leq c(p); \qquad x \perp y, Py, \end{aligned} \tag{18}$$

then M is of constant holomorphic sectional curvature.

Proof. Let $u = y + a$ and $\{x, y, a\}$ span a 3-dimensional totally real space. The first condition of (18) implies $R(x, a, a, x) \geq -c(p)$. Substituting here the vector a by $(a+ty)/\sqrt{1-t^2}$, $|t| < 1$, we get $R(x, u, u, x) \geq 0$. Using, the inequality $R(x, y, y, x) \leq c(p)$ and substituting the vector y by $(y + ta)/\sqrt{1-t^2}$, $|t| < 1$, we obtain $R(x, u, u, x) \leq 0$. Therefore $R(x, u, u, x) = 0$ and the theorem follows now from Theorem 5.

5. Plane axioms

Let M ($\dim M = m \geq 3$) be a differentiable manifold with a linear connection of zero torsion. M is said to satisfy the axiom of r-planes ($2 \leq r < m$), if, for each point p and for any r-dimensional subspace E of T_pM, there exists an r-dimensional totally geodesic submanifold N containing p such that $T_pN = E$.

THEOREM 12. (Axiom of nondegenerate totally real 2-planes of hybrid type) *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. If for any nondegenerate totally real section α in T_pM of hybrid type there exists a 2-dimensional totally geodesic submctnifold N containing p such that $T_pN = \alpha$, then M is of constant holomorphic sectionai curvature.*

Proof. Let $\{x, y, b\}$ be orthogonal and let it span a 3-dimensional totally real space in T_pM. The pair $\{x, y' = (b+ty)/\sqrt{1-t^2}\}$, $|t| < 1$ spans a 2-plane α, which is nondegenerate totally real of hybrid type. By the condition of the theorem, it follows that $R(y', x)x$ is in a and $R(y', x)x \perp y''$, where $y'' = y + tb$. From this, it follows that $R(x, u, u, x) = 0$, where $u = y + b$. Now, the proposition follows from Theorem 5.

Remark. The nondegerate totally real 2-planes of hybrid type in Theorem 12 can be replaced with nondegenerate totaly real 2-planes of pure type.

THEOREM 13. (Axiom of weakly isotropic totally real 2-planes of the I type) *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. If for any weakly isotropic totally real 2 plane α in T_pM of the I type there exists a 2-dimensional totally* *geodesic submanifold N, containing p such that $T_pN = \alpha$, then M has a vanishing Bochner curvature tensor.*

Proof. Let a be an arbitrary weakly isotropic totally real 2-plane of the I type with a basis $\{\xi, x\}$ $\xi \perp x, \xi$ – eigen. By the condition of the theorem, it follows that $R(\xi, x)x$ is in a and therefore, $R(\xi, x, x, \xi) = 0$. Now, the proposition follows from Theorem 4.

THEOREM 14. (Axiom of weakly isotropic totally real 2-planes of the II type) *Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. If for every weakly isotropic totally real 2 plane in T_pM of the II type there exists a 2-dimensional totally geodesic submanifold N containing p such that $T_pM = \alpha$, then M is of constant holomorphic sectional curvature.*

The proof is similar to the proof of Theorem 13 and we omit it.

THEOREM 15. (Axiom of strongly isotropic totally real 2-planes of the I type (II type)) *Let $M(2n \geq 8)$ be a hyperbolic Kaehlerian manifold. If for every strongly isotropic tolally real 2-plane a in T_pM of the I type (II type) there exists a 2-dimensional totally geodesic submanifolod N containing p such that $T_pN = \alpha$, then M has a vanishing Bochner curvature tensor.*

The proof is similar to the proof of Theorem 13 and it is based on Theorem 6 (Theorem 7).

THEOREM 16. (Axiom of nondegenerate holomorphic 2-planes) *Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. If for erery nondegenerate holomorphic 2-plane α in T_pM there exists a 2-dimensional totally geodesic submanifold N containing p surh that $T_pN = \alpha$, then M is of constant holomorphic sectional curvature.*

Proof. Let x be arbitrary and $a \perp x, Px$. If α is the holomorphic section spanned by $\{x, Px\}$, from the condition of the theorem it follows that $R(x, Px)Px$ is in α. Hence,

$$R(x, Px, Px, a) = 0. \tag{19}$$

Substituting the pair $\{x, a\}$ in (19) by $\{(x + ta)/\sqrt{1 - t^2}, (a + tx)/\sqrt{1 - t^2}\}$, $|t| < 1$, we obtain $R(u, Pu, Pu, u) = 0$, where $u = a + x$. Theorem 2 implies $B(R) = 0$. By using (19) and formula (8) we find

$$\rho(x, a) = 0, \tag{20}$$

Substituting the pair $\{x, a\}$ as above, we get $\rho(u, u) = 0$. Now, from Lemma 3 it follows that (20) implies $\rho = (\tau/2n)g$. This condition and $B(R) = 0$ give the proposition.

THEOREM 17. (Axiom of strongly isotropic holomorphic 2-planes) *Let $M(2n \geq 4)$ be a hiperbolic Kaehlerian manifold. If for every strongly isotropic* *holomorphic 2-plane a in T_pM of hybrid type there exists a 2-dimensional totally geodesic submanifold N containing p such that $T_pN = \alpha$, then M a vanishing Bochner currature tensor.*

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