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ON THE CONVERGENCE OF BIORTHOGONAL SERIES CORRESPONDING TO NONSELFADJOINT STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENTS*)

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Abstract. We consider the convergence of the biorthogcnal series corresponding to the nonselfadjoint Sturm-Liouville operator at the points of discontinuity of its coefficients. For any function $f(x) \in L_2$ we construct a function $\tilde{f}_{x_0}(x)$ such that the trigonometrical Fourier series of $\tilde{f}_{x_0}(x)$ is convergent at the point of discontinuity x_0 if and only if the biorthogonal series of $f(x)$ is convergent at this point.

1. Definitions and results. 1. Consider the nonselfadjoint Sturm-Liouville operator

$$L(u) = -(p(x)u')' + q(x)u \tag{1}$$

defined on a finite interval $G = (a, b)$. Let $x_0 \in (a, b)$ be a point of discontinuity of the coefficients of the operator (1). If we introduce the notation

$$p(x) = \begin{cases} p_1(x), & x \in (a, x_0), \\ p_2(x), & x \in (x_0, b), \end{cases}$$

then the following conditions are imposed on the coefficients:

$$\begin{array}{ll} 1) & p_1(x) = p_1 = \text{const.} > 0, x \in (a, x_0]; \;\; p_2(x) = p_2 = \text{const.} > 0, x \in [x_0, b). \\ 2) & q(x) \in L_p^{\text{loc}}(G), 1 < p < +\infty; \;\; q(x) \quad \text{is a complex-valued function.} \end{array} \tag{2}$$

Definition 1. A complex-valued function $u_\lambda^0(x) \not\equiv 0$ is called an eigenfunction of the operator (1) corresponding to the (complex) eigenvalue λ if it satisfies the following conditions:

a) $u_\lambda^0(x)$ is absolutely continuous on any closed subinterval of G.

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b) $u_\lambda^{0'}(x)$ is absolutely continuous on any closed subinterval of the semi-open intervals $(a, x_0]$ and $[x_0, b)$.

c) $u_\lambda^0(x)$ satisfies the equation

$$-p_1 u_\lambda^{0''}(x) + q(x)u_\lambda^0(x) = \lambda u_\lambda^0(x)$$

almost everywhere on (a, x_0), and the equation

$$-p_2 u_\lambda^{0''}(x) + q(x)u_\lambda^0(x) = \lambda u_\lambda^0(x)$$

almost everywhere on (x_0, b),

d) $u_\lambda^0(x)$ and $u_\lambda^{0'}(x)$ satisfy the junction conditions

$$u_\lambda^0(x_0 - 0) = u_\lambda^0(x_0 + 0), \;\; p_1 u_\lambda^{0'}(x_0 - 0) = p_2 u_\lambda^{0'}(x_0 + 0)$$

at the point of discontinuity of coefficients.

Definition 2. A complex-valued function $u_\lambda^i(x)$, $i \in N$, is called an i-th associated function of the operator (1) corresponding to the eigenfunction $u_\lambda^0(x)$ and the eigenvalue λ if it satisfies the following conditions:

a) Conditions a), b) and d) of Definition 1 hold for $u_\lambda^i(x)$.

b) $u_\lambda^i(x)$ satisfies the equation

$$-p_1 u_\lambda^{i''}(x) + q(x)u_\lambda^i(x) = \lambda u_\lambda^i(x) - u_\lambda^{i-1}(x)$$

almost everywhere on (a, x_0), and the equation

$$-p_2 u_\lambda^{i''}(x) + q(x)u_\lambda^i(x) = \lambda u_\lambda^i(x) - u_\lambda^{i-1}(x)$$

almost everywhere on (x_0, b).

We shall suppose that for every eigenvalue λ both the corresponding eigenfunction $u_\lambda^0(x)$ and the first associated function $u_\lambda^1(x)$ exist. Let $\{u_\lambda^i(x) | n \in N, i = 0, 1\}$ be an arbitrary minimal and complete system in $L_2(G)$ of eingenfunctions and associated functions of the operator (1), and let $\{\lambda_n | n \in N\}$ be the corresponding system of eigenvalues. Assuming that finite limit points of the set $\{\sqrt{\lambda_n} | n \in N\}$ do not exist, we can enumerate the numbers λ_n so that the sequence $\nu_n = |\sqrt{\lambda_n}|$[1] does not decrease. Denote by $\{\nu_n^i(x) | n \in N, i = 0, 1\}$ the system of functions biorthogonally dual in $L_2(G)$ to $\{u_n^i(x) | n \in N, i = 0, 1\}$, i.e. such a system that $\nu_n^i(x) \in L_2(G)$ and

$$(u_n^i, \nu_m^j) \stackrel{\text{def}}{=} \int_G u_n^i(x) \cdot \overline{\nu_m^j(x)} dx = \begin{cases} 1, & \text{if } n = m, \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $f(x)$ be an arbitrary function from the class $L_2(G)$ and μ be a positive number. Form the partial sum of the expansion of $f(x)$ in biorthgonal series:

$$\sigma_\mu(x, f) = \sum_{\substack{1 \le n \le \mu \\ i = 0, 1}} (f, \nu_n^i) \cdot u_n^i(x).$$

[1] If $\lambda_n = r_n \cdot e^{i\varphi_n}$, then $\sqrt{\lambda_n} = \sqrt{r_n} \cdot e^{i \cdot \varphi_n / 2}$ where $-\pi/2 < \varphi_n \le 3\pi/2$.

Let R be an arbitrary positive number such that the number $R \cdot \max\{1, \sqrt{p_1}, \sqrt{p_2}\}$ is less than the distance $\rho(x_0, \partial G)$ from x_0 to the boundary ∂G of G. Consider the function

$$S_\mu(x,f) = \frac{1}{\pi} \cdot \int\limits_{x-R}^{x+R} \frac{\sin \mu(x-y)}{x-y} \cdot f(y)dy,$$

where $x \in K_R = [a+R, b-R]$. It is known that the function $S_\mu(x,f)$ differs on K_R from the partial sum of the trigonometrical Fourier series of $f(x)$ by a function converging to 0 when $\mu \to +\infty$, uniformly with respect to $x \in K_R$.

Define the function

$$\tilde{f}_{x_0}(x) = \begin{cases} \dfrac{2\sqrt{p_2}}{\sqrt{p_1}+\sqrt{p_2}} \cdot f(x_0 + \sqrt{p_2}(x-x_0)), & x \in [x_0, x_0+\sqrt{p_2}R) \\ \dfrac{2\sqrt{p_1}}{\sqrt{p_1}+\sqrt{p_2}} \cdot f(x_0 - \sqrt{p_1}(x_0-x)), & x \in (x_0-\sqrt{p_1}R, x_0) \\ 0, \quad x \in (a, x_0-\sqrt{p_1}R] \ \text{or} \ x \in [x_0+\sqrt{p_2}R, b), \end{cases}$$

where $f(x)$ is a given function from $L_2(G)$.

Our main result is the following.

THEOREM. *Let the coefficients of the operator* (1) *satisfy the conditions* (2), *and let the eigenvalues λ_n satisfy the following ones: there exist constants A and B independent of the numbers λ_n such that*

1) $|\mathrm{Im}\sqrt{\lambda_n}| \le A$, $n \in N$.

2) $\sum\limits_{\left||\mathrm{Re}\sqrt{\lambda_n}|-\mu\right| \le 1} 1 \le B$ *for* $\mu \le 0$,

where B does not depend on μ.

Then for every function $f(x)$ from $L_2(G)$ the following holds:[2]

$$\lim_{\mu\to+\infty} (\sigma_\mu(x_0,f) - S_{\nu_{[\mu]}}(x_0, \tilde{f}_{x_0}) = 0. \tag{3}$$

2. This Theorem is an extension of the known result of Il'in [**1**] to the case of nonselfadjoint Sturm-Liouville operator. In [**1**] the convergence of generalized Fourier series corresponding to an arbitrary nonegative selfadjoint extension of the operator (1) was considered. The case of the fourth order selfadjoint ordinary differential operator was considered by Budak in [**2**].

Proof of the theorem. Define the function

$$t(x_0,y,\mu,R) = \begin{cases} \sqrt{p_2} \cdot a(x_0) \dfrac{\sin \mu p_2^{-1/2}(y-x_0)}{y-x_0}, & x_0 \le y \le x_0+\sqrt{p_2}R, \\ \sqrt{p_1} \cdot a(x_0) \dfrac{\sin \mu p_1^{-1/2}(x_0-y)}{x_0-y}, & x_0-\sqrt{p_1}R \le y \le x_0, \\ 0, \quad x \in (a, x_0-\sqrt{p_1}R) \ \text{or} \ x \in (x_0+\sqrt{p_2}R, b), \end{cases}$$

[2] We denote by $[\mu]$ the integer part of μ. If $\mu \to \infty$, then $\nu_{[\mu]} \to \infty$.

where $a(x_0) = \dfrac{2}{\pi(\sqrt{p_1} + \sqrt{p_2}}$ $\mu > 0$, and R is a positive number defined in **1**.

Then for the spectral function

$$\theta(x, y, \mu) = \sum_{\substack{1 \le n \le \mu \\ i = 0,1}} u_n^i(x) \cdot \overline{\nu_n^i(y)}$$

of the operator (1) one has the estimate [**3**, Lemma 3]

$$\int_G |\theta(x_0, y, \mu) - t(x_0, y, \nu_{[\mu]}, R)|^2 dy = O(1), \quad \mu \to \infty. \tag{4}$$

By the Cauchy-Schwartz inequality, from (4) it follows that the estimate

$$\begin{aligned} &\int_G f(y) \Bigg(\sum_{\substack{1 \le n \le \mu \\ i = 0,1}} u_n^i(x) \cdot \overline{\nu_n^i(y)} - t(x_0, y, \nu_{[\mu]}, R) \Bigg) dy = \\ &= O(1) \|f\|_{L_2(G)}, \quad \mu \to +\infty, \end{aligned} \tag{5}$$

is valid for every function $f(x) \in L_2(G)$. On the other hand,

$$\begin{aligned} &\int_G f(y) \Bigg(\sum_{\substack{1 \le n \le \mu \\ i = 0,1}} u_n^i(x) \cdot \overline{\nu_n^i(y)} - t(x_0, y, \nu_{[\mu]}, R) \Bigg) dy = \\ &= \sigma_\mu(x_0, f) - \int_a^b t(x_0, y, \nu_{[\mu]}, R) f(y) dy = \\ &= \sigma_\mu(x_0, f) - \sqrt{p_2} a(x_0) \cdot \int_{x_0}^{x_0 + \sqrt{p_2} R} \frac{\sin \nu_{[\mu]} p_2^{-1/2} (y - x_0)}{y - x_0} f(y) dy - \\ &- \sqrt{p_1} a(x_0) \cdot \int_{x_0 - \sqrt{p_1} R}^{x_0} \frac{\sin \nu_{[\mu]} p_1^{-1/2} (x_0 - y)}{x_0 - y} f(y) dy = \sigma_\mu(x_0, f) - \\ &- \sqrt{p_2} a(x_0) \cdot \int_{x_0}^{x_0 + R} \frac{\sin \nu_{[\mu]} (x - x_0)}{x - x_0} f(x_0 + \sqrt{p_2}(x - x_0)) dx - \\ &- \sqrt{p_1} a(x_0) \cdot \int_{x_0 - R}^{x_0} \frac{\sin \nu_{[\mu]} (x_0 - x)}{x_0 - x} f(x_0 - \sqrt{p_1}(x_0 - x)) dx = \\ &= \sigma_\mu(x_0, f) - S_{\nu_{[\mu]}}(x_0, \tilde{f}_{x_0}). \end{aligned}$$

Thus the estimate (5) has the form

$$\sigma_\mu(x_0, f) - S_{\nu_{[\mu]}}(x_0, \tilde{f}) = O(1)\|f\|_{L_2(G)}, \quad \mu \to +\infty.$$

Let ε be any positive number. By the completeness of the system $\{u_n^i(y) \mid n \in N,\ i = 0, 1\}$, there exist numbers $n_0(\varepsilon) \in N$ and $c_n^i, i = 0, 1,\ 1 \leq n \leq n_0(\varepsilon)$ such that

$$\left\| f(y) - \sum_{\substack{n=1 \\ i=0,1}}^{n_0(\varepsilon)} c_n^i u_n^i(y) \right\|_{L_2(G)} \leq \varepsilon. \tag{7}$$

Let

$$P_{n_0}(y) = \sum_{\substack{n=1 \\ i=0,1}}^{n_0(\varepsilon)} c_n^i u_n^i(y).$$

Then $\sigma_\mu(y, P_{n_0}) = P_{n_0}(y)$ if $\mu \geq n_0(\varepsilon)$. Applying (6) to $f(y) - P_{n_0}(y)$, by (7) we obtain that

$$(\sigma_\mu(x_0, f) - S_{\nu_{[\mu]}}(x_0, \tilde{f}_{x_0}) - (P_{n_0}(x_0) - S_{\nu_{[\mu]}}(x_0(\tilde{P}_{n_0})_{x_0}) = O(1) \cdot \varepsilon \tag{8}$$

holds for $\mu \geq n_0(\varepsilon)$. It is not difficult to see that

$$\lim_{\mu \to +\infty} S_{\nu_{[\mu]}}(x_0, (\tilde{P}_{n_0})_{x_0}) = P_{n_0}(x_0). \tag{9}$$

Now the equality (3) follows from (8)–(9). The theorem is proved.

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