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ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS

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Abstract. We construct a sequence (J_n) of linear positive operators defined on the space $C(K)$, $K = [a,b]$, with the properties: a) $J_nf(f \in C(K))$ is a polynomial of degree $\leq n$; b) if $f \in C(K)$ then there exists a positive constant C_0 such that $\|f - J_nf\| \leq C_0 \cdot \omega(f; 1/n)$, $n = 1, 2, \dots$, where $\|\cdot\|$ is the uniform norm and $\omega(f;\cdot)$ is the modulus of continuity; c) for $f \in C(K)$ there exists a $C_1 > 0$ such that

$$|f(x) - (J_nf)(x)| \leq C_1 \cdot \omega\ (f; \Delta_n(x)), \quad x \in K$$

where

$$\Delta_n(x) = \sqrt{(x-a)(b-x)/n} + n^{-2}, \quad n = 1, 2, \dots ;$$

d) if $\Delta_n^*(x) = \sqrt{(x-a)(b-x)/n}$ and

$$(J_n^*f)(x) = (J_nf)(x) + \frac{b-x}{b-a}[f(a) - (J_nf)(a)] + \frac{x-a}{b-a}[f(b) - (J_nf)(b)],$$

then for every continuous function $f : [a,b] \to R$ there exists a positive constant C_2 such that

$$|f(x) - (J_n^*f)(x)| \leq C_2 \cdot \omega(f; \Delta_n^*(x)), \quad x \in [a,b], \quad n = 1, 2, \dots .$$

In this manner are presented constructive proofs of the well-known theorems of Jackson [8], Timan [14] and Teljakovskiĭ [13]. Likewise, some other approximation properties of the operators (J_n) are investigated.

1. Introduction and definitions. Let K be a compact interval of the real axis and denote by $C(K)$ the normed linear space of continuous real-valued functions on K. As usually, the space $C(K)$ is normed by means of the uniform norm, that is $\|f\| = \max_{t \in K} |f(t)|$, $f \in C(K)$. We will use the notation $\|\cdot\|_K$ to indicate that the maximum is taken over K whenever it is necessary to make it clear which interval the norm is taken over. Likewise, by Π_n is denoted the linear space of polynomials, with real coefficients, of degree at most n.

For $f \in C(K)$ let (P_n^*f), $P_n^*f \in \Pi_n$ be the sequence of polynomials of best approximation to f; more precisely

$$\|f - P_n^*f\| \leq \|f - p_n\|$$

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for all polynomials p_n, $p_n \in \Pi_n$. It is known that the operator $P_n^* : C(K) \to \Pi_n$ which maps f into $P_n^* f$ is not a linear transformation. At the same time, if $f \in C(K)$ and $\omega(f;\cdot)$ is modulus of continuity defined, for $\delta \geq 0$, by

$$\omega(f;\delta) = \max_{\substack{|t-x|\leq\delta \\ t,x\in K}} |f(x) - f(t)|,$$

then according to the well-known theorem of Jackson ([5], [8], [10]) the sequence $(P_n^* f)$ satisfies the inequalities

$$\|f - P_n^* f\| \leq C_0 \cdot \omega(f; 1/n), \quad C_0 \in (0, 1 + \pi^2/2], \quad n = 1, 2, \dots .$$

Several authors (see [2], [3], [7]) have constructed explicitly sequences of polynomials $(A_n f)$ which have essentially the same degree of precision of aproximation to f, as $P_n^* f$. These polynomials $A_n f$, $n = 1, 2, \dots, f \in C(K)$, have the properties:

i) the operator $A_n : f \to C_n f$ is linear on $C(K)$;

ii) $A_n(C(K)) \subseteq \Pi_{m(n)}$, $m(n) \geq n$;

iii) there exists an interval $[c, d]$, $a < c < d < b$, $K = [a, b]$, such that for $f \in C(K) : \|f - A_n f\|_{[c,d]} \leq C \cdot \omega(f; 1/n)$, $C > 0$, $n = 1, 2, \dots$. Therefore, these kinds of polynomial operators $A_n : C(K) \to \Pi_{m(n)}$, $n = 1, 2, \dots$, cannot be used to approximate on all of $K = [a, b]$. They are only efficient on subintervals $[c, d]$ with $[c, d] \subset K$.

In 1951, Timan [14] has proved that if $f \in C[a, b]$, then for every n there exists an algebraic polynomial $\tau_n f$ of degree at most n such that for all $x \in [a, b]$

$$|f(x) - (\tau_n f)(x)| \leq C_1 \cdot \omega(f; \sqrt{(x-a)(b-x)/n} + n^{-2}) \quad n = 1, 2, \dots$$

where C_1, is a positive constant. The characteristic peculiarity of this inequality is the improvement of the order of approximation near the endpoints in comparison to the usual Jackson theorem. This motivates the following:

Definition. A sequence of operators (J_n) defined on $C(K)$, $K = [a, b]$, is said to be of Jackson-type, if

a) $J_n(C(K)) \subseteq \Pi_n$, $n = 1, 2, \dots$;

b) $J_n : C(K) \to \Pi_n$ is a linear positive operator;

c) for every f, $f \in C(K)$, there exists a positive constant C_0 such that $\|f - J_n f\| \leq C_0 \cdot \omega(f; 1/n)$, $n = 1, 2, \dots$, where $\|\cdot\| = \|\cdot\|_K$;

d) if $f \in C(K)$, then for all $x \in [a, b]$ and $n = 1, 2, \dots$ $|f(x) - (J_n f)(x)| \leq C_1 \cdot \omega(f; \sqrt{(x-a)(b-x)/n} + n^{-2})$, C_1 being a positive constant.

Takin into account that we will be concerned with the approximation of continuous functions $f : K \to R$, $K = [a, b]$, by elements from Π_n, and since the space Π_n remains invariant under the transformation $x = (2t - a - b)/(b - a)$, $t \in [a, b]$, it suffices to carry out the analysis for the interval $[-1, 1]$. Throughout this paper, C will denote positive constants which are, in general, different. Likewise, I denotes the interval $[-1, 1]$.

2. A quadrature formula. Let $C^{(j)}(I)$ be the linear space of all functions $f : I \to R$ which have a continuous j^{th} derivative on the interval I. In order to prove some identities we need the following proposition.

LEMMA 1. *Let n be a natural number and $s = s(n) = 1 + [n/2]$. If $f \in C^{(n+2)}(I)$, then there exists a point $\theta = \theta(n, f)$, $\theta \in (-1, 1)$, such that*

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{2\pi}{n+2} \left[\frac{1-(-1)^n}{4} f(-1) + \sum_{k=1}^{s} f(x_{kn}) \right] + R_n(f) \tag{1}$$

where

$$R_n(f) = \frac{\pi}{2^{n+1}} \cdot \frac{f^{(n+2)}(\theta)}{(n+2)!} \quad \text{and} \quad x_{kn} = \cos \frac{(2k-1)\pi}{n+2}. \tag{2}$$

Proof. Let us suppose that n is an even natural number, $n = 2m - 2$, $m \geq 1$. Then (1) may be written as

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{m} \sum_{k=1}^{m} f\left(\cos \frac{(2k-1)\pi}{2m} \right) + R_{2m-2}(f) \tag{3}$$

where $R_{2m-2}(f) = \frac{\pi}{2^{2m-1}} \frac{f^{(2m)}(\theta)}{(2m)!}$, $\theta \in (-1, 1)$.
This is the Mehler-Hermite formula with remainder term [9, p. 111, (7.3.6.)].

Now let n be an odd natural number, $n = 2m - 1$. Then (1) is the same as

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{2m+1} f(-1) + \frac{2\pi}{2m+1} \sum_{k=1}^{m} f\left(\cos \frac{(2k-1)\pi}{2m+1} \right) + R_{2m-1}(f).$$

$$R_{2m-1}(f) = \frac{\pi}{2^{2m}} \frac{f^{2m+1}(\theta)}{(2m+1)!}, \quad \theta \in (-1, 1),$$

which is a quadrature formula attributed to Bouzitat. We note that the remainder-term $R_n(f)$ from (1) may be represented on the space $C(I)$ as

$$R_n(f) = \pi 2^{-n-1} [\theta_1, \theta_2, \ldots, \theta_{n+3}; f]$$

where $[\theta_1, \theta_2, \ldots, \theta_{n+3}]$ denotes the divided difference at a system of distinct points $\theta_1, \theta_2, \ldots, \theta_{n+3}$ from I (see [11]–[12]).

3. A sequence of Jackson type operators. Let $w(t) = 1/\sqrt{1-t^2}$, $t \in (-1, 1)$, and L_w^p, $1 \leq p \leq \infty$, be the class of measurable functions on I which satisfy $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_{-1}^{1} |f(t)|^p w(t) dt \right)^{1/p}, \quad 1 \leq p < \infty,$$

and $\|f\|_\infty$ is the sup-norm. Further, by X we denote one of the following function spaces: $C(I)$ or L_w^p.

Also we use the following notation:

$$T_m(x) = \cos m(\arccos x)$$

$$\varphi_n^*(x) = a_n \cdot \frac{1+T_{n+2}(x)}{(x-\cos\pi/(n+2))^2}, \quad \varphi_n^* \in \Pi_n, \quad a_n = \frac{1}{\pi(n+2)}\sin^2\frac{\pi}{n+2},$$

$$t_k(f) = \int_{-1}^{1} f(t)T_k(t)w(t)dt, \qquad f \in X, \tag{4}$$

$$\omega_k = \frac{1}{t_k(T_k)} = \begin{cases} 2/\pi, & k = 1,2,\dots \\ 1/\pi, & k = 0. \end{cases}$$

Functions from X can be expanded in terms of Chebyshev polynomials. Every $f \in X$ has the expansion

$$f(x) \sim \sum_{k=1}^{\infty} \omega_k t_k(f) T_k(x), \quad x \in I, \tag{5}$$

where $t_k(f)$ are the Chebyshev coefficients defined as above.

In order to try to give a simple and unified approach to the theory of approximation by algebraic polynomials on a compact interval, Butzer and Stens [4] have introduced the translation operator τ_x, $x \in I$, defined on X by:

$$(\tau_x f)(t) = 1/2 \cdot [f(xt + \sqrt{1-x^2}\sqrt{1-t^2}) + f(xt - \sqrt{1-x^2}\cdot\sqrt{1-t^2})], \quad t \in I.$$

If $f, g \in L_w^1$, then their convolution product is defined by means of the equality

$$(f * g)(x) = \int_{-1}^{1} (\tau_x f)(t) g(t) w(t) dt.$$

This convolution has the following properties [1]: if $f, g, h \in L_w^1$, then $f * g \in L_w^1$ and:

i) $f * g = g * f$; ii) $f * (g * h) = (f * g) * h$; iii) $t_k(f * g) = t_k(f)t_k(g)$;

iv) if $f \in L_w^1$, $g \in L_w^p$, $1 \le p \le \infty$, then $f * g \in L_w^p$ and $\|f * g\|_p \le \|f\|_1 \cdot \|g\|_p$.

Taking into account that for $k \ge 1$

$$\begin{aligned} T_k(xt + \sqrt{1-x^2}\cdot\sqrt{1-t^2}) &= T_k(x)T_k(t) + k^{-2}\sqrt{1-x^2}\cdot\sqrt{1-t^2}T_k'(x)T_k'(t) \\ T_k(xt - \sqrt{1-x^2}\cdot\sqrt{1-t^2}) &= T_k(x)T_k(t) - k^{-2}\sqrt{1-x^2}\cdot\sqrt{1-t^2}T_k'(x)T_k'(t), \end{aligned} \tag{6}$$

it follows that $(\tau_x T_k)(t) = T_k(x)T_k(t)$. Therefore, if $f \in X$ has the expansion (5), then

$$(\tau_x f)(t) \sim \sum_{k=0}^{\infty} \omega_k t_k(f) T_k(x) T_k(t), \tag{7}$$

$$(f \cdot g)(x) \sim \sum_{k=0}^{\infty} \omega_k t_k(f) t_k(g) T_k(x).$$

Before proving the main results we need the following simple proposition.

LEMMA 2. *Let φ_n^* be defined as in* (4) *and* $p(t) = At^2 + Bt + C$. *Then*

$$\int_{-1}^{1} \varphi_n^*(t)p(t)w(t)dt = p\left(\cos\frac{\pi}{n+2}\right) + \frac{A}{n+2}\sin^2\frac{\pi}{n+2}. \tag{8}$$

Moreover

$$\int_{-1}^{1} \frac{\varphi_n^*(t)}{\sqrt{1+t}}dt < \frac{\pi\sqrt{2}}{2n}, \quad \int_{-1}^{1} \varphi_n^*(t)dt \le \frac{\pi}{n}. \tag{9}$$

Proof. We first observe that

$$\varphi_n^*\left(\cos\frac{(2k-1)\pi}{n+2}\right) = \begin{cases} (n+2)/2\pi, & k=1 \\ 0, & k=2,3,\dots,n, \end{cases}$$

$$\varphi_n^*(-1) = \frac{1+(-1)^n}{\pi(n+2)}\sin^2\frac{\pi}{2(n+2)}.$$

Now, let $q \in \Pi_{2n+2}$ be defined by

$$q(t) = \varphi_n^*(t)p(t) = c_{0n}t^{n+2} + Q(t), \quad Q \in \Pi_{n+1}.$$

It is easy to see that $c_{0n} = 2^{n+1}a_nA$, where a_n is defined as in (4). Using Lemma 1 we observe that

$$R_n(q) = \frac{A}{n+2}\sin^2\frac{\pi}{n+2},$$

and from (1) we have

$$\int_{-1}^{1} q(t)w(t)dt = \frac{2\pi}{n+2}q(x_{1n}) + R_n(q) = p(x_{1n}) + \frac{A}{n+2}\sin^2\frac{\pi}{n+2}.$$

If $p_1(t) = \sqrt{1-t}$, $p_2(t) = \sqrt{1-t^2}$, then according to (8) we find

$$\begin{aligned} \int_{-1}^{1} \varphi_n^*(t)w(t)dt = 1, \quad & \int_{-1}^{1} \varphi_n^*(t)|p_1(t)|^2w(t)dt = 2\cdot\sin^2\frac{\pi}{2(n+2)} \\ & \int_{-1}^{1} \varphi_n^*(t)|p_2(t)|^2w(t)dt = \frac{n+1}{n+2}\sin^2\frac{\pi}{n+2}. \end{aligned} \tag{10}$$

Let $\Phi_n : C(I) \to R$ be the linear positive functional defined by

$$\Phi_n(f) = \int_{-1}^{1} \varphi_n^*(t)f(t)w(t)dt, \quad f \in C(I).$$

Since $\Phi_n(e_0) = 1$, $e_0(t) = 1$, we have $|\Phi_n(f)|^2 \le |\Phi_n(f^2)$. Therefore, we obtain

$$\Phi_n(p_1) \le \sqrt{2\cdot\sin^2\frac{\pi}{2(n+2)}} < \frac{\pi\sqrt{2}}{2n}, \quad \Phi_n(p_2) \le \sqrt{\frac{n+1}{n+2}\sin^2\frac{\pi}{n+2}} < \frac{\pi}{n}.$$

If $t_{kn}^* = t_k(\varphi_n^*)$, i.e. $\varphi_n^*(x) = \sum_{k=0}^n t_{kn}^* \omega_k T_k(x)$, then from (8)

$$t_{0n}^* = 1, \quad t_{1n}^* = \cos\frac{\pi}{n+2}, \quad t_{2n}^* = \frac{2(n+1)}{n+2}\cos^2\frac{\pi}{n+2} - \frac{n}{n+2}.$$

Next we consider the kernel $L_n : I \times I \to R$, where

$$L_n(x,t) = \sum_{k=0}^n t_{kn}^* \omega_k T_k(x) T_k(t).$$

Taking into account (7), we obtain $L_n(x,t) = (\tau_x \varphi_n^*)(t)$, that is $L_n(x,t) \geq 0$ for $(x,t) \in I \times I$.

Using the kernel we define the linear positive operators $J_n : C(I) \to \Pi_n$, $n = 1, 2, \ldots$, by

$$(J_n f)(x = (\varphi_n^* * f)(x) = \int_{-1}^1 L_n(x,t) f(t) w(t) dt. \tag{11}$$

The main result of this section is the following:

THEOREM 1. *The sequences of operators (J_n) defined in (11) is of Jackson type. If $f \in C(I)$, then*

i) $|f(x) - (J_n f)(x)| \leq C \cdot \omega(f; \Delta_n(x)), \quad x \in I,$

where

$$\Delta_n(x) = \sqrt{1-x^2}/n + n^{-2}, \;\; C \in (0, 1 + \pi\sqrt{2} + \pi^2/2); \tag{12}$$

ii) $\|f - J_n f\| \leq C_1 \cdot \omega(f; 1/(n+2))$, $C_1 \in (0, 8)$.

Proof. If

$$z_1(t,x) = |x - tx - \sqrt{1-x^2}\sqrt{1-t^2}|, \;\; z_2(t,x) = |x - tx + \sqrt{1-x^2}\sqrt{1-t^2}|, \tag{13}$$

then it may be proved that for $(t,x) \in I \times I$

$$z_j(t,x) \leq \Delta_n(x) Q_n(t), \quad j = 1, 2,$$

where $Q_n(t) = 2n\sqrt{1-t} + n^2(1-t) = 2np_1(t) + n^2|p_1(t)|^2$.
From (9)–(10) we have

$$k_n = 1 + \int_{-1}^1 \varphi_n^*(t) Q_n(t) w(t) dt < 1 + \pi\sqrt{2} + \pi^2/2. \tag{14}$$

On the other hand, if $f \in C(I)$, $(t,x) \in I \times I$, we have

$$\begin{aligned} &|f(x) - (\tau_x f)(t)| \leq 1/2 \cdot |f(x) - f(xt + \sqrt{1-x^2}\sqrt{1-t^2})| \\ &+1/2 \cdot |f(x) - f(xt - \sqrt{1-x^2}\sqrt{1-t^2})| \leq 1/2 \cdot \omega(f; z_1(t,x)) + 1/2 \cdot \omega(f; z_2(t,x)) \\ &\leq \omega(f; \Delta_n(x) Q_n(t)). \end{aligned}$$

The well-known inequality $\omega(f;\lambda\delta) \leq (1+[\lambda])\omega(f;\delta)$ makes it possible to write

$$|f(x) - \tau_x f(t)| \leq (1+Q_n(t))\omega(f;\Delta_n(x)), \quad (t,x) \in I \times I, \tag{15}$$

$\Delta_n(x)$ being defined in (12). Using the commutativity of the convolution product, for $f \in C(I)$ and $x \in I$ we have

$$\begin{aligned} |f(x) - (J_n f)(x)| &= |f(x)(e_0 * \varphi_n^*)(x) - (f * \varphi_n^*)(x)|) \\ &\leq \int_{-1}^{1} |f(x) - (\tau_x f)(t)|\varphi_n^*(t)w(t)dt. \end{aligned}$$

In this manner, from (14)–(15) we obtain

$$|f(x) - (J_n f)(x)| \leq k_n \cdot \omega(f;\Delta_n(x)) \leq C \cdot \omega(f;\Delta_n(x))$$

where $C \leq C_0$, $C_0 = 1 + \pi\sqrt{2} + \pi^2/2$ and $x \in I$. From $\omega(f;\Delta_n(x)) \leq \omega(f;(1+1/n)/n) \leq 2\cdot\omega f;1/n)$ it follows that for every $x \in I : |f(x) - (J_n f)(x)| \leq 2C_0\omega(f;1/n)$. Therefore

$$\|f - J_n f\| = \max_{x\in I} |f(x) - (J_n f)(x)| \leq 2C_0\omega(f;1/n).$$

A sharper inequality may be obtained in the formula way: if $Q_x(t) = (x-t)^2$, then from (8)–(11)

$$(J_n Q_x)(x) = 4\left[x^2 + \frac{n+1}{n+2}(1-2x^2)\cos^2\frac{\pi}{2(n+2)}\right]\cdot\sin^2\frac{\pi}{2(n+2)},$$

i.e. $W_n = \max_{x\in I} |(J_n Q_x)(x)| = \frac{n+1}{n+2}\sin^2\frac{\pi}{n+2} < \frac{\pi}{(n+2)^2}$.
It is well-known that for a positive linear operator $J : C(I) \to C(I)$, $Je_0 = e_0$, the inequality

$$\|f - Jf\| \leq (1 + W/\delta^2)\omega(f;\delta), \quad \delta > 0, \quad f \in C(I), \quad W = \max_{x\in I} |(JQ_x)(x)|$$

is verified [5]. In our case, with $\frac{\delta=\pi}{(n+2)}$ we obtain

$$\frac{\|f - J_n f\| \leq 2\cdot\omega(f;\pi}{(n+2)) \leq 8\cdot\omega(f;1/(n+2)).}$$

Next we investigate the local degree of approximation by means of the polynomial operators $J_n^* : C(I) \to \Pi_n$, $n = 1,2,\ldots$, where

$$\begin{aligned} (J_n^* f)(x) = (J_n f)(x) &+ (1-x)/2\cdot[f(-1) - (J_n f)(-1)] \\ &+ (1+x)/2\cdot[f(1) - (J_n f)(1)], \quad x \in I, \end{aligned} \tag{16}$$

J_n being defined in (11).

Theorem 2. *If $J_n^* : C(I) \to \Pi_n$ is defined as in (16), then for $f \in C(I)$ there exists a positive constant C^* such that*

$$|f(x) - (J_n^* f)(x)| \leq C^*\omega(f;\sqrt{1-x^2}/n), \quad x \in I, \quad n = 1,2,\ldots,$$

Proof. Let us denote $\Delta_n^*(x) = \sqrt{1-x^2}/n$, $(\varepsilon_n f)(x) = f(x) - (J_n f)(x)$ and suppose that $x \in I_2 = (-\sqrt{1-n^{-2}}, \sqrt{1-n^{-2}})$, i.e., $n^{-2} < \Delta_n^*(x)$. According to Theorem 1, for $x \in I_2$ we have

$$\begin{aligned}|f(x) - (J_n^* f)(x)| &= |(\varepsilon_n f)(x) - (1-x)/2 \cdot (\varepsilon_n f)(-1) - (1+x)/2 \cdot (\varepsilon_n f)(1)| \\ &\le C \cdot \omega(f; \Delta_n(x)) + C \cdot \omega(f; n^{-2}) \le C_0 \omega(f; 2\Delta_n^*(x)) + C_0 \cdot \omega(f; \Delta_n^*(x)).\end{aligned}$$

More precisely

$$|f(x) - (J_n^* f)(x)| \le 3C_0 \omega(f; \Delta_n^*(x)) \quad x \in I_2. \tag{17}$$

Next we suppose that $\Delta_n^*(x) \le n^{-2}$, i.e., $x \in I_1 \cup I_3$ where

$$I_1 = [-1, -\sqrt{1-n^{-2}}], \quad I_3 = [\sqrt{1-n^{-2}}, 1].$$

If z_1, z_2 are defined as in (13), then for $(x,t) \in U = I_3 \times I$ we have

$$z_j(x,t) \le \Delta_n^*(x) S_n(t), \quad j = 1, 2 \tag{18}$$

where $S_n(t) = 1 + n\sqrt{1-t^2} = 1 + np_2(t)$. Indeed

$$z_j(x,t) \le p_2(x)p_2(t) + |t|(1-x) \le p_2(x)p_2(t) + (1-x^2) \le \Delta_n^*(x) S_n(t).$$

From (9)

$$\bar{a}_n = 1 + \int_{-1}^{1} \varphi_n^*(t) S_n(t) w(t) dt < 2 + \pi.$$

At the same time, for $(x,t) \in U$

$$|(\tau_x f)(t) - f(t)| \le 1/2 \cdot \omega(f; z_1(x,t)) + 1/2 \cdot \omega(f; z_2(x,t))$$

which together with (18) implies

$$|(\tau_x f)(t) - f(t)| \le (1 + S_n(t)) \omega(f; \Delta_n^*(x)).$$

Likewise, for $(x,t) \in U$

$$|(\tau_{-x} f)(t) - f(-t)| \le (1 + S_n(t)) \omega(f; \Delta_n^*(x)).$$

Therefore, in case $x \in I_3$,

$$\begin{aligned}|(J_n f)(x) - (J_n f)(1)| &= |(f * \varphi_n^*)(x) - (f * \varphi_n^*)(1)| \\ &\le \int_{-1}^{1} \varphi_n^*(t) |(\tau_x f)(t) - f(t)| w(t) dt \le \bar{a}_n \omega(f; \Delta_n^*(x))\end{aligned}$$

and $|(J_n f)(-x) - (J_n f)(-1)| \le \bar{a}_n \omega(f; \Delta_n^*(x))$. In other words there exists a $C_1 \in (0, 2+\pi)$ such that for $x \in I_3$:

$$\begin{aligned}|(J_n f)(x) - (J_n f)(1)| &\le C_1 \omega(f; \Delta_n^*(x)), \\ |(J_n f)(-x) - (J_n f)(-1)| &\le C_1 \omega(f; \Delta_n^*(x)).\end{aligned} \tag{19}$$

Let us suppose $x \in I_3$; from (19) and Theorem 1:

$$\begin{aligned}|f(x)-(J_n^*f)(x)| &= |[f(x)-f(1)]-[(J_nf)(x)-(J_nf)(1)] \\ &\quad + (1-x)/2\cdot[(\varepsilon_nf)(1)-(\varepsilon_nf)(-1)]| \\ &\le \omega(f;1-x)+C_1\omega(f;\Delta_n^*(x))+(1-x)C_0\omega(f;n^{-2}) \\ &\le \omega(f;1-x^2)+C_1\omega(f;\Delta_n^*(x))+(1-x^2)C_0\omega(f;n^{-2}) \\ &\le (1+C_1)\omega(f;\Delta_n^*(x))+C_0\Delta_n^*(x)\omega(f;n^{-2}).\end{aligned}$$

It is known that for $0 \le \delta_1 \le \delta_2$ one has $\delta_1\omega(f;\delta_2) \le 2\delta_2\omega(f;\delta_1)$. If we select $\delta_1 = \Delta_n^*(x)$, $\delta_2 = n^{-2}$, $x \in I_3$, then

$$\Delta_n^*(x)\omega(f;n^{-2}) \le 2n^{-2}\omega(f;\Delta_n^*(x)).$$

In conclusion, for $x \in I_3$:

$$|f(x)-(J_n^*f)(x)| \le (1+C_1+2n^{-2}C_0)\omega(f;\Delta_n^*(x))$$

that is

$$|f(x)-(j_n^*f)(x)| \le C^*\omega(f;\Delta_n^*(x)), \quad n=1,2,\dots, \tag{20}$$

with $0 < C^* < 5+(1+2\sqrt{2})\pi+\pi^2$. Using the second inequality from (19) it may be shown that (20) is verified for $x \in I_1$ too. Taking into account (17) we conclude that (20) is true for all x, $x \in I$.

THEOREM 3. *Let J_n be defined as in (11) and x fixed in I. Then to each function $f \in C(I)$ corresponds a system Θ_{1n}, Θ_{2n}, Θ_{3n} of distinct points from I such that*

$$(J_nf)(x) = f(c\cdot\cos\pi/(n+2)) + V_n(x)[\Theta_{1n},\Theta_{2n},\Theta_{3n};f] \tag{21}$$

where $V_n(x) = \frac{n(1-x^2)+1}{n+2}\sin^2\frac{\pi}{n+2}$,

Proof. In [11]–[12] it is proved that if (L_n) is a sequence of positive linear operators defined on $C(K)$ and $L_ne_0 = e_0$, $L_ne_k = a_{kn}$, $e_k(t) = t^k$, then for $f \in C(K)$ and $x \in K$:

$$(L_nf)(x) = f[a_{1n}(x)] + [a_{2n}(x)-a_{1n}^2(x)][\Theta_{1n},\Theta_{2n},\Theta_{3n};f] \tag{22}$$

where $\Theta_{in} = \Theta_{in}(f,x)$, $i=1,2,3$, are distinct points from K. In our case, taking into account that $J_nT_k = t_{kn}^*T_k$, $k=0,1,2$, one finds

$$a_{1n}(x) = x\cdot t_{1n}^* = x\cdot\cos\pi/(n+2),$$
$$a_{2n}(x) = e_2(x)-\frac{1}{2}(1-t_{2n})T_2(x) = x^2+(1-2x^2)\frac{n+1}{n+2}\sin^2\frac{\pi}{n+2},$$

and (22) proves the theorem.

In the case when $f \in C^{(2)}(I)$ the equality (21) makes it possible to show that the remainder-term may be written as

$$f(x)-(J_nf)(x) = Z(n,f,x)\sin^2\pi/2(n+2)$$

where for x fixed in I

$$Z(n,f,x) = 2\left[xf'(\xi_{1n}) + \frac{n(1-x^2)+1}{n+2}f''(\xi_{2n})\cos^2\frac{\pi}{2(n+2)}\right],$$

$\xi_{in} = \xi_{in}(f,x)$ being points from $I = [-1,1]$.

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