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ON THE CUT LOCUS AND THE FOCAL LOCUS OF A SUBMANIFOLD IN A RIEMANNIAN MANIFOLD II

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Abstract. Let M be a compact connected Riemannian manifold and let L be a compact connected submanifold of M. We show that if a point x is a closest cut point of L which is not a focal point of L, then two different minimizing geodesics meet at an angle of π at x. We also generalize some of the results of [9].

1. Introduction

Let M be a compact connected n-dimensional Riemannian manifold of class C^∞ and let L be a C^∞ m-dimensional connected submanifold of M. Let $N(L)$ be the normal bundle of L which is a subbundle of tangent bundle $T(M)$ of M. The exponential map of the Riemannian manifold M restricted to $N(L)$ is a map $\varepsilon : N(L) \to M$ of class C^∞. Let $d : M \times M \to R$ be the distance function of the Riemannian manifold M; then for any point $x \in M$ there is at least one point $x' \in L$ such that $d(x, x') = \inf\{d(x, z) | \, z \in L\}$ holds and x' is said to be a point nearest to x in L. Let $x \in L$ and consider a geodesic $c : R \to M$ of the Riemennian manifold such that $c(0) = x'$, $c(t) = x$ for some $t > 0$ and such that the restriction of c to $[0, t]$ yields a minimal geodesic from x' to x. Then the tangent vector $\dot{c}(0)$ of c is in the normal space $N_{x'}L$ of L at x' by a basic observation [1, pp. 151-152]. Since M is complete such a geodesic c always exists and consequently the map ε is surjective.

When considering the injectivity of the map e some further concepts are essential which can be summarized as follows. If the tangent linear map $T_\nu\varepsilon$: $T_\nu N(L) \to T_{\varepsilon(\nu)}M$ of ε at $\nu \in N(L)$ is not injective then ν is called a focal point of L in the normal bundle $N(L)$ and $\varepsilon(\nu)$ is said to be a focal point of L in M. The set of focal points ν of L is said to be a focal locus of L in the normal bundle $N(L)$ and the set of focal points $\varepsilon(\nu)$ of L is called the focal locus of L in M. In the special case when the submanifold L reduces to a single point $y \in M$ and consequently the normal bundle $N(L)$ coincides with the tangent space T_yM, the focal points of L

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are said to be points conjugate to y and the focal locus of L is called the conjugate locus of y.

Consider now the general case of a submanifold L of M. Fix a point $\in L$ and a unit vector $w \in N_z = L$ and consider the geodesic $c : R \to M$ such that $c(0) = z \in L$, $\dot{c}(0) = w$. Since the tangent linear map $T_x\varepsilon$ is injective at $x = c(t_0)$ for $0 < t_0 \leq t$, where t has sufficiently small positive value. Let S_w be the supremum of such values t, which is always possible, since M is complete. If S_w is finite then $c(S_w)$ is obviously a focal point of L which will be called the first focal point of L on the geodesic c.

Assume now that the submanifold L is compact; then the restricted exponential map ε is injective in a sufficiently small neighborhood of the zero section in the normal bundle $N(L)$ of the submanifold [1, pp. 151-152] and consequently z is the unique nearest point of L to $x = c(t_0)$ for $0 < t_0 \leq t$ where t is sufficiently small positive value. Let S'_w be the supremum of such value t. If S'_w, is finite then S'_w is called a cut point of L in the normal bundle and $c(S'_w)$ is said to be a cut point of L in M. The set of cut points of L in $N(L)$ is called the cut locus of L in $N(L)$ and the set of cutpoints of L in M is said to be the cut locus of L in M. A straightforward generalization of some basic facts established in the special case when L reduces to a single point [1 pp. 237-241] yields the following lemma.

LEMMA 1. *If $\nu = S'_w w \in N(L)$ is a cut point of the submanifold L then at least one of the following two assertions is true:*
1. *the point $\nu = S'_w$ is a first focal point of the submanifold L on the ray $tw, t > 0$,*
2. *there are at least two different points of the subyrtanifold L which are nearest to the cut point $\varepsilon(S'_w)$.*

2. Closest point of the cut lucus

First we shall prove the following lemma.

LEMMA 2. *Let M be a C^∞ compact connected Riemannian manifold and let L be a C^∞ compact connected submanifold of M. Let $c : [0, a] \to M$ be a minimal geodesic from $c(a)$ to L. If c' is the part of c then c' minimizes the distance uniquely from its end point $c'(b)$ to the points of L for any value of the parameter $b < a$.*

Proof. Let c' does not minimize its distance uniquely for $b < a$, then there exists another minimal geodesic c'' from a point $z_1 \in L$ to the point $c'(b) = x'$. But then c is the union of c' from $c(0) = z \in L$ to x' and minimal geodesic c^* from x' to $c(a) = x$. Since angle between c'' and c^* is not equal to π, therefore $c'' \cup c^*$ can not be minimizing geodesic. But $L'(c'' \cup c^*) = L'(c' \cup c^*) = L'(c)$ where L' denotes the length. This means that c can not be a minimal geodesic, which is a contradiction. Hence the lemma.

Now we prove the following theorem.

THEOREM 1. *Let M be a C^∞ complete connected Riemannian manifold and let L be a C^∞ compact connected submanifold of M. Let the cut locus of L be non-empty and let $x = \varepsilon(\nu)$ be a closest point of the cut locus to L. Let c_1 and c_2 be* *two different minimizing geodesics from x to L. If x is not a focal point of L, then the geodesics c_1 and c_2 meet at an angle of π.*

Proof. Let $\nu \in N_{z_1}L$ be a non-zero vector where $z_1 \in U$ and U is a neighborhood of z in the zero-section of $N(L)$. Then the locus of the end points of such ν with fixed length will be a sphere of dimension $n-m-1$. Consider with ν the family of vectors of the same length as ν in $N(L)$; then corresponding to these vectors there is a union of the spheres which forms a piece of a hypersurface, say K, and hence a tangent space $T_n uK$ at ν orthogonal to ν with respect to the induced metric $\overline{g}$ of $N(L)$ [5], as proved in [8]. Now we define geodesic $c_1 : [0,1] \to M$ such that $c_1(0) = z_1 \in L$, $\dot{c}(0) \in N_{z_1}L$, $c_1(1) = r = \varepsilon(\nu)$. Consider for c_1 a family of neighboring geodesics each orthogonal to L, then under the restricted exponential map ε each member of this family is the image of non-zero vectors taken in $N(L)$ corresponding to ν and hence they are of the same length by the Generalized Gauss lemma [8]. As $x = \varepsilon(\nu)$ is not a focal point of L in M the image $\varepsilon(K)$ will be a piece of hypersurface containing x in M. Since $T_\nu K$ is orthogonal to v, the hypersurface $\varepsilon(K)$ will be orthogonal to c_1 by the Generalized Gauss lemma [8]. Similar result holds for the geodesic c_2 passing orthogonally through the point $z_2 \in L$ to x. Assume that c_1 and c_2 meet at x with an angle not equal to π. Then the two tangent hyperplanes at x intersect, as do the two hypersurfaces in each neighborhood of x. Let x' be a point in $\varepsilon(K) \cup \varepsilon(K')$ near x, where $\varepsilon(K')$ is corresponding to geodesic c_2. Then x' is joined by two orthogonal geodesics, one neighboring to c_1 and the other neighboring to c_2 and each being shorter that c_1 and c_2. Thus x' is a cut point of L closer to L than the point x, which contradicts the choice of x. Therefore c_1 and c_2 meet at x with angle π.

3. Focal points under some restrictions

In this section we will generalize some of the results of [9].

THEOREM 2. *Let M be a complete connected Riemannian manifold of class C^∞ and let L be a C^∞ compact connected submanifold of M such that the restricted exponential map has no focal points in $U(b\pi) - U(a\pi)$, where $0 \leq a < b$ and $U(b\pi)$ is the tube of radius $b\pi$ around the zero section in $N(L)$. Let $x \in M$ and assume that c_0 and c_1 are different geodesic segments joining x orthogonally to L and that there is a family h_t, $t \in [0,1]$ of curves joining x orthogonally to L such that $h_0 = c_0$, $h_1 = c_1$ and $L'(h_1) \leq L'(c_1)$ for all $t \in [0,1]$, then $L'(c_0) + L'(c_1) \geq 2b\pi$ or $L'(c_1) + 2a\pi - b(x,L) > 2b\pi$, where $L'(c)$ denotes the length of a path c in M.*

Proof. We assume that $L'(c_0) < b\pi$. Since c_1 has neighboring curves h_t with $L'(h_t) \leq L(c_1)$, c_1 must have index ≥ 1 and length $L'(c_1) > b\pi$. Let $U(b\pi) - U(a\pi) = U'$. Since U' does not contain focal points, the tangent linear map $T_\nu\varepsilon$ is everywhere non-singular in U'. Then the restricted exponential map ε is a covering map [4]. But every covering map has the curve lifting property [2, pp. 25]. Hence for each path h the initial geodesic part of length $a\pi$ can be lifted by the preimage ε^{-1} of ε restricted to U' into U', and this gives a straight segment going from zero-section of $N(L)$ to the inner boundary of U'. In this manner $c_1 = h_1$ can be lifted into a straight segment h_1 of length $> b\pi$ starting from the zero-section of $N(L)$ and leaving U' at its outer boundary. It follows that for all t sufficiently close to 1, the initial part of h_1 can be lifted into $U(b\pi)$ so as to give a straight segment of length $a\pi$, starting from the zero-section and followed by a curve passing from the inner boundary of U' towards the outer boundary of U' and containing in the limit, a point of this outer boundary at distance $b\pi$ from the zero-section. Now we claim that for each sufficiently small r, there exists a t_1, $0 < t_1 < 1$ such that lifting of h_{t_1} after the initial straight segment of length $a\pi$, a curve which runs through U' until it reaches a point with distance $\leq r$ from the outer boundary of U' and then continues to run trough U' until one of the following two possibilities occurs:

(1) we reach with the lifted curve, the inner boundary of U';

(2) we reach with lifted curve, the end point x' of h_{t_1} which gives a point x' in U'.

The implication of the case (1) is

$$L'(c_1) \geq L'(h_{t_1}) \geq 2b\pi - a\pi - 2\varepsilon \geq 2b\pi - L'(c_0) - 2\varepsilon.$$

This gives the result, since *varepsilon* is arbitrary.

In the case (2) the image under ε of the straight segment from the zero-section to x' gives a geodesic c_0' which is different from c_0. Moreover, the lifting of h_{t_1}, into $U(b\pi)$ shows that h_{t_1} can be deformed into c_0' with curves of length $\leq L'(h_{t_1}) \leq L'(c_1)$. Therefore by combining the homotopy (h_t), $t \in [0, t_1]$ with this homotopy from h_{t_1}; inco c_0' we obtain a homotopy (j_t), $t \in [0, 1]$ from $c_0 = j_0$ to $c_0' = j_1$ with $L'(j_1) \leq L'(c_1)$ for all $t \in [0, 1]$. By applying (if necessary) a deformation, we can assume that a curve j_{t_0} of maximal length among the j_t is a geodesic of index 1 and $L'(j_{t_0}) > L'(c_0')$. Now applying to the pair c_0, j_{t_0} with the homotopy (j_t), $t \in (0, t_0]$, the same reasoning as for the original pair c_0, c_1. Since $L'(j_{t_0}) \leq L'(c_1)$ and since there are only finitely many of geodesics of length $\leq L'(c_1)$, we finally get the result.

THEOREM 3. *Let M be a complete simply connected Rimannian: manifold and L be a compact connected submanifold of M. Let $c : R \to M$ be a normal geodesic to L and $c(t_0) = x \in M$ be a point which is not a focal point of L. Let a, b, c, be real numbers satisfying*

(i) $$1 < a < b < c \text{ and } 2(b-a)\pi + d(x, L) \geq c\pi.$$

Assume that on geodesic c starting orthogonally from L, there are no focal points in $[0, \pi]$, p focal points in $(\pi, a\pi]$, $p \geq 1$, no focal points in $[a\pi, b\pi)$ and q focal points in $[b\pi, c\pi)$, ≥ 2. Then the length $L'(c)$ of geodesic c satisfies

(ii) $$L'(c) < 2a\pi - d(x, L) \text{ or } L'(c) > 2(b-a)\pi + d(x, L).$$

Proof. Let c_1 be any geodesic. Let (h_t), $t \in [0, 1]$ be a homotopy from $c_0 = h_0$ to $c_1 = h_1$. If $L'(h_t) \leq L'(c_1)$ for all $t \in [0, 1]$ then we can apply theorem 2 and obtain (ii). Otherwise we assume that there is a t_1, $0 < t_1 < 1$, h_{t_1} is a geodesic of index 1 and $L'(h_{t_1}) > L'(h_t)$ for all $t \in [0,1]$. If h_{t_1} is of broken type we have $L'(c_1) = L'(h_{t_1}) \leq 2a\pi - d(x,L)$ which is (ii). If h_{t_1} is of the unbroken type we can apply Theorem 2 and (i) and obtain $L'(h_{t_1}) > 2(b-a)\pi + d(x,L) \geq c\pi$. Then from our assumptions that h_{t_1} has index $g \geq 2$, we find a contradiction. Hence the theorem.

THEOREM 4. *Let M be a complete simply connected Riemannian manifold and let $L \subset M$ be a compact connected submanifold of M. Let also a, b, c be real numbers satisfying*

$$1 < a < b < c \quad \textit{and} \quad a \leq 2 \quad \textit{and} \quad 2(b-a)+1 \geq c. \tag{1}$$

Let x be a point in L such that on each orthogonal geodesic to L at x there are no focal points in $[0,\pi)$, there are p focal points in $[\pi, a\pi)$, $p \geq 1$, there are no focal points in $[a\pi, b\pi)$ and there are q focal points in $[b\pi, c\pi)$ $q \geq 2$. Then the following holds: (A) *M is compact,* (B) *Let $z \in M$ be a point on the geodesic c which is not a focal point to L and distance $d(z,L)$ is sufficiently close to π. Then a geodesic c either has length $L'(c) < 2a\pi - d(z,L) \sim 2a\pi - \pi$ and index $\leq p$, or has length*

$$L'(c) > 2(b-a)\pi + d(z,L) \sim 2(b-a)\pi + \pi \geq c\pi$$

and index $\geq p+q$.

Proof. To prove (A), it is sufficient to note that a geodesic segment of length $a\pi$ starting from L and being orthogonal to L contains focal points in its interior and, therefore it is not a curve of minimal length from its end point to L. Consequently a tube of radius $a\pi$ about L covers M. Since L is assumed to be compact, this implies that M is compact.

To prove (B), we first remark that there is an $l > 0$ such that the focal points in the interval $[\pi, a\pi)$ of an orthogonal geodesic c starting from $x \in L$, already occur in the interval $[\pi, (a-l)\pi)$. Assume that $z \in M$ is chosen such that $21\pi + d(z,L) \geq \pi$ and z is not focal point of L. Then

$$2(b-(a-l))\pi + d(z,L) \geq 2(b-a)\pi + \pi \geq c\pi.$$

Thus the assumptions of theorem 3 are satisfied with $(a-l)$ instead of a. From Theorem 3, (B) then follows with $2a\pi - \pi < 2b\pi - c\pi < b\pi$ such that

$$L'(c) < 2(a-l)\pi - d(z,L) \leq 2a\pi - \pi < b\pi$$

and hence index $c > p$, or

$$L'(c) > 2(b-(a-l))\pi + d(zL) \geq 2(b-a)\pi + \pi \geq c\pi$$

and hence index $c > p+q$.

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