PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 141–144

A CHARACTERIZATION OF FORMALLY SYMMETRIC UNBOUNDED OPERATORS

Danko Jocić

Abstract. We give necessary and sufficient conditions for an operator in a Hilbert space to be formally symmetric, symmetric or self-adjoint. This generalizes the well-known fact that a bounded operator T is self-adjoint if and only if $T^*T \leq (\operatorname{Re} T)^2$. The proof is based on a well-behaved extension of the corresponding symmetric operator.

0. Introduction

Fong and Istratescu [**1**] and also Kittaneh [**2**] have proved the following:

THEOREM A. *A bounded operator T is self-adjoint if and only if $T^*T \leq (\operatorname{Re} T)^2$.*

They used Theorem A to investigate some classes of bounded operators — θ, WN and hyponormal operators. A large number of well-known and important operators, for example $x+i\,d/dx$, belongs to similar classes of unbounded operators. The aim of this note is to extend Theorem A to unbounded operators and to make it suitable for dealing with such situations. Our main result is Theorem 1 in which we present characterizations for an operator to be formaly symmetric, symmetric or self-adjoint (Theorems 2, 3).

1. Preliminaries

Suppose that $(H, \langle \cdot | \cdot \rangle)$ is a separable, complex, infinite dimensional Hilbert space and let $(H \oplus H, \langle \cdot | \cdot \rangle)$ denote the usual product space. Thoughout this paper we assume that all operators are linear. Let $D(A)$ denote the domain of an operator A. The operators $(A+A^*)/2$ and $(A+A^*)/2i$ (with $\Delta(A) = D(A) \cap D(A^*)$) as their domains) will be denoted by $\operatorname{Re} A$ and $\operatorname{Im} A$ respectively. If A is a restriction of B on $D(A)$, we will write $A \subset B$. Whenever $\Delta(A)$ is dense in H, we will denote the domains of $(\operatorname{Re} A)^*$ and $(\operatorname{Im} A)^*$ by $D(\operatorname{Re} A)^*$ and $D(\operatorname{Im} A)^*$ respectively. We recall

---

AMS Subject Classification (1980): Primary 47 B 25, Secondary 47 B 15

that a densely defined operator A is said to be symmetric iff $\langle Ax|y\rangle = \langle x|Ay\rangle$ for all $x, y \in D(A)$, i.e. if $A \subset A^*$. It is said to be formally symmetric iff $A^*x = Ax$ for all $x \in \Delta(A)$ i.e. iff $\operatorname{Im} A \subset 0$. Note that $\operatorname{Re} A$ and $\operatorname{Im} A$ are symmetric whenever $\Delta(A)$ is dense in H.

2. The construction

Lemma 1. *For a closed, symmetric operator A in H we define the operator $\tilde{A}$ by $\tilde{A}(x,y) = (\tilde{A}x, \tilde{A}y)$. If the domain of $\tilde{A}$ is given by $D(\tilde{A}) = \{(x,y) \in D(A^*) \times D(A^*) \colon x - y \in D(A)\}$ then $\tilde{A}$ is one self-adjoint extension of $A \oplus (-A)$.*

Proof. For all (x,y) and (f,g) in $D(\tilde{A})$ we have that

$$\langle \tilde{A}(x,y)|(f,g)\rangle = \langle A^*x|f\rangle - \langle A^*y|g\rangle = \langle A^*(x-y)|f\rangle + \langle A^*y|(f-g)\rangle.$$

Since $x - y$ and $f - g$ are in $D(A)$, it follows that

$$\begin{aligned}\langle A^*(x-y)|f\rangle + \langle A^*y|(f-g)\rangle &= \langle A(x-y)|f\rangle + \langle y|A(f-g)\rangle \\ &= \langle (x,y)|\tilde{A}(f,g)\rangle\ .\end{aligned}$$

So $\tilde{A}$ is symmetric.

Suppose that $\lim_{n\to\infty}(x_n, y_n) = (x,y)$ and $\lim_{n\to\infty}(A^*x_n, -A^*y_n) = (u,v)$ for some $(x_n, y_n) \in D(\tilde{A})$ and some $x, y, u, v \in H$. This implies that $\lim_{n\to\infty}(x_n - y_n) = x - y$ and $\lim_{n\to\infty} A(x_n - y_n) = \lim_{n\to\infty} A^*(x_n - y_n) = u + v$. Since A^* and A are closed and $x_n - y_n \in D(A)$, it follows that $x - y \in D(A)$ and $x, y \in D(A^*)$. Moreover, $A^*x = u$ and $A^*y = -v$. Therefore $(x,y) \in D(\tilde{A})$ and also $\tilde{A}(x,y) = (A^*x - A^*y) = (u,v)$ is closed.

Finally, suppose that $(x,y) \in R(\tilde{A} + iI)^\perp$. Then it follows that $\langle x|(A^* + iI)f\rangle = \langle y|(A^* - iI)g\rangle$ for all $(f,g) \in D(\tilde{A})$ and, in particular $\langle x|(A^* + iI)f\rangle = 0$ for all $f \in D(A)$. Therefore $x \in (A^{**}) = D(A)$ and, moreover, $x \in \operatorname{Ker}(A - iI)$. It now follows that $2\|x\|^2 = \langle (A + iI)x|x\rangle = \langle x|(A - iI)x\rangle = 0$, and hence $x = 0$. Analogously, we can prove that $y = 0$ and thus $R(\tilde{A} + iI)^\perp = \{0\}$. The equality $R(A^* - iI)^\perp = \{0\}$ follows similarly, and hence $\tilde{A}$ is self-adjoint.

Remark 1. An alternative proof of Lemma 1 can be obtained by using von Neumann's formulae for self-adjoint extensions of $A \oplus (-A)$. The corresponding partial isometry V is given by

$$\begin{aligned} &V(x,y) = -(y,x), && \text{for all}(x,y) \in \operatorname{Cl}\left(R\big(A \oplus (-A) + iI\big)\right), \\ &V(x,y) = 0, && \text{for all}(x,y) \in \operatorname{Ker}\big(A^* \oplus (-A^*) - iI\big). \end{aligned}$$

Lemma 2. *Let A and B be closed symmetric operators and assume that $D(A) \subset D(B)$ and $D(A^*) \subset D(B^*)$. Then there exist selfadjoint extensions $\tilde{A}$ and $\tilde{B}$ of $A \oplus (-A)$ and $B \oplus (-B)$ respectively, such that $D(\tilde{A}) \subset D(\tilde{B})$.*

Proof. It is sufficient to take the extension constructed in Lemma 1. Then the required inclusion can be shown by a straightforward computation.

3. Main results

THEOREM 1. *Let A and B be symmetric operators and assume that $D(A) \subset D(B)$, $D(A^*) \subset D(B^*)$ and also*

$$\|(A^* - iB^*)x\| \le \|A^*x\| \tag{a}$$

for all $x \in D(A^*)$. Then $B \subset 0$.

Proof. Without loss of generality we may assume that A and B are closed. To see this, note that (a) implies $\|Bx\| \le 2\|Ax\|$ for all $x \in D(A)$ and hence $D(A^-) \subset D(B^-)$. Because of $A^{-*} = A^*$ and $B^{-*} = B^*$ it follows that $D(A^{-*}) \subset D(B^{-*})$ and $\|(A^{-*} - iB^{-*})x\| \le \|A^{-*}x\|$ for all $x \in D(A^{-*})$. So, according to Lemma 2, let $\breve{A}$ and $\breve{B}$ be the corresponding self-adjoint extensions of $A \oplus (-A)$ and $B \oplus (-B)$, respectively. A simple calculation gives

$$\|(\breve{A} - i\breve{B})(x,y)\|_\sim \le \|\breve{A}(x,y)\|_\sim \tag{a$'$}$$

for all $(x,y) \in D(\breve{A})$. Let E be the spectral measure induced by $\breve{A}$ and let $\gamma \subset \delta \subset \mathbf{R}$, for some measurable bounded set γ and δ. We define $A(\delta) = E(\delta)\breve{A}E(\delta)$ and $B(\delta) = E(\delta)\breve{B}E(\delta)$. Since $E(\delta)h \in D(\breve{A})$, it follows by Lemma 2 that $E(\delta)h \in D(\breve{B})$, for an arbitrary $h \in H \oplus H$. Hence $D\big(B(\delta)\big) = H \oplus H$. Obviously $B(\delta)$ is symmetric and therefore self-adjoint. Then there exists a sequence $\{h_n\}_{n\in\mathbf{N}}$ of unit vectors in $H \oplus H$ such that $\lim_{n\to\infty}\big(B(\delta) - \lambda\big)h_n = 0$ for some $\lambda \in \mathbf{R}$ satisfying $|\lambda| = \|B(\delta)\|$. It follows from (a$'$) that

$$\|B(\delta)h_n\| \ \le -2\,\mathrm{Re}\, i\langle A(\delta)h_n|\big(B(\delta) - \lambda\big)h_n\rangle\ . \tag{a”}$$

Letting $n \to \infty$ we get $\|B(\delta)\|^2 \le 0$, and consequently $E(\delta)\breve{B}E(\delta) = 0$. Since $\gamma \subset \delta$ we conclude that $E(\delta)\breve{B}E(\gamma) = 0$. If $\bigcup\{\gamma_n : n \in \mathbf{N}\} = \bigcup\{\delta_n : n \in \mathbf{N}\} = \mathbf{R}$ for some increasing sequences $\{\gamma_n\}_{n\in\mathbf{N}}$ and $\{\delta_n\}_{n\in\mathbf{N}}$, it follows that $\breve{B}E(\gamma) = s\text{-}\lim_{n\to\infty} E(\delta_n)\breve{B}E(\gamma) = 0$ because $s\text{-}\lim E(\delta_n) = I$. Moreover, $s\text{-}\lim_{n\to\infty} E(\gamma_n) = I$ implies $\breve{B} = s\text{-}\lim_{n\to\infty} \breve{B}E(\gamma_n) = 0$, since $\breve{B}$ is closed. Consequently, $B \subset 0$ as required.

Remark 2. If, in addition, A is (essentially) self-adjoint, then the assumption $D(A^*) \subset D(B^*)$ can be omitted and the proof of Theorem 1 simplified. Also, the use of lemmas becomes unnecessary.

As a consequence of Theorem 1, we give the following characterization.

THEOREM 2. *If $\Delta(T)$ is dense in H, then T is formally symmetric if and only if:* (1) $D(\mathrm{Re}\,T)^* \subset D(\mathrm{Im}\,T)^*$, (2) $\|(\mathrm{Re}\,T)^*x - i(\mathrm{Im}\,T)^*x\| \le \|(\mathrm{Re}\,T)^*x\|$ *for all $x \in D(\mathrm{Re}\,T)^*$.*

Proof. If (1) and (2) are true, then $\mathrm{Im}\,T \subset 0$ by Theorem 1, and hence T is formally self-adjoint. The necessity of (1) is obvious.

LEMMA 3. *If $D(T) \subset D(T)^*$ for an operator T, then the following are equivalent:*

(1) $D(\mathrm{Re}\,T)^* \subset D(\mathrm{Im}\,T)^*$; (1$'$) $D(\mathrm{Re}\,T)^* \subset D(T^*)$.

If the assumption (1) *is satisfied, then* $T^*x = (\operatorname{Re} T)^*x - i(\operatorname{Im} T)^*x$ *for every* $x \in D(\operatorname{Re} T)^*$.

Proof. Since $D(\operatorname{Re} T) = D(\operatorname{Im} T) = D(T)$ it follows that $D(\operatorname{Re} T)^* \cap D(\operatorname{Im} T)^* \subset D(T^*)$ and $D(\operatorname{Re} T)^* \cap D(T^*) \subset D(\operatorname{Im} T)^*$ and therefore the equivalence of (1) and (1′) is obvious. Because of $T = \operatorname{Re} T + i \operatorname{Im} T$ it follows that $T^* \supset (\operatorname{Re} T)^* - i(\operatorname{Im} T)^*$ from which we derive the rest of the statement.

THEOREM 3. *An operator* T *is symmetric* (*resp. self-adjoint*) *iff*

$$(0') \quad D(T) \subset D(T^*), \quad (\textit{resp. } D(T) = D(T^*))$$

$$(1') \quad D(\operatorname{Re} T)^* \subset D(T^*);$$

$$(2') \quad \|T^*x\| \leq \|(\operatorname{Re} T)^*x\|$$

for all $x \in D(\operatorname{Re} T)^*$.

Proof. If (0′), (1′) and (2′) are true, then T is formally symmetric by Lemma 3 and Theorem 2. Because of (0′), T is symmetric (resp. self-adjoint). The necessity of (0′), (1′) and (2′) is obvious.

REFERENCES

[**1**] C. K. Fong, V. I. Istratescu, *Some characterizations of Hermitian operators and related classes of operators. I*, Proc. Amer. Math. Soc. **76** (1979), 107–112.

[**2**] F. Kittaneh, *Some characterizations of self-adjoint opeators*, Acta Sci. Math. **47** (1984), 441–445.

[**3**] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1980.

Institut za primenjenu matematiku i elektroniku
Kneza Miloša 37
11000 Beograd
Yugoslavia

(Received 23 12 1988)