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ON RIEMANNIAN 4-SYMMETRIC MANIFOLDS

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Abstract. If M is a Riemannian 4-symmetric manifold, then it is known that M has three complex differentiable distributions D_{-1}, D_1 and $\overline{D}_1$ on it. We shall prove that there are three differentiable complementry projection operators P, P_1 and $\overline{P}_1$ on M that project on D_{-1}, D_1 and $\overline{D}_1$ respectively. Some useful relations containing Nijenhuis tensor are found. Necessary and sufficient conditions for D_{-1}, D_1, and $\overline{D}_1$ to be integrable are studied.

1. Introduction

An isometry s_p on a C^∞ Riemannian manifold (M, g) for which $p \in M$ is the only isolated fixed point is called a symmetry at p. (M, g) is called a Riemannian s-manifold if M is connected, and to each point $p \in M$ a symmetry s_p can be assigned.

If $S_p^k = \mathrm{id}_M$, where $k \geq 2$ is the least positive integer with this property, then M is called a Riemannian k-symmetric manifold. See Graham and Ledger [**2**] and Kowalsky [**3**], [**4**].

In a Riemannian k-symmetric manifold, we have

$$S^k = I \tag{1.1,}$$

where S is the C^∞ tensor field of type (1), on M, such that $S_p = (ds_p)_p$, and I is the identity tensor field. S is real, orthogonal and nonsingular. The eigenvalues of S_p, $p \in M$ are k^{th} roots of unity. Since S is continuous on M, each root is constant over M. Note that 1 is not an eigenvalue because s_p does not fix points except p, therefore, the possible eigenvalues are -1, and pairs of complex conjugates $w_1, \overline{w}_1, \dots, w_r, \overline{w}_r$. Due to the orthogonality of S, we shall have a collection of mutually orthogonal differentiable distributions $M_{-1}, M_1, \dots, M_r$ on M such that

$$M_x = M_{-1x} \oplus M_{1x} \oplus \cdots \oplus M_{rx} \qquad \text{(direct sum)} \tag{1.2}$$

and S decomposes into

$$S = S_{-1} \oplus S_1 \oplus \cdots \oplus S_r \qquad \text{(direct sum)}$$

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where $S_j M_j \subset M_j$.

At each point $p \in M$, let us denote by $H_{-1}, H_1, \overline{H}_1, \dots, H_r$, and $\overline{H}_r$ respectively (-1)-eigenspace, w_1-eigenspace, $\overline{w}_1$-eigenspace, ... w_r-eigenspace, and $\overline{w}_r$-eigenspace on the complexification M_p^c of the tangent space M_p. Let $D_{-1}, D_1, \overline{D}_1, \dots, D_r, \overline{D}_r$ be complex C^∞ distributions on M which assign $H_{-1}, H_1, \overline{H}_1, \dots, H_r, \overline{H}_r$ to p.

If w_1 and $\overline{w}_1$, are the only eigenvalues of S on a Riemannian k-symmetric manifold M then M is a Riemannian 3-symmetric manifold with $w_1^2 = \overline{w}_1$, or the underlying manifold M is a symmetric space (see Ledger and Obata [**5**]).

A distribution B on a manifold M is said to be involutive if $[X, Y] = 0$, whenever $X, Y \in B$. The distribution B is said to be integrable if each point of M lies on the domain of a flat chart. It is well known that a distribution is integrable if and only if it is involutive, Brickell and Clark [**1**].

Nijenhuis tensor of a C^∞ tensor field A of type $(1,1)$ is defined by

$$N(X,Y) = [AX, AY] + A^2[X,Y] - A[AX,Y] - A[X,AY] \tag{1.4}$$

Nijenhuis [**6**].

2. Complementary projection operators on a Riemannian 4-symmetric manifold

Let (M, g) be a C^∞ connected Riemannian manifold and suppose that for each point $p \in M$, we have an isometry s_p on$M$ such that $s_p(p) = p$, but p is not the only isolated fixed point of s_p, i.e. s_p fixes points beside x. Then we shall call s_p a p-isometry, and M is called a ps-manifold. If $s_p^k = \mathrm{id}_M$ and $k \geq 2$, is the least positive integer with this property, then s_p is called a p-isometry of order k, and M is called a ps-manifold of order k. The tensor field S of type $(1,1)$ on M such that

$$S_p = (ds_p)_p \tag{2.1}$$

will have the property

$$S^k = I. \tag{2.2}$$

The eigenvalues of S are $\pm 1, w_1, \overline{w}_1, \dots, w_r, \overline{w}_r$, consequently (1.2) and (1.3) are replaced by

$$M_x = M_{1x}^* \oplus M_{-1x} \oplus M_{1x} \oplus \dots \oplus M_r \qquad \text{(direct sum)} \tag{2.3}$$

and

$$S = S_1^* \oplus S_{-1} \oplus S_1 \oplus \dots \oplus S_r. \tag{2.4}$$

We shall also have complex distributions $D^*, D_{-1}, D_1, \dots, D_r, \overline{D}_r$ on M corresponding to the eigenvalues $\pm 1, w_1, \overline{w}_1, \dots, w_r, \overline{w}_r$.

THEOREM 1. *Let M be a ps-manifold of order 4, such that the eigenvalues of S are ± 1, $\pm i$. Then*

$$(2.4)\quad \text{(a)}\quad P^* = (S^3+S^2+S+I)/4, \quad \text{(b)}\quad P = (-S^3+S^2-S+I)/4$$
$$\text{(c)}\quad P_1 = (iS^3-S^2-iS+I)/4, \quad \text{(d)}\quad \overline{P}_1 = (-iS^3-S^2+iS+I)$$

are complementary projection operators on $D^*, D_{-1}, D_1, \overline{D}_1$ respectively.

Proof. If X is any complex vector field on M, then

$$S(P^*X) = S(S^3+S^2+S+I)X/4 = (I+S^3+S^2+S)X/4 = P^*X,$$

i.e. $P^*X \in D^*$. Similarly, $S(PX) = -PX$, $S(P_1X) = iP_1X$, and $S(\overline{P}_1X) = i\overline{P}_1X$. Also $P^*+P+P_1+\overline{P}_1 = I$. Hence P^*, P, P_1 and $\overline{P}_1$ are complementary projection operators on D^*, D_{-1}, D_1, and $\overline{D}_1$ respectively. □

THEOREM 2. *On a ps-manifold of order 4, such that the eigenvalues of S are ± 1, $\pm i$ we have*

$$(2.5)\qquad \text{(a)}\ P^{*2} = P^*, \quad \text{(b)}\ P^2 = P, \quad \text{(c)}\ P_1^2 = P_1, \quad \text{(d)}\ \overline{P}_1^2 = \overline{P}_1.$$

$$(2.6)\quad \begin{array}{lll} \text{(a)}\ P^*P = PP^* = 0, & \text{(b)}\ P^*P_1 = P_1P^* = 0 & \text{(c)}\ P^*\overline{P}_1 = \overline{P}P^* = 0 \\ \text{(d)}\ PP_1 = P_1P = 0, & \text{(e)}\ P\overline{P}_1 = \overline{P}_1 = 0, & \text{(f)}\ P_1\overline{P}P_1 = 0. \end{array}$$

Proof.

$$(2.5)\quad \begin{aligned} \text{(a)}\ P^{*2} &= (S^3+S^2+S+I)P^*/4 = (S^3P^*+S^2P^*+SP^*+P^*)/4 \\ &= (P^*+P^*+P^*+P^*)/4 = P^*. \end{aligned}$$

Similarly we can prove (2.5) (b), (c), (d).

$$(2.6)\quad \begin{aligned} \text{(a)}\ P^*P &= (S^3+S^2+S+I)P/4 = (S^3P+S^2P+SP+P)/4 \\ &= (-P+P-P+P)/4 = 0. \end{aligned}$$

Similarly $PP^* = 0$ (2.6) (b), (d), (e) and (f) are proved in a similar way.

Suppose that we have a Riemannian 4-symmetric manifold. Then 1 is not going to be an eigenvalue of S, since s_p, for all $p \in M$, has p as the only isolated fixed point. Two possibilities arise

(i) The eigenvalues of S are -1, $\pm i$, in this case the underlying manifold is a symmetric manifold, and we are not interested in this case.

(ii) The eigenvalues of S are -1, $\pm i$, and we shall investigate this case.

From now on for every Riemannian 4-symmetric manifold we assume that the symmetry tensor field S has -1, $\pm i$ i as eigenvalues.

THEOREM 3. *Let (M, g) be a Riemannian 4-symmetric manifold; then*

$$(2.7)\quad \begin{array}{c} \text{(a)}\ P = (-S^3+S^2-S+I)/4, \quad \text{(b)}\ P_1 = (iS^3-S^2-iS+I)/4 \\ \text{(c)}\ \overline{P}_1 = (-iS^3-S^2+iS+I)/4 \end{array}$$

are complementary projection operators on D, D_1 *and* $\overline{D}_1$ *respectively.*

Proof. Let X be any complex vector field on M: then $S(PX) = -PX$, $S(P_1X) = iPX$, $S(\overline{P}_1X) = -iPX$. Also $P + P_1 + \overline{P}_1 = (-S^3 - S^2 - S + 3I)/4$. Since 1 is not an eigenvalue, we have $P^* = (S^3 + S^2 + S_I)/4 = 0$. Hence $P + P_1 + \overline{P}_1 = (I + 3I)/4 = I$. □

COROLLARY 1. *On a Riemannian* 4*-symmetric manifold, we have*

$$(2.8)\quad \text{(a)}\ P^2 = P, \qquad \text{(b)}\ P_1^2 = P_1, \qquad \text{(c)}\ \overline{P}_1^2 = \overline{P}_1$$

$$(2.9)\quad \text{(a)}\ PP_1 = P_1P = 0, \quad \text{(b)}\ P\overline{P}_1 = \overline{P}_1P = 0, \quad \text{(c)}\ P_1\overline{P}_1 = \overline{P}_1P_1 = 0.$$

Proof. Obvious. □

3. Nijenhuis Tensor

THEOREM 4. *On a Riemannian* 4*-symmetric manifold we have*

$$(3.1)\qquad \text{(a)}\ -64dP_1[PX, PY] = (S^3 - iI)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}N(S^kX, S^jY)$$

$$\text{(b)}\ -64d\overline{P}_1[PX, PY] = (S^3 + iI)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}N(S^kX, S^jY).$$

Proof. From (2.9) (a), we have

$$\begin{aligned}
&\text{(a)}\quad -64dP_1[PX, PY] = 64P_1[PX, PY]\\
&= (iS^3 - S^2 - iS + I)[-S^3X + S^2X - SX + X,\ -S^3Y + S^2Y - SY + Y]\\
&\big((I - S^2) + i(S^3 - S)\big)\big([X, Y] - [X, SY] + [X, S^2Y] - [X, S^3Y] - [SX, Y]\\
&+[SX, SY] - [SX, S^2Y] + [SX, S^3Y] + [S^2X, Y] - [S^2X, SY] + [S^2X, S^2Y]\\
&-[S^2X, S^3Y] - [S^3X, Y] + [S^3X, SY] - [S^3X, S^2Y] + [S^3X, S^3Y]\big)
\end{aligned}$$

$$(1)\qquad = \big((I - S^2) + (S^3 - S)\big)\sum_{k=0}^{3}\sum_{k=0}^{3}(-1)^{k+j}[S^kX, S^jX]$$

Using $S^4 = I$, we have

$$\begin{aligned}
N(X, Y) &= [SX, SY] + S^2[X, Y] - S[SX, Y] - S[X, SY]\\
-N(X, SY) &= -[SX, S^2Y] - S^2[X, SY] + S[SX, SY] + S[X, S^2Y]\\
N(X, S^2Y) &= [SX, S^3Y] + S^2[X, S^2Y] - S[SX, S^2Y] - S[X, S^3Y]\\
-N(X, S^3Y) &= -[SX, Y] - S^2[X, S^3Y] + S[SX, S^3Y] + S[X, Y]\\
-N(SX, Y) &= -[S^2X, SY] - S^2[SX, Y] + S[S^2X, Y] + S[SX, SY]\\
N(SX, SY) &= [S^2X, S^2Y] + S^2[SX, SY] - S[S^2X, SY] - S[SX, S^2Y]
\end{aligned}$$

$$
\begin{aligned}
-N(SX,S^2Y) &= -[S^2X,S^3Y]-S^2[SX,S^2Y]+S[S^2X,S^2Y]+S[SX,S^3Y]\\
N(SX,S^3Y) &= [S^2X,Y]+S^2[SX,S^3Y]-S[S^2X,S^3Y]-S[SX,Y]\\
N(S^2X,Y) &= [S^3X,SY]+S^2[S^2X,Y]-S[S^3X,Y]-S[S62X,SY]\\
-N(S^2X,SY) &= -[S^3X,S^2Y]-S^2[S^2X,SY]+S[S^3X,SY]+S[S^2X,S^2Y]\\
N(S^2X,S^2Y) &= [S^3X,S^3Y]+S^2[S^2X,S^2Y]-S[S^3X,S^2Y]-S[S^2X,S^3Y]\\
-N(S^2X,S^3Y) &= -[S^3X,Y]-S^2[S^2X,S^3Y]+S[S^3X,S^3Y]+S[S^2X,Y]\\
-N(S^3X,Y) &= -[X,SY]-S^2[S^3X,Y]+S[X,Y]+S[S^3X,SY]\\
N(S^3X,SY) &= [X,S^2Y]+S^2[S^3X,SY]-S[X,SY]-S[S^3X,S^2Y]\\
-N(S^3X,S^2Y) &= -[X,S^3Y]-S^2[S^3X,S^2Y]+S[X,S^2Y]+S[S^3X,S^3Y]\\
N(S^3X,S^3Y) &= [X,Y]+S^2[S^3X,S^3Y]-S[X,S^3Y]-S[S^3X,Y]
\end{aligned}
$$

Adding, we have

$$
\begin{aligned}
\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}N(S^kX,S^jY) &= (S^2+2S+i)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}[S^kX,S^jY]\\
&= S(I-S^2)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}[S^kX,S^jY]. \qquad (2)
\end{aligned}
$$

From (1) and (2) we have

$$
\begin{aligned}
-64dP_1[PX,PY] &= (S^3-iI)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}N(S^kX,S^jY)\\
-64d\overline{P}_1[PX,PY] &= (-iS^3-S^2+iS+I)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}[S^kX,S^jY]\\
&= \big((I-S^2)-i(S^3-S)\big)\sum_{k=0}^{3}\sum_{j=0}^{3}(-1)^{k+j}[S^kX,S^jY] \qquad (3)
\end{aligned}
$$

From (2) we have

$$
-64dP_1[PX,PY] = (S^3+iI)\sum_{k=0}^{3}\sum_{k=0}^{3}(-1)^{k+j}N(S^kX,S^kY).\ \square
$$

THEOREM 5. *On a Riemannian 4-symmetric manifold, we have*

$$
\begin{aligned}
(I+S^2)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big(N(S^{2k+1}X,S^{2j+1}Y)-N(S^{2k}X,S^{2j}Y)\big) \qquad (3.2)\\
= 2(S^3+s)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big([S^{2k+1}X,S^{2j}Y]+[S^{2k}X,S^{2j+1}]\big).
\end{aligned}
$$

Proof. We have

$$\begin{aligned} N(SX,SY) &= [S^2X,S^2Y]+S^2[SX,SY]-S[S^2X,SY] \\ &\quad - S[S^2X,SY]-S[SX,S^2Y] \\ -N(SX,S^3Y) &= -[S^2X,Y]-S^2[SX,S^3Y]+S[S^2X,S^3Y]+S[SX,Y] \\ -N(S^3X,SY) &= -[X,S^2Y]-S^2[S^3X,SY]+S[X,SY]+S[S^3X,S^2Y] \\ N(S^3X,S^3Y) &= [X,Y]+S^2[S^3X,S^3Y]-S[X,S^3Y]-S[S^3X,Y] \\ -N(X,Y) &= -[SX,SY]-S^2[X,Y]+S[SX,Y]+S[X,SY] \\ N(X,S^2Y) &= [SX,S^3Y]+S^2[X,S^2Y]-S[SX,S^2Y]-S[X,S^3Y] \\ N(S^2X,Y) &= [S^3X,SY]+S^2[S^2X,Y]-S[S^3X,Y]-S[S^3X,SY] \\ -N(S^2X,S^2Y) &= -[S^3X,S^3Y]-S^2[S^2X,S^2Y] \\ &\quad + S[S^3X,S^2Y]+S[S^2X,S^3Y] \end{aligned}$$

And we get

$$\begin{aligned} &\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big(N(S^{2k+1}X,S^{2j+1}Y)-N(S^2kX,S^2jY)\big) \\ &= (I-S^2)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big([S^{2k}X,S^{2j}Y]-[S^{2k+1}X,S^{2j+1}]\big) \\ (1)\qquad &+2S\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big([S^{2k+1}X,S^{2j}Y]+[S^{2k}X,S^{2j+1}Y]\big) \end{aligned}$$

Multiply (1) by S^2 and add to (1) and we get the result. □

THEOREM 6. *The following are equivalent*

$$(3.3)\qquad \text{(a)}\ \sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{k+j}\big([S^{2k}X,S^{2j}]-[S^{2k+1}X,s^{2j+1}Y]\big)=0,$$

$$\text{(b)}\ \sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{k+j}\big([s^{2k+1}X,S^{2j}Y]+[S^{2k}X,S^{2j+1}Y]\big)=0.$$

Proof. Equation (3.6) (a) is

$$(1)\qquad \begin{aligned} &[X,Y]+[S^3X,SY]+[SX,S^3]+[S^2Y,S^2Y]-[SX,SY] \\ &-[X,S^2Y]-[S^2X,Y]-[S^3X,S^3Y]=0. \end{aligned}$$

Equation (3.6) (b) is

$$(2)\qquad \begin{aligned} &[SX,Y]+[X,SY]+[S^3X,S^2Y]+[S^2X,S^3Y]-[SX,S^2Y] \\ &-[X,S^3Y]-[S^3X,Y]-[S^2X,SY]=0. \end{aligned}$$

If we replace X by SX in (1) we get (2). If we replace X by S^3X in (2) we get (1). □

THEOREM 7. *On a Riemannian* 4*-symmetric manifold we have*

$$(S^2 - I)\sum_{j=0}^{3}\sum_{k=0}^{3}[S^{k+j}S^kX, S^jY] = -\sum_{j=0}^{3}\sum_{k=0}^{3}S^{k+j+2}N(S^kX, S^jY) \tag{3.4}$$

Proof.

$$\begin{aligned}
S^2N(X,Y) &= S^2[SX,SY] + [X,Y] - S^3[SX,Y] - S^3[X,SY]\\
S^3N(X,SY) &= S^3[SX,S^2Y] + S[X,SY] - [SX,SY] - [X,S^2Y]\\
N(X,S^2Y) &= [SX,S^3Y] + S^2[X,S^2Y] - S[SX,S^2Y] - S[X,S^3Y]\\
SN(X,S^3Y) &= S[SX,Y] + S^3[X,S^3Y] - S^2[SX,S^3Y] - S^2[X,Y]\\
S^3N(SX,Y) &= S^3[S^2X,SY] + S[SX,Y] - [S^2X,Y] - [SX,SY]\\
N(SX,SY) &= [S^2X,S^2Y] + S^2[SX,SY] - S[S^2X,SY] - S[SX,S^2Y]\\
SN(SX,S^2Y) &= S[S^2X,S^3Y] + S^3[SX,S^2Y] - S^2[S^2X,S^2Y] - S^2[SX,S^2Y]\\
S^2N(SX,S^3Y) &= S^2[S^2X,Y] + [SX,S^3Y] - S^3[S^2X,S^3Y] - S^3[SX,Y]\\
N(S^2X,Y) &= [S^3X,SY] + S^2[S^2X,Y] - S[S^3X,Y] - S[S^2X,SY]\\
SN(S^2X,SY) &= S[S^3X,S^2Y] + S^3[S^2X,SY] - S^2[S^3X,SY] - S^2[S^2X,S^2Y]\\
S^2N(S^2X,S^2Y) &= S^2[S^3X,S^3Y] + [S^2X,S^2Y] - S^3[S^3X,S^2Y] - S^3[S^2X,S^3Y]\\
S^3N(S^2X,S^3Y) &= S^3[S^3X,Y] + S[S^2X,S^3Y] - [S^3X,S^3Y] - [S^2X,Y]\\
SN(S^3X,Y) &= S[X,SY] + S^3[S^3X,Y] - S^2[X,Y] - S^2[S^3X,SY]\\
S^2N(S^3X,SY) &= S^2[X,S^2Y] + [S^3X,SY] - S^3[X,SY] - S^3[S^3X,S^2Y]\\
S^3N(S^3X,S^2Y) &= S^3[X,S^3Y] + S[S^3X,S^2Y] - [X,S^2Y] - [S^3X,S^3Y]\\
N(S^3X,S^3Y) &= [X,Y] + S^2[S^3X,S^3Y] - S[X,S^3Y] - S[S^3X,Y]
\end{aligned}$$

and we get the result. □

4. Integrability Conditions

THEOREM 8. *In order that D be integrable, it is necessary and sufficient that*

$$\sum_{j=0}^{3}\sum_{k=0}^{3}(-1)^{k+j}N(S^kX, S^jY) = 0. \tag{4.1}$$

Proof. D is integrable if and only if

$$[PX,PY] \in D \Longleftrightarrow P_1[PX,PY] = 0 \quad \text{and} \quad \overline{P}_1[PX,PY] = 0.$$

From theorem $(4)(a)$ we have

$$-64dP_1[PX,PY] = 64P_1[PX,PY] = (S^3 - iI)\sum_{j=0}^{3}\sum_{k=0}^{3}(-1)^{k+j}N(S^kX, S^jY).$$

Since S^3 is nonsingular, therefore,

$$P_1[PX, PY] = 0 \Longleftrightarrow \sum_{j=0}^{3}\sum_{k=0}^{3}(-1)^{k+j} N(S^k X, S^j Y) = 0$$

Also from theorem 4(b), we have

$$P_1[PX, PY] = 0 \Longleftrightarrow \sum_{j=0}^{3}\sum_{k=0}^{3}(-1)^{k+j} N(S^k X, S^j Y) = 0.$$

Hence, that result. □

THEOREM 9. *In order that D_1 be integrable, it is necessary and sufficient that*

(4.2)

$$\text{(a)}\quad \sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{k+j}\big(N(S^{2k+1}X, S^{2j+1}Y) - N(S^{2k}X, S^{2j}Y)\big) = 0$$

$$\text{(b)}\quad \sum_{j=0}^{3}\sum_{k=0}^{3} S^{k+j} N(S^k X, S^j Y) = 0$$

Proof. We have, by using (2.9) (a)

$$\begin{aligned}
&-64dP[P_1X, P_1Y] = 64P[P_1x, P_1y]\\
&= (-S^3 + S^2 - S + I)[iS^3X - S^2X - iSX + X,\, iS^3Y - S^2Y - iSY + Y]\\
&= 2(S^2 + I)\big([X, Y] - [X, S^2Y] - [S^2X, Y] + [S^2X, S^2Y] - [SX, SY]\\
&\quad + [SX, S^3Y] + [S^3X, SY] - [S^3X, S^3Y] - 2i(S^2 + I)\big([SX, Y]\\
&\quad - [SX, S^2Y] - [S^3X, Y] + [S^3X, S^2Y] + [X, SY] - [X, S^3Y]\\
&\quad - [S^2X, SY] + [S^2X, S^3Y]\big)\\
&= 2(S^2 + I)\sum_{j=0}^{1}\sum_{k=0}^{1}(-1)^{k+j}\big([S^{2k}X, S^{2j}Y] - [S^{2k+1}X, S^{2j+1}Y]\big)\\
&\quad - 2i(S^2 + I)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big([S^{2k+1}X, S^{2j+1}Y] + [S^{2k}X, S^{2j}Y]\big).
\end{aligned}$$

Using (2.9) (c), and (3.4), we have

$$\begin{aligned}
&-64d\overline{P}_1[P_1X, P_1Y] = 64\overline{P}_1[P_1X, P_1Y]\\
&= (-iS^3 - S^2 + iS + I)[iS^3X - S^2X - iSX + X,\, iS^3Y - S^2Y - iSY + Y]\\
&= \big((I - S^2) + i(S - S^3)\big)[iS^3 - S^2X - iSX + X,\, iS^3Y - S^2Y - iSY + Y]\\
&= (I - S^2)\sum_{j=0}^{3}\sum_{k=0}^{3} S^{k+j}[S^kX, S^jY] + i(S^3 - S)\sum_{j=0}^{3}\sum_{k=0}^{3} S^{k+j}[S^kX, S^jY]
\end{aligned}$$

$$= \sum_{j=0}^{3}\sum_{k=0}^{3} S^{k+j+2}N(S^kX, S^jY) - i\sum_{j=0}^{3}\sum_{k=0}^{3} N(S^kX, S^jY).$$

D_1 is integrable if and only if

$$[P_1X, P_1Y] \in D_1 \iff P[P_1X, P_1Y] = 0 \text{ and } \overline{P}_1[P_1X, P_1Y] = 0.$$

Using theorems (5), (6) and (7), and that $S^2 + I$ is nonsingular, we have

$$P[P_1X, P_1Y] = 0 \iff \sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{j+k}\big([S^{2k+1}X, S^{2j}Y] + [S^{2k}X, S^{2j+1}Y]\big) = 0$$

$$\iff \sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big(N(S^{2k+1}X, S^{2j+1}Y) - N(S^{2k}X, S^{2j}Y)\big) = 0,$$

$$\overline{P}_1[P_1X, P_1Y] = 0 \iff \sum_{k=0}^{3}\sum_{k=0}^{3} S^{k+j}N(S^kX, S^jY) = 0. \square$$

THEOREM 10. *In order that $\overline{D}_1$ be integrable, it is necessary and sufficient that*

(4.3)

(a) $$\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\big(N(S^{2k+1}X, S2j+1Y) - N(S^{2k}X, S^{2j}Y)\big) = 0$$

(b) $$\sum_{k=0}^{3}\sum_{j=0}^{3} S^{k+1}N(S^kX, S^jY) = 0$$

Proof. We have, using (2.9) (a) (c)

$$- 64dP[\overline{P}_1X, \overline{P}_1Y] = 64P[\overline{P}_1X, \overline{P}_1Y]$$

$$= 2(S^2 + I)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\left([S^{2k}X, S^{2j}Y] - [S^{2k+1}X, S^{2j+1}Y]\right)$$

$$+ 2i(S^2 + I)\sum_{k=0}^{1}\sum_{j=0}^{1}(-1)^{k+j}\left([S^{2k+1}X, S^{2j}Y] + [S^{2k}X, S^{2j+1}Y]\right)$$

$$- 64dP_1[\overline{P}_1X, \overline{P}_1Y] = 64P_1[\overline{P}_1X, \overline{P}_1Y]$$

$$= (I - S^2)\sum_{k=0}^{3}\sum_{j=0}^{3} S^{k+j}[S^kX, S^jY] - i(S^3 - S)\sum_{k=0}^{1}\sum_{j=0}^{1} S^{k+j}[S^kX, S^jY]$$

Therefore $\overline{D}_1$ is integrable if and only if

$$[\overline{P}_1X, \overline{P}_1Y] \in \overline{D}_1, \; P[\overline{P}_1X, \overline{P}_1Y] = 0, \text{ and } P_1[\overline{P}_1X, \overline{P}_1Y] = 0.$$

Using theorems (5), (6) and (7), the proof follows the pattern of the proof of the Theorem 9. □

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