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ON $\mathcal{P}$-CHAIN-NET SPACES

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Abstract. $\mathcal{P}$-chain-net spaces (with some variations) were introduced and studied in [**5**]. In this note we give several further characterizations of these spaces in terms of continuous mappings.

1. Introduction. Let $p \in \beta\omega \setminus \omega$ be a free ultrafilter on ω, the (discrete) space of positive integers. For a topological space X and a sequence $(x_n : n \in \omega)$ in X, a p-limit of (x_n), denoted by $x = p\text{-lim } x_n$, is a point $x \in X$ such that for each neighbourhood U of x the set $\{n \in \omega : x_n \in U\}$ belongs to p. A space X is called p-compact if each sequence in X has a p-limit point in X. These notions were introduced by Bernstein in [**1**].

Kombarov [**7**] defined P-compactness and P-sequentiality, where $P \subset \beta\omega \setminus \omega$ is a non-empty set of free ultrafilters on ω. In [**2**], [**4**] and [**8**] one can find several results concerning P-sequentiality and P-compactness.

Saks [**9**] generalizes the notion of p-limit to transfinite sequences: let τ be an infinite cardinal; if $p \in \beta\tau \setminus \tau$ is a free ultrafilter on τ (with the discrete topology) and $(x_\alpha : \alpha \in \tau)$ is a τ-sequence in a space X, then a point $x \in X$ is a p-limit point of (x_α), $x = p\text{-lim}\, x_\alpha$, if for each neighbourhood U of x, $\{\alpha \in \tau : x_\alpha \in U\} \in p$.

In [**5**], the author introduced and studied some chain-net generalizations of the notion of P-sequentiality (see also [**10**]). For any cardinal λ, $\mu(\lambda) = \{p \in \beta\lambda : |A| = \lambda \text{ for each } A \in p\}$ will denote the set of all uniform ultrafilters on λ; if $x = p\text{-lim}\, x_\alpha$ we say that (x_α) p-converges to x. Let λ be a cardinal and $P_\lambda \subset \mu(\lambda)$. We say that a λ-sequence $(x_\alpha : \alpha \in \lambda)$ in a space X:

(a) strongly P_λ-converges (to x) (= sP_λ-converges) if it p-converges to (the same point) x for every $p \in P_\lambda$;

(b) P_λ-converges if it p-converges to a point $x(p)$ for every $p \in P_\lambda$;

(c) weakly P_λ-converges (= wP_λ-converges) (to x) if it p-converges to a point x for some $p \in P_\lambda$;

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(d) very weakly P_λ-converges (= vwP_λ-converges) if there is a point x with the property that for every neighbourhood U of x there exists some $p(U) \in P_\lambda$ such that $\{\alpha \in \lambda : x_\alpha \in U\} \in p(U)$.

Let τ be a regular cardinal and for every regular cardinal $\lambda \leq \tau$ let us select a set $P_\lambda \subset \mu(\lambda)$. Put $\mathcal{P} = \{P_\lambda : \lambda \leq \tau\}$. If A is a subset of a space X, we define s$\mathcal{P}(A)$ (resp., w$\mathcal{P}(A)$, vw$\mathcal{P}(A)$) to be $A \cup \{x \in X :$ there is some $\lambda \leq \tau$ and a λ-sequence $(x_\alpha : \alpha \in \lambda)$ in A which sP_λ- (resp., wP_λ-, vwP_λ-) converges to $x\}$. A space X is called *s$\mathcal{P}$-radial* (resp., *w$\mathcal{P}$-radial*, *vw$\mathcal{P}$-radial*) if s$\mathcal{P}(A) = \overline{A}$) (resp., w$\mathcal{P}(A) = \overline{A}$, vw$\mathcal{P}(A) = \overline{A}$) for every $A \subset X$. X is called ***s$\mathcal{P}$-pseudo-radial*** (resp., *w$\mathcal{P}$-pseudo-radial*, *vw$\mathcal{P}$-pseudo-radial*) if s$\mathcal{P}(A) \subset A$ (resp., w$\mathcal{P}(A) \subset A$, vw$\mathcal{P}(A) \subset A$) implies A is closed in X. A space X is called ***s$\mathcal{P}$-compact*** (resp., *$\mathcal{P}$-compact*, *w$\mathcal{P}$-compact*, *vw$\mathcal{P}$-compact*) if for every $\lambda \leq \tau$ each λ-sequence (x_α) s$\mathcal{P}$-converges (resp., $\mathcal{P}$-converges, w$\mathcal{P}$-converges, vw$\mathcal{P}$-converges) in X (see [**5**]).

In this note we give several further characterizations of these $\mathcal{P}$-chain-net spaces in terms of continuous mappings.

Throughout the paper all spaces are Hausdorff and all mappings are continuous (except in Theorem 2.2) and onto. Usual topological notation and terminology are used. $\mathcal{P}$ is $\{P_\lambda \subset \mu(\lambda) : \lambda \leq \tau\}$.

2. $\mathcal{P}$-pseudo-radial spaces. We shall consider s$\mathcal{P}$-pseudo-radial spaces only, because the consideration concerning w$\mathcal{P}$ and vw$\mathcal{P}$ variants is quite similar.

Definition 2.1. Call a mapping $f : X \to Y$ $\mathcal{P}$-continuous if for every $\lambda \leq \tau$ and every λ-sequence $(x_\alpha : \alpha \in \lambda)$ in X which sP_λ-converges to x, the λ-sequence $\big(f(x_\alpha) : \alpha \in \lambda\big)$ vwP_λ-converges to $f(x)$.

The next theorem is similar to Proposition 4 in [**3**] (concerning pseudo-radial spaces and given without proof) and Theorem 1 in [**6**] (about almost radial spaces).

Theorem 2.2. *A space X is s$\mathcal{P}$-pseudo-radial if and only if every $\mathcal{P}$-continuous mapping* (*defined*) *on X is continuous.*

Proof. ($\Rightarrow$) Suppose $f : X \to Y$ is $\mathcal{P}$-continuous but not continuous. There is a closed subset B of Y such that $f^{-1}(B)$ is not closed. Since X is $\mathcal{P}$-pseudo-radial, there exist $x \in \overline{f^{-1}(B)} \setminus f^{-1}(B)$ and a λ-sequence (x_α), $\lambda \leq \tau$ in $f^{-1}(B)$ which sP_λ-converges to x. By assumption, the λ-sequence $\big(f(x_\alpha)\big) \subset B$ vwP_λ-converges to $f(x)$. As B is closed, $f(x) \in B$ and thus we have $x \in f^{-1}(B)$. This contradiction shows that f must be continuous.

($\Leftarrow$) Denote by Y the topological sum of all s$\mathcal{P}$-convergent transfinite sequences of lenght $\leq \tau$ in X. Let $f : X \to Y$ be the identity mapping. Clearly, f is $\mathcal{P}$-continuous (because s$\mathcal{P}$-convergent sequences of X and Y are the same and every s$\mathcal{P}$-convergent sequence is vw$\mathcal{P}$-convergent). By assumption f is continuous. Let us prove that X is s$\mathcal{P}$-pseudo-radial. Let $A \subset X$ be a non-closed set. Then $f(A)$ is non-closed in Y and, since Y is obviously s$\mathcal{P}$-pseudo-radial, there are $y \in \overline{f(A)} \setminus f(A)$ and a λ-sequence $(y_\alpha) \subset f(A)$ sP_λ-converging to y. The corresponding λ-sequence $\big(x_\alpha = f^{-1}(y_\alpha)\big) \subset A$ sP_λ-converges to $x = f^{-1}(y)$. The theorem is proved.

Definition 2.3. Let $f : Y \to X$ be a (continuous) mapping. Denote by (π) the following property: for every $\lambda \in \tau$ and every λ sequence $(x_\alpha : \alpha \in \lambda)$ in X which sP_λ-converges to x, the set $f^{-1}(\{x_\alpha : \alpha \in \lambda\} \cup \{x\})$ is vwP_λ-compact (i.e., the inverse image of s$\mathcal{P}$-convergent sequences in X are vw$\mathcal{P}$-compact).

THEOREM 2.4. *A space X is s$\mathcal{P}$-pseudo-radial if and only if every mapping $f : Y \to X$ onto X which has the property (π) is closed.*

Proof. ($\Rightarrow$) Suppose that a mapping $f : Y \to X$ which has the property (π) is not closed. Let F be a closed subset of Y such that $f(F)$ is not closed in X. Since X is s$\mathcal{P}$-pseudo-radial, there are $x \in \overline{f(F)} \setminus f(F)$ and a λ-sequence $(x_\alpha : \alpha \in \lambda)$, $\lambda \leq \tau$, in $f(F)$ which sP_λ-converges to x. Choose arbitrary points $y_\alpha \in f^{-1}(x_\alpha) \cap F$, $\alpha \in \lambda$. Since $f^{-1}(\{x_\alpha : \alpha \in \lambda\} \cup \{x\})$ is vwP_λ-compact, this λ-sequence (in F) vwP_λ-converges to a point $y \in f^{-1}(x)$. As F is closed $y \in F$, so that $f(y) = x \in f(F)$. We have a contradiction which shows that f must be closed.

($\Leftarrow$) Let Y be the topological sum of all s$\mathcal{P}$-convergent sequences of X and let $f : Y \to X$ be the identity mapping. Clearly, f is continuous. It also has the property (π) because s$\mathcal{P}$-convergent sequences of X and Y are the same and every sP_λ-compact set is vwP_λ-compact. By assumption f is closed (and consequently quotient). Closed mappings preserve s$\mathcal{P}$-pseudo-radiality [**5**] and therefore X is s$\mathcal{P}$-pseudo-radial (since Y is obviously such a space). The theorem is proved.

The theorem above should be compared with a result of Tanaka [**12**] about sequential spaces (see also Theorem 2 in [**6**]).

Remark 2.5. The first part of Theorem 2.4 remains valid if both in Definition 2.3 and Theorem 2.4 we replace s$\mathcal{P}$ by vw$\mathcal{P}$ and vw$\mathcal{P}$ by s$\mathcal{P}$.

Definiton 2.6. A mapping $f : Y \to X$ is called $\mathcal{P}$-sequence covering if whenever $(x_\alpha : \alpha \in \lambda)$ is a λ-sequence, $\lambda \leq \tau$, in X which sP_λ-converges to a point $x \in X$, then there are points $y_\alpha \in f^{-1}(x_\alpha)$, $\alpha \in \lambda$, and $y \in f^{-1}(x)$ such that $(y_\alpha : \alpha \in \lambda)$ vwP_λ-converges to y.

(This definition is completely analogous to that of a sequence covering mapping defined by Siwiec in [**11**].)

THEOREM 2.7. *A space X is s$\mathcal{P}$-pseudo-radial if and only if every $\mathcal{P}$-sequence covering mapping onto X is quotient.*

3. $\mathcal{P}$-radial spaces. Recall that a mapping $f : Y \to X$ is pseudo-open if for every $A \subset X$ and every $x \in \overline{A}$ there exists $y \in f^{-1}(x)$ with $y \in \overline{f^{-1}(A)}$.

THEOREM 3.1. *A space X is s$\mathcal{P}$-radial if and only if every $\mathcal{P}$-sequence covering mapping onto X is pseudo-open.*

Proof. (The proof of Theorem 2.7 is almost the same.) Let X be an s$\mathcal{P}$-radial space and let $f : Y \to X$ be a $\mathcal{P}$-sequence covering mapping onto X. Let A be a subset of X and $x \in \overline{A}$. As X is s$\mathcal{P}$-radial there exists a λ-sequence $(x_\alpha : \alpha \in \lambda)$, $\lambda \le \tau$, in A which sP_λ-converges to x. Pick some $y_\alpha \in f^{-1}(x_\alpha)$, $\alpha \in \lambda$, and $y \in f^{-1}(x)$ so that (y_α) vwP_λ-converges to y. Then $y \in f^{-1}(A)$, i.e., f is pseudo-open.

Denote now by Y the topological sum of all s$\mathcal{P}$-convergent sequences of X. Let $f : Y \to X$ be the identity mapping. Clearly, f is $\mathcal{P}$-sequence covering and thus it is pseudo-open. Obviously, Y is s$\mathcal{P}$-radial an since s$\mathcal{P}$-radiality is preserved by pseudo-open mappings [**5**] we have that X is s$\mathcal{P}$-radial. The theorem is proved.

In connection with Theorem 2.4 we have the following

THEOREM 3.2. *If X is an s$\mathcal{P}$-radial space then every mapping f onto X which has the property (π) (in Def. 2.3) is closed on all subspaces of the form $f^{-1}(A) \cup \{y\}$, where A is any subset of X and $f(y) \notin vw\mathcal{P}(A)$.*

Proof. Let $f : Y \to X$ satisfy (π). Take $A \subset X$ and $y \in Y$ with $f(y) \notin$ vw$\mathcal{P}(A)$. Suppose that F is closed in $f^{-1}(A) \cup \{y\}$. Let $x \in \overline{f(F)} \cap (A \cup \{f(y)\})$. The subspace $A \cup \{f(y)\}$ of X is s$\mathcal{P}$-radial, so that there exists a λ-sequence, $\lambda \le \tau$, $\{x_\alpha : \alpha \in \lambda\}$ in $f(F)$ which sP_λ-converges to x. Choose $y_\alpha \in f^{-1}(x_\alpha) \cap F$ for every $\alpha \in \lambda$. By assumption the set $f^{-1}(\{x_\alpha : \alpha \in \lambda\} \cup \{x\})$ is vwP_λ-compact and thus the λ-sequence (y_α) (in F) vwP_λ-converges to a point $z \in f^{-1}(x)$. As $x \in$ s$\mathcal{P}(A) \subset$ vw$\mathcal{P}(A)$ we have $x \ne f(y)$ and hence $z \in f^{-1}(A)$. On the other hand, F is closed in $f^{-1}(A) \cup \{y\}$ and one has $z \in F$. Therefore, $f(z) \equiv x \in f(F)$ which means that $f(F)$ is closed in $A \cup \{f(y)\}$, i.e., f is closed on $f^{-1}(A) \cup \{y\}$.

If $f : Y \to X$ is a mapping, then (π') denotes property that the inverse image of s$\mathcal{P}$-convergent sequences in X are s$\mathcal{P}$-compact sets in Y.

THEOREM 3.3. *If every mapping $f : Y \to X$ onto X which has the property (π') is closed on all subspaces of the form $f^{-1}(A) \cup \{y\}$, where A is any subset of X and $f(y) \notin s\mathcal{P}(A)$, then X is s$\mathcal{P}$-radial.*

Proof. Suppose, on the contrary, that X is not s$\mathcal{P}$-radial. Then there are some $A \subset X$ and a point $x \in \overline{A} \setminus A$ with $x \notin$ s$\mathcal{P}(A)$. Denote by Y the topological sum $\big(X \times \{0\}\big) \oplus \big(X \times \{1\}\big)$. Let $f : Y \to X$ be the mapping defined by $f\big((x,0)\big) = f\big((x,1)\big) = x$, $x \in X$. We have $f\big((x,1)\big) = x \notin$ s$\mathcal{P}(A)$ and, obviously, f satisfies (π'). But f is not closed on the set $f^{-1}(A) \cup \{(x,1)\}$ because $A \times \{0\}$ is closed in this set and $f\big(A \times \{0\}\big) = A$ is not closed in $A \cup \{f((x,1))\} = A \cup \{x\}$. So, we have a contradiction and the theorem is proved.

COROLLARY 3.4. *A space X is s$\mathcal{P}$-radial if and only if every mapping $f : Y \to X$ onto X which has the property (π') is closed on all subspaces described in Theorem 3.3.*

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