## FIXED POINT THEOREMS IN BANACH SPACES

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**Abstract**. A class of self-maps on Banach spaces which have at least one fixed-point is presented and investigated.

We shall prove three fixed point theorems in Banach spaces.

Theorem 1. Let K be a closed and convex subset of a Banach space with the norm ||x|| = d(x,0) and  $f: K \to K$  a mapping which satisfies the condition

$$d[x, f(x)] + d[y, f(y)] \le qd(x, y) \tag{1}$$

for all x, y in K, where  $2 \le q < 4$ . Then f has at least one fixed point.

*Proof.* Let  $x_0$  in K be arbitrary and let a sequence  $\{x_n\}$  be defined by

$$x_{n+1} = [x_n + f(x_n)]/2$$
  $(n = 0, 1, 2, ...).$  (2)

For this sequence we have

$$x_n - f(x_n) = 2[x_n - (x_n + f(x_n))/2] = 2(x_n - x_{n+1})$$
(3)

and hence  $d[x_n, f(x_n)] = ||x_n - f(x_n)|| = 2d(x_n, x_{n+1})$  (n = 0, 1, 2, ...). Therefore, for  $x = x_{n-1}$  and  $y = x_n$  the condition (1) states

$$2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) < qd(x_{n-1}, x_n).$$

Hence we have that  $d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n)$ , where  $0 \leq c = (q-2)/2 < 1$ , as  $2 \leq q < 4$ . Therefore,  $\{x_n\}$  is a Cauchy sequence in K and hence converges to some  $u \in K$ . Since

$$d[u, f(x_n)] \le d(u, x_n) + d[x_n, f(x_n)] = d(u, x_n) + 2d(x_n, x_{n+1}),$$

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we conclude that

$$\lim_{n \to \infty} f(x_n) = u. \tag{4}$$

Now, let us put in (1) x = u and  $y = x_n$ , and use (3). Then we have

$$d[u, f(u)] + 2d(x_n, x_{n+1}) \le qd(u, x_n).$$

If now n tends to infinity one has  $d[u, f(u)] \leq 0$ , which implies f(u) = u, and our theorem is established.

Remark. The identity mapping satisfies (1). Therefore, maps which satisfy (1) may have many fixed points.

Corollary. Let K be as in Theorem 1 and let  $f: K \to K$  be a mapping which satisfies the condition

$$d[x, f(y)] + d[y, f(x)] \le pd(x, y); \qquad 0 \le p < 2.$$
 (5)

Then f has a fixed point.

*Proof*. Using the triangle inequality we have

$$d[x, f(x)] + d[y, f(y)] \le d(x, y) + d[y, f(x)] + d(y, x) + d[x, f(y)].$$

By (5) we now get

$$d[x, f(x)] + d[y, f(y)] \le pd(x, y) + 2d(x, y).$$

Hence we conclude that f satisfies (1) with q = p + 2 < 4.

Theorem 2. Let K be as in Theorem 1 and let  $f:K\to K$  be a mapping such that

$$d[f(x), f(y)] + d[x, f(x)] + d[y, f(y)] \le qd(x, y)$$
(6)

for all x, y in K where  $2 \le q < 5$ . Then f has at least one fixed point.

*Proof*. Consider a sequence  $\{x_n\}$  in K defined by (2). For this sequence the equalities (3) and

$$x_n - f(x_{n-1}) = [x_{n-1} + f(x_{n-1})]/2 - f(x_{n-1}) = [x_{n-1} - f(x_{n-1})]/2$$
 (7)

hold. Then the inequality

$$d[x_n, f(x_n)] - d[x_n, f(x_{n-1})] \le d[f(x_{n-1}), f(x_n)]$$
(8)

becomes

$$2d(x_n, x_{n+1}) - d(x_{n-1}, x_n) \le d[f(x_{n-1}), f(x_n)]. \tag{9}$$

Now, if we put in (6)  $x = x_{n-1}$  and  $y = x_n$ , and use (3), then from (6) and (9) we get

$$2d(x_n, x_{n+1}) - d(x_{n-1}, x_n) + 2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) \le qd(x_{n-1}, x_n)$$
 (10)

and hence  $d(x_n, x_{n+1}) \leq ((q-1)/4)d(x_{n-1}, x_n)$ . Since  $1 \leq q < 5$ , it follows that the sequence  $\{x_n\}$  is a Cauchy sequence which converges to some  $u \in K$ . Since  $f(x_n)$  also converges to u, from the inequality (6) with x = u and  $y = x_n$  we get  $d[f(u), u] + d[u, f(u)] \leq 0$  which implies f(u) = u and our theorem is established.

Now we give a general form of the inequality (6) which includes (1) and (6).

Theorem 3. Let f be a self-map on K, where K is as in Theorem 1. If there exist real numbers a, b and q such that

$$0 \le q + |a| - 2b < 2(a+b); \tag{11}$$

$$ad[f(x), f(y)] + b\{d[x, f(x)] + d[y, f(y)]\} \le qd(x, y)$$
(12)

for all x, y in K, then f has at least one fixed point.

*Proof.* Consider a sequence  $\{x_n\}$  defined by (2) and put in (12)  $x = x_{n-1}$  and  $y = x_n$ . If  $a \ge 0$ , then by (8) and (12) we obtain, similarly as (10) in Theorem 2, the inequality

$$2ad(x_n, x_{n+1}) - |a|d(x_{n-1}, x_n) + 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \le qd(x_{n-1}, x_n), (13)$$

since then -a = -|a|. If a < 0, than we use the inequality

$$d[x_n, f(x_n)] + d[x_n, f(x_{n-1})] \ge d[f(x_{n-1}), f(x_n)]$$
(14)

instead of (8). Then by (14) and (12) (with  $x = x_{n-1}$  and  $y = x_n$ ) we obtain (13), because in this case we can write -|a| instead of a. Therefore, (13) holds for all a, b and q. From (13) we get

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n);$$
  $k = (|a| - 2b + q)/(2(a+b)).$ 

Since (11) implies  $0 \le k < 1$ , it follows that  $\{x_n\}$  is a Cauchy sequence and therefore it converges to some  $u \in K$ . Using (12) with x = u and  $y = x_n$  and letting n to tend to infinity we obtain

$$ad[f(u), u] + bd[u, f(u)] \le 0.$$

Then, as a + b > 0, it follows that f(u) = u, completing the proof.

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