PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 121–126

ON A DENSE G_δ-DIAGONAL

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Abstract. We study topological spaces the diagonal of which contains a dense set which is a G_δ-set in $X \times X$.

We use the usual notation and terminology as in [**6**], [**7**], [**2**]. All spaces are at least T_2.

Let us say that X is a space with a *dense G_δ-diagonal* if there exists a G_δ-subset U of the space $X \times X$ such that $U \subset \Delta_X$ and $\overline{U} = \Delta_X$. Here $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal in $X \times X$.

This notion was introduced in [**11**] under the name "weak G_δ-diagonal" (see also [**12**] about related subjects). In the same paper it was proved that if the space $\exp X$ of all closed subsets of X with the Vietoris topology is weakly perfect, then X has a dense G_δ-diagonal. A space X is called *weakly perfect* [**11**], [**13**] if every closed subset of X contains a dense set which is a G_δ-set in X. Note that there are spaces which are weakly perfect but not perfect [**9**].

PROPOSITION 1. *X is a space with a dense G_δ-diagonal if and only if there exists a subspace $Y \subset X$ such that $\overline{Y} = X$, Y is a G_δ-set in X and Y has a G_δ-diagonal.*

Proof. ($\Longrightarrow$) Let $\{U_n : n \in \mathbf{N}^+\}$ be a family of open subsets in $X \times X$ such that $\bigcap\{U_n : n \in \mathbf{N}^+\} \subset \Delta_X$ and $\bigcap\{U_n : n \in \mathbf{N}^+\}$ is dense in Δ_X. Put $V_n = \{x \in X : (x,x) \in U_n\}$. Clearly, each V_n is open in X and $Y = \bigcap\{V_n : n \in \mathbf{N}^+\}$ is the subspace we are looking for.

($\Longleftarrow$) Let Y be a G_δ-subset of X. Then $Y \times Y$ is a G_δ-subset of $X \times X$. Indeed, let $Y = \bigcap\{V_n : n \in \mathbf{N}^+\}$ where each V_n is open in X. We can choose V_n to satisfy the condition: $V_{n+1} \subset V_n$ for all $n \in \mathbf{N}^+$. Then $Y \times Y = \bigcap\{V_n \times V_n : n \in \mathbf{N}^+\}$.

AMS Subject Classification (1980): Primary 54 D 15, 54 D 30, 54 E 50

The second author was supported by RZN Srbije

If Y is dense in X then Δ_Y is dense in Δ_X. If the diagonal Δ_Y is a G_δ-subset of $Y \times Y$ then Δ_Y is a G_δ-subset of $X \times X$ as $Y \times Y$ is a G_δ-subset of $X \times X$.

THEOREM 1. *Let X be a Čech-complete space. Then X has a dense G_δ-diagonal if and only if it contains a dense subspace metrizable by a complete metric.*

Proof. ($\Longleftarrow$) If Y is dense in X and the space Y is metrizable by a complete metric then Y has a G_δ-diagonal and Y is a G_δ-subset of Y (see [**6**], [**7**]). Then by Proposition 1, X is a space with a dense G_δ-diagonal. (We didn't use in this part of the argument Čech-completeness of X).

($\Longrightarrow$) Assume that X has a dense G_δ-diagonal. By Proposition 1 there exists a G_δ-subset Y of X which is dense in X and is a space with a G_δ-diagonal. As X is Čech-complete and Y is a G_δ in X the space Y is also Čech-complete. By a result of Šapirovskiĭ (see [**15**]), there exists a paracompact Čech-complete subspace Z of Y which is dense in Y. Then Z is also dense in X. The space Z also has a G_δ-diagonal (this property is obviously inherited by arbitrary subspaces). But it is well known that every paracompact Čech-complete space with G_δ-diagonal is metrizable (see [**7**]). Moreover if a metrizable space is Čech-complete then it is metrizable by a complete metric [**6**], [**7**]. It follows that Z is metrizable by a complete metric. The theorem is proved.

Remark 1. From the proof of the first part of Theorem 1 and the fact that countable product of complete metric spaces is complete we have: if a space X contains a dense subspace metrizable by a complete metric, then the spaces X^n, $n \in \mathbf{N}^+$, and X^ω have a dense G_δ-diagonal.

Question 1. Can a space X^ω be weakly perfect?

COROLLARY 1. *Let X be a Čech-complete space with a dense G_δ-diagonal such that the Souslin number of X is countable. Then X has a countable π-base. Hence X is separable and every dense subspace of X is separable.*

Recall that a π-base of a space X is a family $\mathcal{V}$ of non-empty open subsets of X such that every open subset U of X contains some $V \in \mathcal{V}$ (see [**2**], [**6**], [**10**]).

Proof of Corollary 1. By Theorem 1 there exists a dense metrizable subspace Y of the space X. As $\overline{Y} = X$, the Souslin number of Y does not exceed the Souslin number of X (see [**2**], [**10**]). Hence $c(Y) \leq \omega$. As Y is metrizable it follows that Y has a countable base $\mathcal{B}$. For each $U \in \mathcal{B}$ fix an open subset $\widetilde{U}$ of X such that $\widetilde{U} \cap Y = U$. Then the countable family $\{\widetilde{U} : U \in \mathcal{B}\}$ of open subsets of X is a π-base of X — this is shown easily using the fact that Y is dense in X.

COROLLARY 2. *Let X be a Čech-complete space such that the space $X \times X$ is weakly perfect. Then in every closed subspace of X there exists a dense subspace metrizable by a complete metric.*

Proof. Let X_1 be a closed subspace of X. Then X_1 is Čech-complete and weakly perfect — both properties are inherited by closed subspaces. Obviously if the space $X_1 \times X_1$ is weakly perfect, then X_1 has a dense G_δ-diagonal. Hence X_1 satisfies the assumptions in Theorem 1 and thus there exists a dense subspace in X_1 metrizable by a complete metric.

Recall that spread $s(X)$ of a space X is the supremum of cardinalities of discrete subspaces of X.

THEOREM 2. *Let X be a Čech-complete space such that the space $X \times X$ is weakly perfect. Then spread of X is equal to hereditary density of X: $s(X) = \mathrm{hd}(X)$. In particular, if all discrete subspaces of X are countable, then X is hereditarily separable.*

Proof. For metrizable spaces spread is equal to density. We also have $s(Y) \leq s(X)$ for every subspace $Y \subset X$. From Corollary 2 it follows now that density of every closed subspace of X does not exceed spread of X. As X is Čech-complete it is a k-space and for k-spaces the following inequality (of Arhangel'skiĭ-Šapirovskiĭ) holds: tightness is not greater than spread (see [**2**]). Thus $t(X) \leq s(X)$. Put $s(X) = \tau$ and let Y be any subspace of X. Then $t(\overline{Y}) \leq \tau$ and $d(\overline{Y}) \leq \tau$ as $\overline{Y}$ is closed in X. Fix a subset $A \subset \overline{Y}$ such that $\overline{A} = \overline{Y}$ and $|A| \leq \tau$. For each $a \in A$ we can fix a subset $B_a \subset Y$ such that $|B_a| \leq \tau$ and $a \in \overline{B}_a$. Then for the set $M = \bigcup\{B_a : a \in A\}$ we have: $|M| \leq \tau \cdot \tau = \tau$, $M \subset Y$ and $\overline{M} = \overline{Y} \supset Y$. Thus $d(Y) \leq \tau = s(X)$, i.e. $\mathrm{hd}(X) \leq s(X)$. It is always true that $s(X) \leq \mathrm{hd}(X)$. Hence $\mathrm{hd}(X) = s(X)$.

Remark 2. Our results on weakly perfect $X \times X$ remain true under weaker assumption that every closed subspace F of Δ_X contains a subset A which is a G_δ-set in F and is dense in F.

From Corollary 2 we derive

COROLLARY 3. *Let X be a compact non-separable space, the Souslin number of which is countable. Then X does not have a dense G_δ-diagonal. Hence $X \times X$ is not weakly perfect.*

From Theorem 1 we get

COROLLARY 4. *If X is a Čech-complete space with a dense G_δ-diagonal, then X satisfies the first axiom of countability at a dense G_δ-set of points.*

Proof. There exists a dense subspace Y of X metrizable by a complete metric. Then Y is a G_δ-subset of X and X is first countable at every point of Y (as X is regular and Y is dense in X — see [**10**]).

Every dyadic compactum which is first countable at a dense set of points is metrizable — this is the well known result of Efimov (see [**7**]). Now Corollary 4 implies the following assertion:

COROLLARY 5. *If a dyadic compactum X has a dense G_δ-diagonal then X is metrizable.*

Let us recall that a space X is called $\aleph_0$-monolithic if closure of every countable subset $A \subset X$ is a space with a countable network [**1**] (see also [**4**], [**5**]). Every compact space with a countable network is metrizable [**6**], [**7**]. Applying Corollary 1 we get

COROLLARY 6. *If X is an $\aleph_0$-monolithic compact space the Souslin number of which is countable and X has a dense G_δ-diagonal, then X is metrizable.*

Of course the last assertion is also true for Čech-complete spaces.

In connection with Corollary 4 we have the following assertion which can be proved in a similar way as one proves the fact that every space with a G_δ-diagonal has countable pseudo-character.

PROPOSITION 2. *If a space X has a dense G_δ-diagonal, then the set of points of countable pseudocharacter is dense in X.*

From this proposition and the fact that for every topological group G one has $\psi(G) = \Delta(G)$ [**3**] we derive

COROLLARY 7. *If G is a topological group with a dense G_δ-diagonal, then G has a G_δ-diagonal.*

There is an interesting necessarry and sufficient condition for a space X to have a dense G_δ-diagonal.

PROPOSITION 3. *A space $(X, \mathcal{T})$ has a dense G_δ-diagonal if and only if there exist a subset $Y \subset X$ dense in $(X, \mathcal{T})$ and a topology $\mathcal{T}_1$ on X such that $\mathcal{T} \subset \mathcal{T}_1$, the space $(X, \mathcal{T}_1)$ has a G_δ-diagonal and $\mathcal{T}$ is a base of $(X, \mathcal{T}_1)$ at all points $y \in Y$.*

Proof. ($\Longleftarrow$) There exist open sets U_n, $n \in \mathbf{N}^+$, in the product space $(X, \mathcal{T}_1) \times (X, \mathcal{T}_1)$ such that $\bigcap\{U_n : n \in \mathbf{N}^+\} = \Delta_X$. For each $y \in Y$ and each $n \in \mathbf{N}^+$ we can fix a $V(y,n) \in \mathcal{T}$ such that $y \in V(y,n)$ and $V(y,n) \times V(y,n) \subset U_n$. Put $G_n = \bigcup\{V(y,n)^2 : y \in Y\}$ for every $n \in \mathbf{N}^+$. Obviously $\Delta_Y \subset G_n \subset U_n$ and G_n is open in $(X, \mathcal{T}) \times (X, \mathcal{T})$. Hence $\Delta_Y \subset \bigcap\{G_n : n \in \mathbf{N}^+\} \subset \Delta_X$. As Δ_Y is dense in Δ_X, the set $\bigcap\{G_n : n \in \mathbf{N}^+\}$ is the one we were looking for. Thus X has a dense G_δ-diagonal.

($\Longrightarrow$) Let B be a dense subset of Δ_X which is a G_δ-subset in the space $(X, \mathcal{T}) \times (X, \mathcal{T})$. Fix open sets U_n in $(X, \mathcal{T}) \times (X, \mathcal{T})$ for $n \in \mathbf{N}^+$ such that $\bigcap\{U_n : n \in \mathbf{N}^+\} = B$. Put $Y = \{x \in X : (x,x) \in B\}$ and $\mathcal{B}_1 = \mathcal{T} \cup \{\{x\} : x \in X \setminus Y\}$. Then $\mathcal{B}_1$ is a base of a topology $\mathcal{T}_1$ on X. It is clear that $\mathcal{T} \subset \mathcal{T}_1$ and that $\mathcal{T}$ is a base of the space $(X, \mathcal{T}_1)$ at all points of the set Y. It remains to check that the space $(X, \mathcal{T}_1)$ has a G_δ-diagonal.

Let $W_n = U_n \cup \Delta_X$. Then W_n is open in the product space $(X, \mathcal{T}_1) \times (X, \mathcal{T}_1)$ by the definition of $\mathcal{T}_1$. Clearly, $\bigcap\{W_n : n \in \mathbf{N}^+\} = \Delta_X$. Hence $(X, \mathcal{T}_1)$ has a G_δ-diagonal. The proposition is proved.

As every metrizable space has a G_δ-diagonal the following assertion is a direct corollary of Proposition 3.

THEOREM 3. *A space $(X, \mathcal{T})$ has a dense G_δ-diagonal if there exists a metrizable topology $\mathcal{T}_1$ on X such that $\mathcal{T} \subset \mathcal{T}_1$ and the set of all points at which $\mathcal{T}$ is a base of the topology $\mathcal{T}_1$ is dense in the space $(X, \mathcal{T})$.*

The conditions in Theorem 3 are satisfied by every Eberlein compactum (see T.4.3 in [**4**]). Thus we have

COROLLARY 8. *Every Eberlein compactum has a dense G_δ-diagonal.*

One could derive Corollary 8 from Theorem 1 on the following fact — Namioka's theorem (see [**2**]): in every Eberlein compactum there exists a dense subspace metrizable by a complete metric.

Every Gul'ko compact space [**5**] also has a dense subspace metrizable by a complete metric (Leiderman-Gruenhage; see [**14**], [**8**] or [**5**]. Thus applying Theorem 1 we get.

COROLLARY 9. *Every Gul'ko compact space has a dense G_δ-diagonal.*

Remark 3. S. Todorčević has shown that not in each Corson compactum [**5**] there exists a dense metrizable subspace. It follows from Theorem 1 that not every Corson compactum has a dense G_δ-diagonal.

Remark 4. If the set of all isolated points of a space X is dense in X, then X has a dense G_δ-diagonal. This is evident. Thus if X is a scattered space then every subspace of X has a dense G_δ-diagonal while X itself need not have a G_δ-diagonal (take a compact non-metrizable scattered space — for example, the space $T(\omega_1 + 1)$).

We conclude the paper with several questions on weakly perfect spaces and spaces with a dense G_δ-diagonal.

Question 2 [**11**]. What can we say on density of weakly perfect compact spaces? Is it true that density of each such space is $\leq \aleph_1$?

Question 3 [**11**]. Is it true that for every weakly perfect countably compact space X spread of X is countable?

Question 4. Is it true that every symmetrizable space X has a dense G_δ-diagonal? is weakly perfect?

In connection with this question it should be noted that there are symmetrizable spaces without a G_δ-diagonal and non-perfect.

Question 5. Let X be a weakly perfect compact space. Is it true then that X contains a dense metrizable subspace?

Question 6. Is every weakly perfect compact space of countable Souslin number separable?

Question 7. Let X be a compact space such that $X \times X$ is weakly perfect. What about X? Is X perfect?

(This question is suggested by Example in [**11**]).

Question 8. When there exists a countable family $\mathcal{U}$ of open sets in $X \times X$ such that $\bigcap \mathcal{U} \cap \Delta_X$ is dense in Δ_X and for each open neighborhood V of Δ_X in $X \times X$ one can find $U \in \mathcal{U}$ such that $U \subset V$? Such $\mathcal{U}$ will be called *a dense* Δ*-base* of X.

Let us note that if X has a dense discrete subspace then X has a countable dense Δ-base.

Question 9. Let X be a compact space with a countable dense Δ-base. Does there exist a dense open metrizable subspace $Y \subset X$? dense separable metrizable subspace $Z \subset X$?

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(Received 19 05 1989)