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IDEMPOTENT SEPARATING CONGRUENCES ON AN ORTHODOX SEMIGROUP VIA THE LEAST INVERSE CONGRUENCE

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Dedicated to Professor M. Yamada on his 60-th birthday

Abstract. The least inverse congruence Y on an orthodox semigroup S was considered by Yamada [**14**] for the case where the band of idempotents of S is normal. It was considered in the general orthodox case by Schein [**12**] and Hall [**4**]. An explicit construction for idempotent separating congruences on an orthodox semigroup S in terms of idempotent separating congruences on S/Y was given by McAlister [**8**]. In this paper we describe these congruences by inverse congruences contained in $\mu \circ Y$, where μ is the greatest idempotent separating congruence on S. Also, we obtain some mutually inverse complete lattice isomorphisms of intervals $[Y, \mu \circ Y]$ and $[\varepsilon, \mu]$, where ε is the identity relation on S.

1. Preliminaries. In the following we shall use the notation and terminology of [**3**], [**5**] and [**10**]. This will be suplemented with the following.

Let S be a regular semigroup. A congruence ρ on S is uniquely determined by its kernel $\ker \rho = \{x \in S \mid x\rho e \text{ for some } e \in E\}$ and trace $\operatorname{tr} \rho = \rho\,|_E$, where E is the set of idempotents of S [**2**].

RESULT 1. (Lemma 1.3 of [**6**], Lemma 2.5 of [**9**]) *For any family F of congruences on a regular semigroup S,* $\ker \bigcap_{\rho \in F} \rho = \bigcap_{\rho \in F} \ker \rho$.

A congruence ρ on S is idempotent separating if $\operatorname{tr} \rho = \varepsilon\,|_E$. The greatest idempotent separating congruence on S is denoted by μ.

Let $\mathcal{C}$ be a class of semigroups and let ρ be a congruence on S. Then ρ is a *$\mathcal{C}$-congruence* if $S/\rho \in \mathcal{C}$. The least inverse congruence on a regular semigroup is denoted by Y.

RESULT 2. [**1**, Lemma 3.1] *Let ρ and ξ be any congruences on an orthodox semigroup S. Then*

$$(\rho \subseteq \mu \text{ and } \xi \subseteq Y) \Rightarrow \rho \vee \xi = \rho \circ \xi.$$

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If S is a regular semigroup and $a \in S$, then $V(a)$ denotes the set of all inverses of a in S.

RESULT 3. [**7**, Lemma 1] *Let ρ be an inverse congruence on a regular semigroup S. Then*

$$(\forall a, b \in S)(a\rho b \Rightarrow (\forall a' \in V(a))(\forall b' \in V(b))a'\rho b').$$

RESULT 4. ([**4**],[**12**], [**5**, Theorem VI.1.12]) *If S is an orthodox semigroup, then the relation Y defined by $aYb \Leftrightarrow V(a) = V(b)$ $(a, b \in S)$ is the least inverse congruence on S.*

RESULT 5. [**3**] *If f is an order isomorphism of a lattice L onto a lattice L', then f is a lattice isomorphism.*

LEMMA 1. *Let ρ be an idempotent separating congruence on a regular semigroup S and $a, b \in S$. The following conditions are equivalent:*

(i) $a\rho b$,

(ii) $a\mathcal{H}b$ and $(\forall a' \in V(a))(\exists b' \in V(b))a'\rho b'$,

(iii) $a\mathcal{H}b$ and $(\exists a' \in V(a))(\exists b' \in V(b))a'\rho b'$.

Proof. (i) $\Rightarrow$ (ii) Let $a\rho b$ and $a' \in V(a)$. Then $a\mathcal{H}b$ that is $a'a = b'b$ and $aa' = bb'$ for some $b' \in V(b)$ (by [**5**, Proposition 4.1]). Therefore

$a' = a'aa' = b'ba'\rho\ b'aa' = b'bb' = b'$.

(ii)$\Rightarrow$ *(iii)* This is trivial.

(iii)$\Rightarrow$ *(i)* Let $a\mathcal{H}b$ and $a'\rho b'$ for some $a' \in V(a)$ and $b' \in V(b)$. Then $a\mathcal{H}b$ implies $a'a = b''b$ and $bb' = aa''$ for some $a'' \in V(a)$ and $b'' \in V(b)$ (by [**5**, Proposition 4.1]). So we have $a = aa''a = bb'a\rho\ ba'a = bb''b = b$.

The implication (iii)$\Rightarrow$(i) is valid for any semigroup S and any congruence ρ on S [**8**, Lemma 2.1].

2. The congruence $\rho \circ Y$. In the remainder of the paper, S will denote an orthodox semigroup with the band of idempotents E.

Let $a \in S$, $a' \in V(a)$ and $e \in E$. Then by Result 4 we have $aYe \Leftrightarrow V(a) = V(e)$ which yields $a' \in V(e)$. According to [**11**] (see also Theorem VI.1.1 of [**5**]) we have $a' \in E$. Since $a \in V(a')$, by the same argument we get $a \in E$. Therefore, $\ker Y = E$.

LEMMA 2. *Let ρ be an idempotent separating congruence on S and let ξ be a congruence on S such that $\xi \subseteq Y$. Then*

(1) $\operatorname{tr}(\rho \circ \xi) = \operatorname{tr}\xi$, (2) $\ker(\rho \circ \xi) = \ker\rho$.

Proof. According to Result 2, we have $\rho \vee \xi = \rho \circ \xi$.

(1) For $e, f \in E$ we obtain

$$\begin{aligned} e(\rho \circ \xi)f &\Rightarrow e\rho a \text{ and } a\xi f \text{ for some } a \in E && (\text{since } \ker\xi = E)\\ &\Rightarrow e = a \text{ and } a\xi f && (\text{since } \operatorname{tr}\rho = \varepsilon)\\ &\Rightarrow e\xi f. \end{aligned}$$

Therefore $\operatorname{tr}(\rho \circ \xi) \subseteq \operatorname{tr} \xi$ and thus $\operatorname{tr}(\rho \circ \xi) = \operatorname{tr} \xi$.

(2) For $a \in S$ and $e \in E$, we get

$$\begin{aligned} a(\rho \circ \xi)e &\Rightarrow a\rho b \text{ and } b\xi e \text{ for some } b \in E \quad (\text{since} \ker \xi = E) \\ &\Rightarrow a \in \ker \rho. \end{aligned}$$

Therefore $\ker(\rho \circ \xi) \subseteq \ker \rho$ and hence $\ker(\rho \circ \xi) = \ker \rho$.

Notice that (1) is a consequence of Proposition of [**13**], and (2) is a special case of Lemma 2.1 of [**1**].

If ρ and ξ are congruences on S such that $\rho \subseteq \xi$, then the relation ξ/ρ on S/ρ defined by $(a\rho)\xi/\rho(b\rho) \Leftrightarrow a\xi b$ $(a, b \in S)$ is a congruence.

According to Lemma 2 we have that for any idempotent separating congruence ρ on S, $(\rho \circ Y)/Y$ is an idempotent separating congruence on S/Y. In particular, $(\mu \circ Y)/Y$ is an idempotent separating congruence on S/Y [**8**, Lemma 2.2].

The next theorem describes the inverse congruence $\rho \circ Y$, where ρ is an idempotent separating congruence on S.

THEOREM 1. *Let ρ be an idempotent separating congruence on S and $a, b \in S$. Then the following conditions are equivalent:*

(i) $a(\rho \circ Y)b$,

(ii) $(\exists a' \in V(a))(\exists b' \in V(b))aa' = ab'ba',\ bb' = ba'ab',\ ab' \in \ker \rho$,

(iii) $(\forall a' \in V(a))(\exists b' \in V(b))a'\rho\, b'$,

(iv) $(\exists a' \in V(a))(\exists b' \in V(b))a'\rho\, b'$.

Proof. (i)$\Leftrightarrow$(ii) Since $\rho \circ Y$ is an inverse congruence on S, we have

$$\begin{aligned} a(\rho \circ Y)b &\Leftrightarrow (\exists a' \in V(a))(\exists b' \in V(b))a'a(\rho \circ Y)b'b,\ ab' \in \ker \rho \\ &\quad\ \text{(by Theorem 1 of [7])} \\ &\Leftrightarrow (\exists a' \in V(a))(\exists b' \in V(b))a'aYb'b,\ ab' \in \ker \rho \\ &\quad\ \text{(by Lemma 2)} \\ &\Leftrightarrow (\exists a' \in V(a))(\exists b' \in V(b))a'a = a'ab'ba'a,\ b'b = b'ba'ab'b, \\ &\quad\ ab' \in \ker \rho \\ &\Leftrightarrow (\exists a' \in V(a))(\exists b' \in V(b))aa' = ab'ba',\ bb' = ba'ab',\ ab' \in \ker \rho \end{aligned}$$

(i)$\Rightarrow$(iii) Let $a, b \in S$ and $a' \in V(a)$. Then

$$\begin{aligned} a(\rho \circ Y)b &\Rightarrow a\rho c \text{ and } cYb \text{ for some } c \in S \\ &\Rightarrow (\exists c' \in V(c))a'\rho c' \text{ and } V(c) = V(b) \quad \text{(by Lemma 1 and Result 4)} \\ &\Rightarrow (\exists b' \in V(b))a'\rho b'. \end{aligned}$$

(iii)$\Rightarrow$(iv) This is trivial.

(iv)⇒(i) Let $a' \in V(a)$ and $b' \in V(b)$. Then

$$\begin{aligned} a'\rho b' &\Rightarrow a'(\rho \circ Y)b' \\ &\Rightarrow a(\rho \circ Y)b \qquad \text{(by Result 3).} \end{aligned}$$

Let ρ be an idempotent separating congruence on S and let ξ be a congruence on S such that $\xi \subseteq Y$. Let $a, b \in S$. The proof of Theorem 1 shows that $a(\rho \circ \xi)b \Rightarrow$ (iii) $\Rightarrow$ (iv). But (iv)⇒(i) follows from the fact that $\rho \circ Y$ is an inverse congruence on S. In this context, the following result is of interest.

PROPOSITION 1. *Let ρ be an idempotent separating congruence on S and let ξ be a congruence on S such that $\xi \subseteq Y$. Then*

$$(1) \quad Y \subseteq \rho \circ \xi \Leftrightarrow \xi = Y, \qquad (2) \quad \mu \subseteq \rho \circ \xi \Leftrightarrow \rho = \mu.$$

Proof. It is evident that $\xi = Y$ implies $Y \subseteq \rho \circ \xi$ and $\rho = \mu$ implies $\mu \subseteq \rho \circ \xi$.

$$\begin{aligned} Y \subseteq \rho \circ \xi &\Rightarrow \operatorname{tr} Y \subseteq \operatorname{tr}(\rho \circ \xi) \\ &\Rightarrow \operatorname{tr} Y \subseteq \operatorname{tr} \xi \qquad \text{(by Lemma 2)} \\ &\Rightarrow \operatorname{tr} Y = \operatorname{tr} \xi \qquad \text{(since } \xi \subseteq Y). \end{aligned} \tag{1}$$

Also, $\xi \subseteq Y \Rightarrow \ker \xi \subseteq \ker Y \Rightarrow \ker \xi = E$ (since $\ker Y = E$). Therefore $\operatorname{tr} Y = \operatorname{tr} \xi$ and $\ker Y = \ker \xi$, so by [**2**] we have $Y = \xi$.

$$\begin{aligned} \mu \subseteq \rho \circ \xi &\Rightarrow \ker \mu \subseteq \ker(\rho \circ \xi) \\ &\Rightarrow \ker \mu \subseteq \ker \rho \qquad \text{(by Lemma 2)} \\ &\Rightarrow \ker \mu = \ker \rho \qquad \text{(since} \rho \subseteq \mu). \end{aligned} \tag{2}$$

Since $\operatorname{tr} \rho = \operatorname{tr} \mu$, by [**2**] we have $\rho = \mu$.

LEMMA 3. *Let ρ be an idempotent separating congruence on S.*

(1) *Then $\rho = (\rho \circ Y) \cap \mathcal{H} = (\rho \circ Y) \cap \mu$. In particular, $Y \cap \mathcal{H} = Y \cap \mu = \varepsilon$.*

(2) *If ξ is a congruence on S such that $\xi \subseteq \rho \circ Y$, then $\xi \cap \rho = \xi \cap \mathcal{H} = \xi \cap \mu$.*

Proof. (1) From Lemma 1 and Theorem 1 we have $\rho = (\rho \circ Y) \cap \mathcal{H}$. Hence $\mu = (\mu \circ Y) \cap \mathcal{H}$. So we have $(\rho \circ Y) \cap \mu = (\rho \circ Y) \cap (\mu \circ Y) \cap \mathcal{H} = (\rho \circ Y) \cap \mathcal{H}$. From the preceding equalities for $\rho = \varepsilon$ we get $Y \cap \mathcal{H} = Y \cap \mu = \varepsilon$.

(2) Let $\xi \subseteq \rho \circ Y$. By (1) we get $\xi \cap \rho = \xi \cap (\rho \circ Y) \cap \mathcal{H} = \xi \cap \mathcal{H}$, and also $\xi \cap \rho = \xi \cap (\rho \circ Y) \cap \mu = \xi \cap \mu$.

The equality $Y \cap \mathcal{H} = \varepsilon$ can be found in [**5**] and [**8**].

3. Description of $[Y, \mu \circ Y]$. In this section we describe inverse congruences on S contained in $\mu \circ Y$. This leads to a characterization of idempotent separating congruences on S (Theorem 2).

LEMMA 4. *Let ξ be a congruence on S such that $\xi \in [Y, \mu \circ Y]$. Then $\xi = (\xi \cap \mu) \circ Y$.*

Proof.
$$\begin{aligned} Y \subseteq \xi \subseteq \mu \circ Y &\Rightarrow \operatorname{tr} Y \subseteq \operatorname{tr} \xi \subseteq \operatorname{tr}(\mu \circ Y) \\ &\Rightarrow \operatorname{tr} Y \subseteq \operatorname{tr} \xi \subseteq \operatorname{tr} Y && \text{(by Lemma 2)} \\ &\Leftrightarrow \operatorname{tr} Y = \operatorname{tr} \xi \\ &\Leftrightarrow \operatorname{tr}((\xi \cap \mu) \circ Y) = \operatorname{tr} \xi && \text{(by Lemma 2).} \end{aligned}$$

Also,
$$\begin{aligned} \xi \subseteq \mu \circ Y &\Rightarrow \ker \xi \subseteq \ker(\mu \circ Y) \\ &\Rightarrow \ker \xi \subseteq \ker \mu && \text{(by Lemma 2)} \\ &\Leftrightarrow \ker(\xi \cap \mu) = \ker \xi && \text{(by Result 1)} \\ &\Leftrightarrow \ker((\xi \cap \mu) \circ Y) = \ker \xi && \text{(by Lemma 2).} \end{aligned}$$

Hence, by [**2**] we get $\xi = (\xi \cap \mu) \circ Y$.

PROPOSITION 2. *Let $\xi \in [Y, \mu \circ Y]$ and let ζ be any inverse congruence on S. Then $\xi \cap \mu \subseteq \zeta \cap \mu \Rightarrow \xi \subseteq \zeta$.*

Proof. Let $a, b \in S$ and let $\xi \cap \mu \subseteq \zeta \cap \mu$. Then

$$\begin{aligned} a\xi b &\Rightarrow a'\xi b' \text{ and } a'\mu b' \text{ for some } a' \in V(a),\ b' \in V(b) \\ &\qquad \text{(by Result 3 and Theorem 1)} \\ &\Rightarrow a'(\xi \cap \mu)b' \text{ for some } a' \in V(a),\ b' \in V(b) \\ &\Rightarrow a'(\zeta \cap \mu)b' \text{ for some } a' \in V(a),\ b' \in V(b) \\ &\Rightarrow a'\zeta b' \Rightarrow a\zeta b \qquad \text{(by Result 3).} \end{aligned}$$

The following theorem describes idempotent separating congruences on S by means of inverse congruences contained in $\mu \circ Y$.

THEOREM 2. *Let ρ be an idempotent separating congruence on S. Then $\xi = \rho \circ Y$ is the unique inverse congruence on S contained in $\mu \circ Y$ and for which $\rho = \mathcal{H} \cap \xi$. Conversely, if ξ is a congruence on S such that $\xi \subseteq \mu \circ Y$, then $\mathcal{H} \cap \xi$ is an idempotent separating congruence on S.*

Proof. Let ρ be an idempotent separating congruence on S and let $\xi = \rho \circ Y$. Then by Lemma 3, $\rho = \mathcal{H} \cap \xi$. Clearly, ξ is an inverse congruence on S contained in $\mu \circ Y$. Let $\zeta \in [Y, \mu \circ Y]$ such that $\mathcal{H} \cap \xi = \mathcal{H} \cap \zeta$. Then by Lemma 3 we have $\mu \cap \xi = \mu \cap \zeta$. According to Lemma 4 we have $\zeta = (\zeta \cap \mu) \circ Y = (\xi \cap \mu) \circ Y = \xi$.

Now suppose that ξ is a congruence on S such that $\xi \subseteq \mu \circ Y$. Then by Lemma 3 we have $\mathcal{H} \cap \xi = \mu \cap \xi$. Hence $\mathcal{H} \cap \xi$ is an idempotent separating congruence on S.

If ρ is a congruence on S and α is a congruence on S/ρ, then the relation $\bar{\alpha}$ on S defined by $a\bar{\alpha}b \Leftrightarrow (a\rho)\alpha(b\rho)$, $(a, b \in S)$ is a congruence.

COROLLARY 1. [**8**, Theorem 2.4] *Let ρ be an idempotent separating congruence on S. Then $\alpha = (\rho \circ Y)/Y$ is the unique congruence on S/Y contained in $(\mu \circ Y)/Y$ and for which $\rho = \mathcal{H} \cap \bar{\alpha}$. Conversely, if α is a congruence on S/Y such that $\alpha \subseteq (\mu \circ Y)/Y$, then $\mathcal{H} \cap \bar{\alpha}$ is an idempotent separating congruence on S.*

Proof. Let ρ be an idempotent separating congruence on S and let $\alpha = (\rho \circ Y)/Y$. Then $\alpha \subseteq (\mu \circ Y)/Y$ and $\bar{\alpha} = \rho \circ Y$. By Theorem 2 we have $\rho = \mathcal{H} \cap \bar{\alpha}$. Let γ be a congruence on S/Y such that $\gamma \subseteq (\mu \circ Y)/Y$ and $\rho = \mathcal{H} \cap \bar{\gamma}$. It is clear that $Y \subseteq \bar{\gamma} \subseteq \mu \circ Y$. According to Theorem 2 we have $\bar{\alpha} = \bar{\gamma}$, that is $\alpha = \gamma$.

Conversely, let α be a congruence on S/Y such that $\alpha \subseteq (\mu \circ Y)/Y$. Then $\bar{\alpha} \subseteq \mu \circ Y$. By Theorem 2, $\mathcal{H} \cap \bar{\alpha}$ is an idempotent separating congruence on S.

4. An isomorphism theorem. The preceding characterizations lead to the following theorem.

THEOREM 3. *For S, the mappings φ and ψ defined by*

$$\varphi : \rho \longmapsto \rho \circ Y \qquad (\rho \in [\varepsilon, \mu]),$$
$$\psi : \xi \longmapsto \xi \cap \mu \qquad (\xi \in [Y, \mu \circ Y])$$

are mutually inverse complete lattice isomorphisms between $[\varepsilon, \mu]$ and $[Y, \mu \circ Y]$.

Proof. Let $\rho \in [\varepsilon, \mu]$ and $\xi \in [Y, \mu \circ Y]$. Then

$\rho(\varphi\psi) = (\rho\varphi)\psi = (\rho \circ Y)\psi = (\rho \circ Y) \cap \mu = \rho$ (by Lemma 3), and

$\xi(\psi\varphi) = (\xi\psi)\varphi = (\xi \cap \mu)\varphi = (\xi \cap \mu) \circ Y = \xi$ (by Lemma 4).

So we have $\varphi\psi = I_{[\varepsilon,\mu]}$ and $\psi\varphi = I_{[Y,\mu\circ Y]}$. Since $[\varepsilon, \mu]$ and $[Y, \mu \circ Y]$ are complete lattices, and φ and ψ are order preserving, they are both complete lattice isomorphisms [**3**].

COROLLARY 2. (1) $(\cap_{\rho \in F} \rho) \vee Y = \cap_{\rho \in F}(\rho \vee Y) \qquad (F \subseteq [\varepsilon, \mu])$,
(2) $(\vee_{\rho \in F} \rho) \cap \mu = \vee_{\rho \in F}(\rho \cap \mu) \qquad (F \subseteq [Y, \mu \circ Y])$.

Notice that the first part of Corollary 2 is a special case of Theorem 2.4 of [**1**].

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