PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 55 (69), 1994, 18–22

ON SUBHARMONIC BEHAVIOUR AND OSCILATION OF FUNCTIONS ON BALLS IN R^n

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Abstract. We give sufficient conditions for a nonnegative function to behave like a subharmonic function. If f is a C^1-function on a domain $D \subset R^n$ such that $|\bigtriangledown f(a)| \le Kr^{-1}\,\omega_f(a,r)$ (K = const.), where $\omega_f(a,r)$ is the oscillation of f on the ball $B_r(a) \subset D$, then both $|f|^p$ and $|\bigtriangledown f|^p$ ($p > 0$) have a weakened sub-mean-value property.

Let D be a domain in the Euclidean space R^n. If f is a function harmonic in D, then the function $|f|^p$ ($p > 0$), although need not be subharmonic when $p < 1$, yet behaves like a subharmonic function. This fact was established by Hardy and Littlewood [**2**] for $n = 2$ and generalized by Fefferman and Stein [**1**, Section 9, Lemma 2] to several variables.

THEOREM (HLFS). *Let $p > 0$. If f is harmonic in D, then*

$$|f(a)|^p \le Kr^{-n} \int_{B_r(a)} |f|^p dm$$

whenever $B_r(a) := \{x : |x - a| < r\} \subset D$, where K is a constant depending only on p and n.

Here dm denotes the Lebesgue measure normalized so that $m(B) = 1$, $B = \{x : |x| < 1\}$.

The theorem romains true if $|f|$ is replaced by $|\operatorname{grad} f|$ (f harmonic) or, more generally, by a nonnegative subharmonic function.

In this paper we prove two results which, via the simplest properties of harmonic functions, imply Theorem HLFS and can be applied to wider classes of functions. We start with two observations. If f is harmonic in D, then (for $K = 1$)

$$(\mathrm{sh}_K) \qquad F(a) \le Kr^{-n} \int_{B_r(a)} F dm \qquad \text{whenever } B_r(a) \subset D,$$

AMS Subject Classification (1991): Primary 31B05
Key words and phrases: harmonic functions, sub-mean-value property, oscillation

where $F = |f|$; and (for $K = n$)

$$(h_K) \qquad L(a,f) \le K r^{-1} \sup_{B_r(a)} |f| \qquad \text{whenever } B_r(a) \subset D,$$

where $L(\cdot, f) = |\mathrm{grad}\, f|$. (The letter is verified, for example, by differentiation of Poisson's integral.)

In the general case $L(\cdot, f)$ is defined by

$$L(a,f) = \limsup_{x \to a} \frac{|f(x) - f(a)|}{|x-a|}.$$

If f is differentiable at a, then $L(a,f) = |\mathrm{grad}\, f(a)|$. If f is continuous, then the function $a \to L(a,f)$ is Borel measurable. (See **[5]** for further information on the operator L.)

THEOREM 1. *If a nonnegative, locally integrable function F on D satisfies condition* (sh_K) *for some K, then the function F^p $(p > 0)$ satisfies condition* (sh_C) *for some C depending only on K, p and n.*

Observe that the hypotheses of the theorem imply that F is locally bounded and consequently F^p is locally integrable.

THEOREM 2. *If a locally bounded function f on D satisfies condition* (h_K) *for some K, then the function $|f|^p$ $(p > 0)$ satisfies condition* (sh_C) *for some C depending only on K, p and n.*

Although the general case of Theorem 2 can be deduced from the case $p = 1$ by using Theorem 1, we will give an independent proof, and this will be a new proof of Theorem HLFS.

Note that the hypothesis of Theorem 2 implies that f is continuous.

We will apply Theorems 1 and 2 to a class of "regularly oscillating" functions. The oscillation of f on the ball $B_r(a)$ is defined by

$$\omega_f(a,r) = \sup\{|f(x) - f(a)| : x \in B_r(a)\}.$$

We have

$$L(a,f) = \limsup_{r \to 0} \omega_f(a,r)/r.$$

If f is convex or concave on D, then the function $r \to \omega_f(a,r)$, where a is fixed, is convex, and since $\omega_f(a, 0^+) = 0$, this implies that

$$(h_K^+) \qquad L(a,f) \le K r^{-1} \omega_f(a,r) \qquad \text{whenever } B_r(a) \subset D,$$

with $K = 1$. As noted above, a harmonic function satisfies (h_n), and applying this to $f - f(a)$we see that it satisfies (h_n^+) as well. There are many other examples of functions satisfying (h_K^+), for some $K \ge 1$. In a separate paper we shall discuss certain relations, sufficient for the validity of (h_K) or (h_K^+), between the Lapplacian and the gradient of a C^2-funcition.

THEOREM 3. *Let $p > 0$. If a continuous function f on D satisfies condition* (h_K^+) *for some K, then $L(\cdot, f)^p$ satisfies* (sh_C) *for some $C = C(K, n, p)$.*

Proofs. *Proof of Theorem 1.* Let F satisfy (sh_K) for some $K \geq 1$, and let $p < 1$. (If $p > 1$, we apply Jensen's inequality.) By considering the functions $x \to F(a + rx)$, defined on the unit ball if $B_r(a) \subset D$, we see that the proof reduces to proving that

$$F(0)^p \leq C \int_B F^p dm \tag{1}$$

provided that $B \subset D$, where C depends only on K, p and n. In proving this we can also assume that the closed unit ball is in D and

$$(i) \qquad \int_B F^p dm = 1.$$

Since F is locally bounded, then we can choose $a \in B$ so that

$$(ii) \qquad F(x)^p (1 - |x|)^n \leq 2F(a)^p (1 - |a|)^n \quad \text{for all} \quad x \in B.$$

Let $r = (1 - |a|)/2$. It follows from (sh_K) that

$$(iii) \qquad F(a)(1 - |a|)^n \leq 2^n K \int_{B_r(a)} F^p F^{1-p} dm.$$

On the other hand, it follows from (ii) that $F(x)^p \leq 2^{n+1} F(a)^p$ for $x \in B_r(a)$ and therefore, by (i) and (iii),

$$F(a)(1 - |a|)^n \leq C_1 F(a)^{1-p},$$

where C_1 depends only on K, p and n. Hence, by (ii) $(x = 0)$,

$$f(0) \leq 2F(a)^p (1 - |a|)^n \leq 2C_1,$$

which was to be proved. □

Remark. This proof is similar, but simpler, to the proof of Lemma 2.4 of [4], where the complex hyperbolic space was considered.

Proof of Theorem 2. Let $F = |f|$, where f satisfies (h_K) for some $K > 0$, and let $p > 0$. As in the case of Theorem 1, it suffices to prove (1) under the assumption that the closed unit ball is contained in D. Assuming (i) we choose $a \in B$ so that there holds (ii). Then we use the inequality

$$|f(a + h) - f(a)| \leq |h| \sup_{r<1} L(a + rh, f), \tag{2}$$

which is proved by the standard compactness argument, to find that

$$F(a) \leq F(x) + t \sup_{B_t(a)} L(\cdot, f) \quad \text{for} \quad x \in B_t(a) \subset B.$$

From this and (h_K) it follows that

$$F(a) \le F(x) + K(t/r) \sup_{B_s(a)} F \qquad (s = t + r).$$

Now choose t and r so that $s = t + r = (1 - |a|)/2$ and $K(t/r) = 2^{-1-(n+1)/p}$. Since, by (ii), $F(x)^p \le 2^{n+1} F(a)$ for $x \in B_s(a)$, we get that

$$F(a) \le F(x) + (1/2)F(a) \ \text{ for } \ x \in B_t(a) \subset B_s(a),$$

whence

$$F(a)^p \le 2^p F(x)^p \ \text{ for } \ x \in B_t(a).$$

Integrating this inequality over $B_t(a)$ we obtain

$$t^n F(a)^p \le 2^p \int_{B_t(a)} F^p dm \le 2^p.$$

Since $t = c(1 - |a|)$ ($c =$ const.), we finally get

$$F(0)^p \le 2(1 - |a|)^n F(a)^p \le 2^{p+1} c^{-n},$$

and this completes the proof. □

Proof of Theorem 3. Let f satisfy (h_K^+). By theorem 1, it suffices to prove that, for some q and C, the function $L(\cdot, f)^q$ satisfies (sh_C), which is reduced to proving that

$$L(0, f)^q \le C \int_B L(x, f)^q dm(x)$$

provided that $B \subset D$. Since the function $f - f(0)$ satisfies (h_{2K}), we have, by Theorem 2,

$$L(0, f) \le C_1 \int_B |f(x) - f(0)| dm(x).$$

On the other hand, it follows from (2) and the hypotheses of the theorem that f satisfies a Lipschitz condition on balls and therefore the functions $r \to f(a + rh)$ are absolutely continuous. Hence

$$|f(x) - f(0)| \le |x| \int_0^1 L(rx, f)\, dr.$$

Combiming these estimates we get

$$\int_B |f(x) - f(0)| dm(x) \le \int_0^1 dr \int_B L(rx, f) |x|\, dm(x).$$

Hence, by the change $x = y/r$ and Fubini's theorem,

$$L(0,f) \le C_1 \int_B L(y,f)\,dm(y) \int_{|y|}^{1} r^{-n-1}|y|\,dr$$
$$\le C_1 n^{-1} \int_B L(y,f)|y|^{1-n} dm(y).$$

Now the requred inequality is proved by Hölder's inequality with the indices $q = 2n-1$ and $q' = (2n-2)/2(n-1)$, and by using the fact that the function $y \to |y|^{1-n}$ belongs to the space $L^{q'}(B, dm)$. □

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(Received 16 06 1994)