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ON THE NUMBER OF 2-FACTORS IN RECTANGULAR LATTICE GRAPHS

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Abstract. Let $f_m(n)$ and $h_m(n)$ denote the number of 2-factors and the number of connected 2-factors (Hamiltonian cycles) respectively in a $(m-1)\times(n-1)$ grid i.e. in the labelled graph $P_m \times P_n$. We show that for each fixed m $(m > 1)$ the sequences $f_m = (f_m(2), f_m(3), \dots)$ and $h_m = (h_m(2), h_m(3), \dots)$ satisfy difference equations (linear, homogeneous, and with constant coefficients). Furthermore, a computational method is given for finding these difference equations together with the initial terms of the sequence. The generating functions of f_m and h_m are rational functions $\mathcal{F}_m$ and $\mathcal{H}_m$ respectively, and they are given explicitly for some values of m.

1. Introduction. There are exactly three 2-factors (spanning 2-regular subgraphs) of $P_3 \times P_4$ as shown in Fig. 1. Note that two of them are Hamiltonian cycles (connected 2-factors).

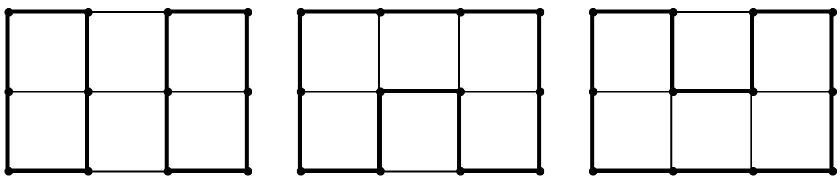

Fig. 1

The approach taken here is to fix m and then find a way to calculate the sequences $f_m = (f_m(2), f_m(3), \dots)$ and $h_m = (h_m(2), h_m(3), \dots)$. Using the so-called *transfer matrix method* [**5**] it will be shown that the sequences f_m and h_m

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satisfy difference equations (linear, homogeneous, and with constant coefficients) i.e. their generating functions

$$\mathcal{F}_m(x) = \sum_{n=0}^{\infty} f_m(n)x^n \quad \text{and} \quad \mathcal{H}_m(x) = \sum_{n=0}^{\infty} h_m(n)x^n$$

(we can take $f_m(0) = f_m(1) = h_m(0) = h_m(1) = 0$) represent rational functions, say $\mathcal{F}_m(x) = \mathcal{P}_m(x)/\mathcal{Q}_m(x)$ with $\mathcal{P}_m, \mathcal{Q}_m$ relatively prime polynomials with integer coefficients and $\mathcal{H}_m(x) = \mathcal{R}_m(x)/\mathcal{S}_m(x)$ with $\mathcal{R}_m, \mathcal{S}_m$ relatively prime polynomials with integer coefficients too and $\mathcal{Q}_m(0) = \mathcal{S}_m(0) = 1$. Algorithms are given for calculating these polynomials and the rational functions $\mathcal{F}_m$(x) are given for $m = 2, \ldots, 7$ and the functions $\mathcal{H}_m$(x) for $m = 2, \ldots, 6$. This makes it easy to calculate $f_m(n)$ and $h_m(n)$ for these values of m, and a few tables are given.

Mathematical considerations

In the labelled graph $P_m \times P_n$ (cartesian product of two paths with m and n vertices, respectively), there are $(m-1) \times (n-1)$ cycles of order 4 (squares), called *windows* (since they look like the windows in an $m \times n$ window frame).

With the graph $P_m \times P_n$ we can associate its *window lattice graph* $W_{m,n}$ whose vertices are the windows of $P_m \times P_n$ (Fig. 2), two vertices being adjacent in $W_{m,n}$ iff the two windows of $P_m \times P_n$ which correspond to those vertices have a common edge. We denote by $w_{i,j}$ $(i = 1, \ldots, m-1;\ j = 1, \ldots, n-1)$ vertices of $W_{m,n}$ as shown in Fig. 2. Obviously, the window lattice graph $W_{m,n}$ associated with $P_m \times P_n$ is isomorphic to the graph $P_{m-1} \times P_{n-1}$.

It is easy to prove the following statement:

$P_m \times P_n$ $(m, n > 1)$ *has a* 2*-factor (Hamiltonian cycle) iff the number of vertices is even.*

We associate with each 2-factor of $P_m \times P_n$ $(m > 2)$ a binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ defining its elements in the following way (Fig. 4):

$$a_{i,j} = \begin{cases} 1, & \text{if } w_{i,j} \text{ belongs to interiors of an odd number} \\ & \text{of cycles of that 2-factor;} \\ 0, & \text{otherwise.} \end{cases}$$

This matrix satisfies the following necessary conditions which are easy to verify:

- *Adjacency of Column Conditions:*
  $(\forall j)(1 \le j \le n-2)$

$$\neg(a_{1,j} = a_{1,j+1} = 0 \quad \vee \quad a_{m-1,j} = a_{m-1,j+1} = 0) \tag{1}$$

$$(\forall i)(1 \le i \le m-2)(\forall j)(1 \le j \le n-2)$$

$$(a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}) \notin \tag{2}$$

$$\{(0,0,0,0),(1,1,1,1),(1,0,0,1),(0,1,1,0)\} \tag{3}$$

- *First and Last Column Conditions:*

$$a_{1,1} = a_{m-1,1} = a_{1,n-1} = a_{m-1,n-1} = 1 \tag{4}$$

$$(\forall i)(1 \le i \le m-2)$$

$$\neg(a_{i,1} = a_{i+1,1} = 0 \quad \vee \quad a_{i,n-1} = a_{i+1,n-1} = 0) \tag{5}$$

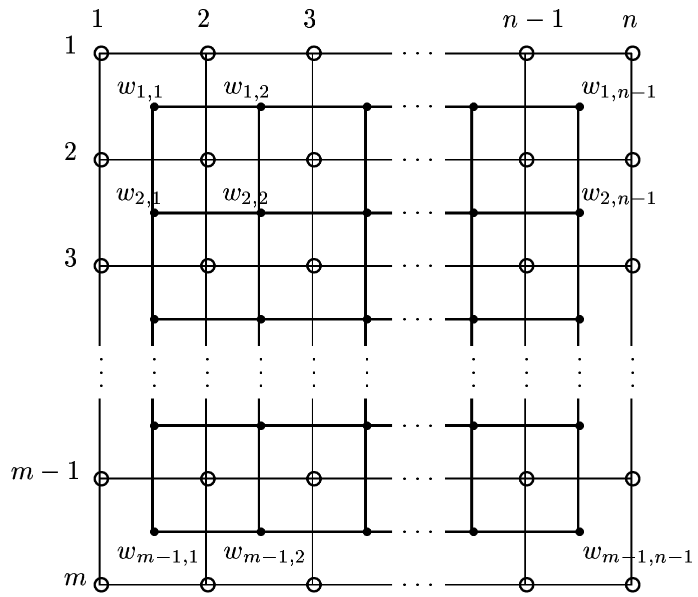


Fig. 2 The labelled graph $P_m \times P_n$ (thin lines) associated with the labelled graph $W_{m,n}$ (thick lines).

Condition (1) says that the first and the last rows do not have two adjacent 0's; the condition (5) says the same things for columns; the condition (4) says that the corners are 1's; and the condition (2) says that 2 by 2 submatrices given in Fig. 3 are forbidden.

0	0
0	0

1	1
1	1

1	0
0	1

0	1
1	0

Fig. 3

If the 2-factor is connected as well, then the window lattice graph $W_{m,n}$ of this matrix satisfies the following condition too:

• *Root Condition:* Each connected component outside a Hamiltonian cycle has a tree structure (we call it *exterior tree (ET)*) with one square $w_{i,j}$ (we call it *root of the exterior tree*) on the edge of the rectangle but not at a corner i.e.

$$(i \in \{1,(m-1)\} \ \wedge \ j \notin \{1,(n-1)\}) \ \vee \ (j \in \{1,(n-1)\} \ \wedge \ i \notin \{1,(m-1)\}) \quad (6)$$

The converse is also satisfied: every binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satisfies adjacency of column conditions and first and last column conditions, determines exactly one 2-factor of the graph $P_m \times P_n$; every such matrix A which satisfies root condition as well determines a connected 2-factor.

Using these conditions some new values of $h_m(n)$ were obtained in [3], but, that algorithm is very slow because it generates each binary matrix which fulfills (1)–(4) and the root condition, one by one.

Enumeration of 2-factors. Let A be an arbitrary binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satisfies adjacency of column conditions and first and last column conditions We create for each number m $(m > 2)$ a graph G_m in the following way: the set of vertices $V(G_m)$ consists of all possible columns in the matrix A (note that it is not the set of all binary words in $\{0,1\}^{m-1}$); a line joins a vertex v to a vertex u $(u, v \in V(G_m))$ iff the vertex v (as a binary word) might be previous column for the vertex u (as a binary word).

The subset of vertices which consists of all possible first (last) columns in the matrix A is called the set of the *emphasized* vertices.

So, in this way, the problem of the enumeration of all 2-factors in $P_m \times P_n$ is reduced to the enumeration of all walks of the length $(n-2)$ in the graph G_m with emphasized initial and final vertices.

Enumeration of Hamiltonian cycles. The values of $h_4(n)$ and $h_5(n)$ were studied in [**1**] and [**2**]. In [**4**] a recurrence relation is given for the sequences h_6 using a new characterization of the Hamiltonian cycles in $P_m \times P_n$. It enables us to determine a special digraph D_m for each number m. In this way, the enumeration of all connected 2-factors in $P_m \times P_n$ is reduced to enumeration of all oriented walks of the length $(n-2)$ in the digraph D_m with the initial and final vertices in the special sets. In [4], the following definition was introduced:

Definition 1. Two windows $w_{i,l}$ and $w_{j,s}$ which satisfy: $a_{i,l} = 0$, $a_{j,s} = 0$ and $l, s \leq k$ are said to be *Surely In the Same Exterior Tree at the k-th level* (i.e. *in relation k-SISET*) iff they belong to the same component in the subgraph of $W_{m,n}$ which is induced by set of all windows $w_{p,t}$ which satisfy $a_{p,t} = 0$ and $t \leq k$.

The relation k-SISET represents a RST-relation in the set of all windows $w_{i,k}$ which satisfy $a_{i,k} = 0$ $(1 \leq i \leq m-1)$ for a fixed k. There are at most $\lceil (m-1)/2 \rceil$ classes of the RST-relation. (It is possible that two different classes belong to the same ET but we can't conclude that if we know only the first k column of the matrix A.) Further, every class belongs to exactly one ET, so it can be in relation k-SISET with at most one root.

Let C denote the set $\{2, 3, \ldots, \lceil (m-1)/2 \rceil\}$. Now, for each Hamiltonian cycle, we associate with binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satysfies adjacency of column conditions, the first and last column conditions and the root condition, the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$, $b_{i,j} \in C \cup \{0,1\}$ in the following way (Fig. 5):

(a) $b_{i,j} = 1$ iff $a_{i,j} = 1$ $(1 \leq i \leq (m-1))$ $(1 \leq j \leq (n-1))$;

(b) if the window $w_{i,j}$ is the root of an ET and ($i = 1$ or $i = (m-1)$ or $j = 1$) then $b_{i,j} = 0$;

(c) if the window $w_{i,j}$ isn't a root of an ET but it is in relation j-SISET with a root then $b_{i,j} = 0$;

(d) we associate with the remaining windows some elements of C considering the ordinal numbers of remaining classes in the fixed j-th column. (Till now, we associated 0's with some classes in the fixed column ((b) and (c)). Thus, the first of the remaining classes in the fixed column (from above) is associated with number 2, the second one with number 3, etc.

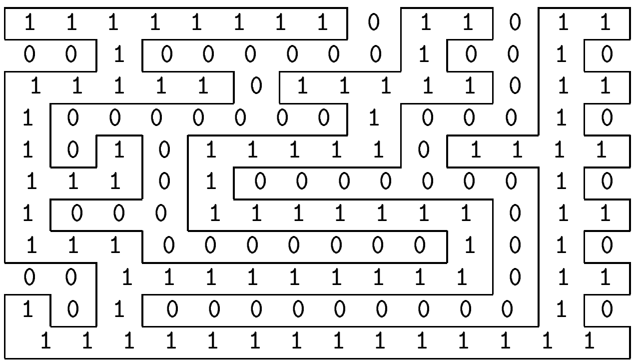


Fig. 4

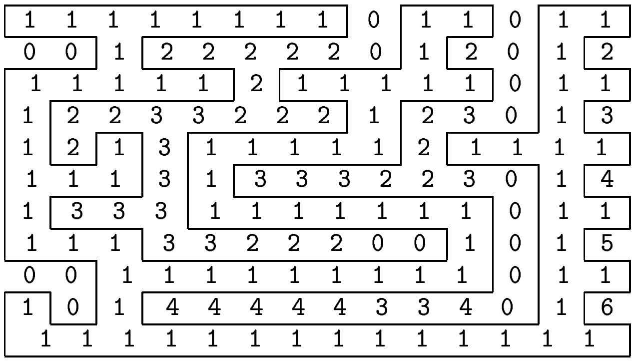


Fig. 5

We need a few additional definitions:

Definition 2. The *base* of the integer word $d_1d_2\dots d_{m-1}$ is the binary word $\bar{d}_1\bar{d}_2\dots\bar{d}_{m-1}$ where

$$\bar{d}_i = \begin{cases} 1, & \text{if} \quad d_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Definition 3. The *base* of the integer matrix $[d_{i,j}]$ is the binary matrix $[\bar{d}_{i,j}]$ of the same format where

$$\bar{d}_{i,j} = \begin{cases} 1, & \text{if} \quad d_{i,j} = 1 \\ 0, & \text{otherwise} \end{cases}$$

Definition 4. A subword u of v such that all letters of u are equal b is said to be a *b-subword* of v. A b-subword u of v is a *maximal b-subword* of v if u is not a proper subword of any b-subword of v.

From the definition of the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$, we can easily obtain the following properties of that matrix:

1. The base of the matrix B i.e. matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ satisfies adjacency of column conditions ((1) and (2)) and first and last column conditions ((4) and (5)).

2. The first column is equal to its base i.e.

$$(\forall i)(1 \le i \le m-1)(b_{i,1} = a_{i,1})$$

3. The last, $(n-1)$-th column doesn't contain any 0s, and if the number p of all 1's is less than $(m-1)$, then the word obtained from $(n-1)$-th column by removing all 1's is the word $23\dots(m-p)$.

4. For every k-th column $(2 \leq k \leq n-1)$ of the matrix B it is satisfied:

   (a) $b_{1,k} = a_{1,k}$; $b_{m-1,k} = a_{m-1,k}$.

   (b) If $b_{i,k} \neq 1$ $(2 \leq i \leq m-2)$ then $b_{i-1,k} \in \{b_{i,k}, 1\}$ and $b_{i+1,k} \in \{b_{i,k}, 1\}$. (Two windows belonging to the same class must be associated with the same number.)

   (c) If $b_{i,k-1} = 0$ $(2 \leq i \leq m-2)$ then $b_{i,k} \in \{0, 1\}$. (If the window $w_{i,k-1}$ is in relation $(k-1)$-SISET with a root (i.e. $b_{i,k-1} = 0$) then it is in relation k-SISET with the same root, as well; and if it is in relation k-SISET with $w_{i,k}$ (i.e. $a_{i,k} = 0$) then $w_{i,k}$ must be in relation k-SISET with the same root.)

   (d) For each number $b \in C$ which appears in the $(k-1)$-th column there must be a window $w_{i,k-1}$ with $b_{i,k-1} = b$ and $b_{i,k} \neq 1$. (There are no ET without root.)

   (e) For each p and l such that $p \neq l$, $2 \leq p, l \leq m-1$, where $b_{p,k-1} = b_{l,k-1} \neq 1$ and $a_{p,k} = a_{l,k} = 0$ we have $b_{p,k} = b_{l,k}$. (If $w_{p,k-1}$ and $w_{l,k-1}$ are in relation $(k-1)$-SISET and $a_{p,k} = a_{l,k} = 0$ then the windows $w_{p,k}$ and $w_{l,k}$ must be in relation k-SISET.)

   (f) If $b_{i,k-1} = b_{j,k-1} \in C$ and $b_{i,k} = b_{j,k} = b \neq 1$ $(i \neq j,\ 2 \leq i, j \leq m-2)$ then there is no sequence of consecutive appearances of number $b \in C \cup \{0\}$ (i.e. b-subword) in the k-th column which contains both $w_{i,k}$ and $w_{j,k}$. (In the opposite case, we would get a cycle in a ET.)

   (g) For every maximal 0-subword v in the k-th column, exactly one of the following two conditions are fulfilled:

      I v is adjacent to exactly one 0-window from the $(k-1)$-th column or contains exactly one of the elements $w_{1,k}$ and $w_{m-1,k}$ (Fig. 6a).

      II There is exactly one sequence $v = v_1, v_2, \ldots, v_p$ $(p \geq 1)$ of different maximal 0-subwords in the same column satisfying the following condition:

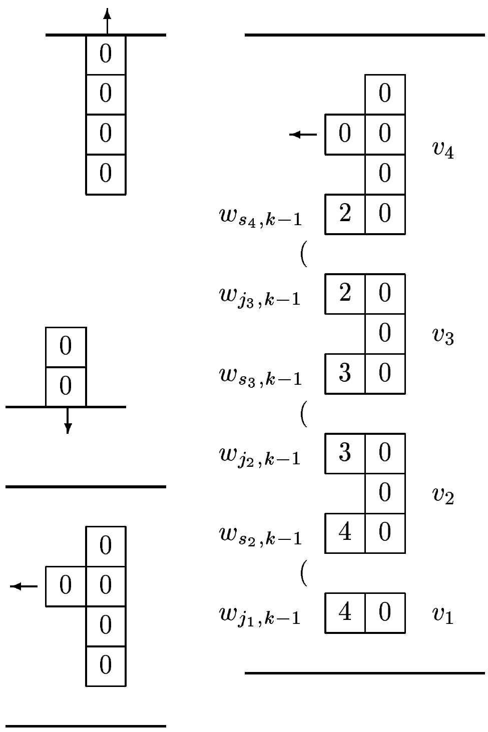
         for every i $(1 \leq i \leq (p-1))$, there is exactly one $w_{j_i,k-1}$ with $b_{j_i,k-1} \in C$ for which $w_{j_i,k} \in v_i$, and there is exactly one $w_{s_{i+1},k-1}$ with $b_{s_{i+1},k-1} \in C$ for which $w_{s_{i+1},k} \in v_{i+1}$ and $b_{j_i,k-1} = b_{s_{i+1},k-1}$;

         the p-th sequence v_p is either adjacent with exactly one 0-th window from the $(k-1)$-th column or contains exactly one of the windows $w_{1,k}$ and $w_{m-1,k}$ (Fig. 6b).

   (h) If v and u are two different maximal b-subwords $b \in C$ in the k-th column (i.e. if we can conclude by knowing the first k columns that v and u are in the same ET), then there is exactly one sequence $v = v_1, v_2, \ldots, v_p = u$ of p $(p > 1)$ different maximal b-subwords in the k-th column which satisfies: for every i $(1 \leq i \leq p-1)$ there is exactly one $w_{j_i,k-1}$ with $b_{j_i,k-1} \in C$ for which $w_{j_i,k} \in v_i$ and there is exactly one $w_{s_{i+1},k-1}$ with $b_{s_{i+1},k-1} \in C$ for which $w_{s_{i+1},k} \in v_{i+1}$ and $b_{j_i,k-1} = b_{s_{i+1},k-1}$ (Fig. 7).

(i) Consider the windows with the first appearances of elements from the set C in the k-th column from above (from $w_{1,k}$ to $w_{m-1,k}$). Let them be $w_{p_1,k}, w_{p_2,k}, \ldots, w_{p_l,k}$ $(l < \lceil (m-1)/2 \rceil)$. Then, $b_{p_i,k} = i+1$. (This follows from definition of matrix B) .

Conversely, it can be easily proved that every matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$ with elements from the set $C \cup \{0,1\}$ which satisfies 1–4 determines exactly one Hamiltonian cycle in the graph $P_m \times P_n$ i.e. the base of the matrix $B = [b_{i,j}]_{(m-1)\times(n-1)}$ fulfills the root condition and also the adjacency of column conditions and the first and last column conditions.



a) b)

Fig. 6

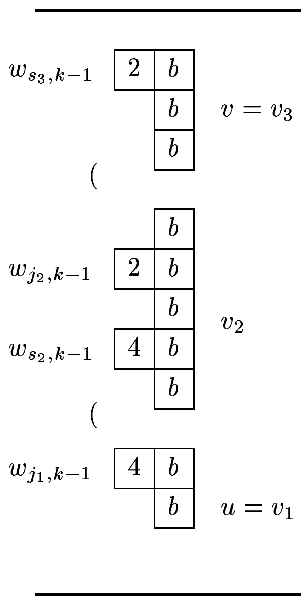


Fig. 7

Now, we can create for each number m $(m \geq 3)$ a digraph D_m in the following way: the set of vertices $V(D_m)$ consists of all possible columns in the matrix B (integer words $d_1 d_2 \dots d_{m-1}$ of the alphabet $C \cup \{0,1\}$); a (directed) line joins the vertex v with the vertex u $(v, u \in V(D_m))$ i.e. $v \to u$ iff the vertex v (as an integer word $b_{1,k-1} b_{2,k-1} \dots b_{m-1,k-1}$) might be the previous column for the vertex u (as a word $b_{1,k} b_{2,k} \dots b_{m-1,k}$) i.e. these words satisfy conditions 1 and 4.

The subset of $V(D_m)$ which consists of all possible first columns in the matrix B (conditions 1 and 2) will be called the set of the *emphasized* vertices. The subset of $V(D_m)$ which consists of all possible last columns in the matrix B (conditions 1 and 3) will be called the set of the *last* vertices. Note that these two subsets of $V(D_m)$ have exactly one element in common (the word $11 \dots 1$). So, in this way, our problem of enumeration of all Hamiltonian cycles in $P_m \times P_n$ is reduced to enumeration of all oriented walks of the length $(n-2)$ in the digraph D_m with the *emphasized* initial vertices and the *last* final vertices. For every m $(m \geq 3)$ we can create a digraph D_m using the properties of the matrix B.

Now, we continue these considerations in order to obtain better results than the ones in [**4**]. It follows from these considerations, in both cases, that there is a one-to-one correspondence between (oriented) walks of length $(n-2)$ which begin at the emphasized vertices and end at the (last) emphasized vertices in (D_m) G_m.

Now we use a well-known result from graph theory. Namely, the number of walks of lenght $(n-2)$ from vertex i to vertex j in a directed graph with vertex set $\{1, \dots, h\}$ is the (i,j)-entry in M_m^n, where $M_m = [M_{ij}]_{h \times h}$ is the incidence matrix of the (di)graph (D_m) G_m. It is a simple consequence of the Cayley-Hamilton Theorem that (h_m) f_m satisfies a difference equation. In fact, from the coefficients of the characteristic polynomial of M_m we obtain the coefficients for a difference equation satisfied by (h_m) f_m.

Let $c_0, c_1, \dots, c_p$ denote the coefficients of the difference equation satisfied by (h_m) f_m with $c_0 = 1$. Thus,

$$\sum_{i=0}^{p} c_i f_m(n-i) = 0 \quad (n \geq p).$$

Computational results

On the base of previous considerations we wrote computer programs for computation of the incidence relations and matrices M_m, incidence matrices of the (di)graph (D_m) G_m. Generation of the vertices (as (integer) binary words of the lenght $(m-1)$ begins from emphasized vertices.

We used the symmetry of some couples of words from sets $V(G_m)$ $(V(D_m))$ in order to simplify computations, by reducing incidence matrix M_m to the matrix $M_m^{'}$ such that the coefficients of its characteristic polynomial the coefficients of a difference equation satisfied by f_m (h_m).

The matrix $M_m^{'}$ is the incidence matrix of the (di)graph $G_m^{'}$ $(D_m^{'})$ obtained by contracting the vertices corresponding to the couples of symmetric words or identical rows in M_m. The contraction of the vertices, for instance v and w , is performed as follows: we reorient all lines going to v $(x \to v)$ to the vertex w $(x \to w)$ and then remove the vertex v. We will say that it is the *contraction of the vertices v and w by removing vertex v.*

The dimension of these reduced matrices $M_m^{'}$ for some values of m are given in Tab. 1 and Tab. 2.

m	3	4	5	6	7
$\mid V(G_m) \mid$	4	6	15	20	56
$\mid V(G_m^{'}) \mid$	3	5	9	14	31

Tab. 1

m	3	4	5	6	7
$\mid V(D_m) \mid$	3	6	19	32	113
$\mid V(D_m^{'}) \mid$	2	4	7	15	43

Tab. 2

We used Leverrier's method to obtain the characteristic polynomial of $M_m^{'}$:

$$g(x) = x^p + c_1 x^{p-1} + c_2 x^{p-2} + \ldots + c_{p-1} x + c_p.$$

We can get the generating function $\mathcal{U}(x)/\mathcal{V}(x)$ for the sequences corresponding to the $M_m^{'}$ in the following way:

$$\mathcal{V}(x) = x^p g(1/x) = 1 + c_1 x + c_2 x^2 + \cdots + c_p x^p$$

$$\mathcal{U}(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_{p+1} x^{p+1};$$

where

$$\begin{array}{l} u_0 = u_1 = 0; \;\; u_2 = f_m(2) \\ u_{i+2} = f_m(i+2) + \sum_{j=1}^{i} c_j f_m(i-j+2), \quad 1 \le i \le p-1. \end{array} \tag{7}$$

(In the case of connected 2-factors it is necessery to put h_m insteed of f_m.)

$m \backslash n$	2	3	4	5	6	7	8	9	10
2	1	1	2	3	5	8	13	21	34
3	1	0	3	0	9	0	27	0	81
4	2	3	18	54	222	779	2953	10771	40043
5	3	0	54	0	1140	0	24360	0	521064
6	5	9	222	1140	13903	99051	972080	7826275	71053230
7	8	0	779	0	99051	0	13049563	0	1729423756

Tab. 3 Values of $f_m(n)$, $2 \le m \le 7$, $2 \le n \le 10$

$m \backslash n$	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	1	0	2	0	4	0	8	0	16
4	1	2	6	14	37	92	236	596	1517
5	1	0	14	0	154	0	1696	0	18684
6	1	4	37	154	1072	5320	32675	175294	1024028
7	1	0	92	0	5320	0	301384	0	17066492
8	1	8	236	1696	32675	301384	4638576	49483138	681728204

Tab. 4 Values of $h_m(n)$, $2 \le m \le 8$, $2 \le n \le 10$

$$\mathcal{F}_2(x) = x^2 \frac{[1]}{[1,-1,-1]} \qquad \mathcal{F}_3(x) = x^2 \frac{[1]}{[1,0,-3]}$$

$$\mathcal{F}_4(x) = x^2 \frac{[2,-1,-2,1]}{[1,-2,-7,2,3,-1]} \qquad \mathcal{F}_5(x) = x^2 \frac{[3,0,-18,0,15]}{[1,0,-24,0,57,0,-26]}$$

$$\mathcal{F}_6(x) = x^2 \frac{[5,-11,-84,101,353,-256,-399,200,135,-45,-19,3,1]}{[1,-4,-54,67,479,-264,-1171,517,928,-397,-217,73,23,-4,-1]}$$

$$\mathcal{F}_7(x) = x^2 \frac{[8,0,-725,0,20295,0,-261639,0,1772203,0,-6715082,0,}{[1,0,-188,0,8462,0,-160189,0,1535495,0,-8158979,0,25253651,0,}$$

$$\frac{14790582,0,-19244327,0,14597627,0,-6125795,0,1266517,0,-97104]}{-46589758,0,51364132,0,-33102019,0,11793011,0,-2068475,0,131784]} .$$

Tab. 5. The generating functions $\mathcal{F}_m(x) = \sum_{n=0}^{\infty} f_m(n)x^n$, $2 \le m \le 7$.
Polynomials $c_0 + c_1x + c_2x^2 + \ldots + c_px^p$ are written as $[c_0, c_1, c_2, \ldots, c_p]$

$$\mathcal{H}_2(x) = x^2 \frac{[1]}{[1,-1]} \qquad \mathcal{H}_3(x) = x^2 \frac{[1]}{[1,0,-2]}$$

$$\mathcal{H}_4(x) = x^2 \frac{[1]}{[1,-2,-2,2,-1]} \qquad \mathcal{H}_5(x) = x^2 \frac{[1,0,3]}{[1,0,-11,0,0,0,-2]}$$

$$\mathcal{H}_6(x) = x^2 \frac{[1,-1,3,-24,24,-3,0,3,-15,9,4,-2,1]}{[1,-5,-14,63,-12,-90,35,66,-118,8,82,-42,-28,4,-2]} .$$

Tab. 6. The generating functions $\mathcal{H}_m(x) = \sum_{n=0}^{\infty} h_m(n)x^n$, $2 \le m \le 6$.
Polynomials $c_0 + c_1x + c_2x^2 + \ldots + c_px^p$ are written as $[c_0, c_1, c_2, \ldots, c_p]$

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