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IN PURSUIT OF COLORED CARATHÉODORY-BÁRÁNY THEOREMS

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Abstract. One of the shortest and the most elegant proofs of the well known Tverberg theorem, due to Karanbir Sarkaria, relies on a generalization of Carathéodory's theorem found by Imre Bárány. Our objective is to set up the stage for studying other Carathéodory-Bárány type results which should open a possibility for similar proofs of related Tverberg type statements. Among the Tverberg type results which motivate this study are colored Tverberg theorems, [**2**], [**6**], [**15**], [**20**] and more recent results related to the Tverberg-Vrećica conjecture, [**17**], [**18**].

Introduction

In every party of six persons there are at least three of them knowing each other or at least three persons who haven't met before. This simple observation is an instance of the well known Ramsey theorem. Note that this is a purely combinatorial statement which can be classified and placed into the extremal theory of finite sets, a branch of combinatorics. Infinite analogs and relatives of this theorem were one of the favorite subjects of the late professor Djuro Kurepa whose enthusiasm and contributions to mathematics inspired generations of his students. If the sets are geometric objects, say finite families of points, lines etc., and if the relations among them have a clear geometric meaning, then we deal with the extremal combinatorial geometry. A noble example of a result which belongs to this area is the Tverberg theorem. In its simplest nontrivial form it claims that every collection C of 7 points in the plane can be partitioned into three nonempty, disjoint sets, $C = C_1 \cup C_2 \cup C_3$, so that $\mathrm{conv}(C_1) \cap \mathrm{conv}(C_2) \cap \mathrm{conv}(C_3) \neq \emptyset$. Tverberg type theorems occupy one of the central places in extremal combinatorial geometry, [**5**], [**8**], [**10**], [**16**], [**19**]. The old problem about three houses and three wells, which goes back to Euler and in a popular form illustrates nonplanarity of the graph $K_{3,3}$, also can be seen as a nonlinear relative of the Tverberg theorem, [**19**].

Majority of known Tverberg type results are proved by topological methods. A recent proof of the original Tverberg theorem due to K. Sarkaria which is based on

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a Bárány's extension of Carathéodory theorem, opens a possibility of proving other "linear" Tverberg type statements by elementary methods. The main objective of our note is to formulate and illustrate by examples a general guiding principle for formulating such Carathéodory-Bárány statements.

As usual, $[q]$ is the set $[q] = \{1, \dots, q\}$, $K * L$ is the join of two simplicial complexes K and L, $K^{*(m)} = K * \dots * K$ is a multiple join of m copies of K, while $K_\delta^{*(m)}$, the m^{th} deleted join of K, is the subcomplex of $K^{*(m)}$ consisting of simplices $\sigma_1 * \dots * \sigma_m$ such that $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$.

1. From Carathéodory-Bárány to Tverberg theorem

We start with a proof of Tverberg's theorem which is a variation on a theme introduced by Sarkaria, [**12**]. The proof is based on an extension of Carathéodory theorem found by Bárány, [**1**].

THEOREM 1. (Bárány [**1**]) *If simplices $\sigma_0, \sigma_1, \dots, \sigma_n$ in n-dimensional affine space R^n have a point in common, then this point is also contained in a simplex of the form $\operatorname{conv}\{x_i\}_{i=0}^n$ where x_i is one of the vertices of the simplex σ_i, $i = 0, \dots, n$. Moreover for one of the indices i_0, the vertex x_{i_0} can be prescribed in advance.*

COROLLARY 1. *Let $\sigma_0, \sigma_1, \dots, \sigma_{n-k}$ be a collection of simplices in R^n and D a k-dimensional affine subspace of R^n. If $D \cap \sigma_i \neq \emptyset$ for all $i = 0, \dots, n-k$, then $\operatorname{conv}\{x_i\}_{i=0}^{n-k} \cap D \neq \emptyset$ for some choice $x_i \in \sigma_i$, where x_i is one of the vertices of the simplex σ_i.*

THEOREM 2. (Tverberg [**13**]) *Every subset $S \subset R^d$ of size $(q-1)(d+1)+1$ admits a partition $S = S_1 \cup \dots \cup S_q$ into nonempty, disjoint pieces such that $\bigcap_{i=1}^q \operatorname{conv}(S_i) \neq \emptyset$.*

Proof. Let $S = \{a_i\}_{i \in \Lambda} \subset R^d$, $\Lambda := \{0, 1, \dots, (q-1)(d+1)\}$, be a collection of points in a d-dimensional euclidean space R^d. This space can be isometrically embedded in a $(qd+q-1)$-dimensional euclidean space $W \cong R^{qd+q-1}$ in q different ways so that the corresponding copies $L_1, L_2, \dots, L_q$ of R^d are in general position, specially they do not intersect. Moreover, it can be assumed that there is a linear map A of W so that $A^q = \mathbb{1}$ and $L_k = A^{k-1}(L_1)$ for all $k = 1, 2, \dots, q$.

There is a convenient way of constructing these spaces. Let

$$E = \{e_1, e_2, \dots, e_{q(d+1)}\}$$

be the usual orthonormal base in the euclidean space $R^{q(d+1)}$ and let $x_1, \dots, x_{q(d+1)}$ be the dual basis of linear forms. A hyperplane W is defined by

$$W := \{v \in R^{q(d+1)} \mid (x_1 + \dots + x_{q(d+1)})(v) = 1\}.$$

Let E_k, $k = 1, \dots, q$, be the partition of E defined by

$$E_k := \{e_j \mid (k-1)(d+1) + 1 \leq j \leq k(d+1)\}.$$

Alternatively, E can be viewed as the set of vertices of a regular $[q(d+1)-1]$-dimensional simplex in W and sets $\mathrm{conv}(E_k)$ are seen as its d-dimensional disjoint faces. The isometry A acts on the basis E by permuting cyclically these faces. More precisely, $A : R^{q(d+1)} \to R^{q(d+1)}$ is defined by $A(e_j) := e_{j+d+1}$ where indices are added modulo $q(d+1)$. Let $L_i := \mathrm{aff}(E_i)$ be the affine hull of E_i.

The ambient space $R^{q(d+1)}$, the isometry A and the collection of affine subspaces L_i, $i = 1, \dots, q$ form our basic background picture for the proof of Tverberg's theorem. As a Z_q-representation space defined by the action of A, $R^{q(d+1)}$ has a very simple structure. It splits as a sum of $d+1$ regular representations, in particular the subspace of fixed points has dimension $d+1$ and consists of vectors of the form $v + A(v) + \dots + A^{q-1}(v)$. The hyperplane W is invariant with respect to this action and its subspace D of fixed points is d-dimensional and consists also of vectors of the form $v + A(v) + \dots + A^{q-1}(v)$ where v can be chosen from the space L_1.

Let us identify the original space R^d with L_1 so the original set $S \subset R^d$ can be seen as a subset of L_1. Let $\{\sigma_j \mid 0 \le j \le (q-1)(d+1)\}$ be the collection of simplices in W defined by $\sigma_j := \mathrm{conv}\{a_j, A(a_j), \dots, A^{q-1}(a_j)\}$. Obviously $\sigma_j \cap D \neq \emptyset$ for all indices j, so by Corollary 1 there exists a function $\nu : \Lambda \to \{1, \dots, q\}$ such that

$$\mathrm{conv}\{A^{\nu_j}(a_j) \mid 0 \le j \le (q-1)(d+1)\} \cap D \neq \emptyset \,. \tag{1}$$

We notice first that the function ν is an epimorphism since (1) would not be true in the opposite case. Hence, ν defines a partition $\Delta_i := \nu^{-1}(i)$ of $\Lambda = \{0, 1, \dots, (q-1)(d+1)\}$ into q nonempty pieces and a corresponding partition of $S \subset R^d \cong L_1$. We claim that this is the desired partition of Λ. Indeed, it follows from (1) that there exists $v \in L_1$ such that

$$\frac{1}{q}(v + A(v) + \dots + A^{q-1}(v)) \in \mathrm{conv}\{A^{\nu(j)}(a_j) \mid 0 \le j \le (q-1)(d+1)\} \,.$$

By construction $A^k(v) \in L_{k+1} := A^k(L_1)$ for all $k = 0, \dots, q-1$. Obviously,

$$\mathrm{conv}\{A^{\nu(j)}(a_j) \mid 0 \le j \le (q-1)(d+1)\} \subset \mathrm{conv}\left(\bigcup_{i=1}^{q} L_i\right) .$$

Since the general position assumption guarantees that the topological join $L_1 * \dots * L_q$ coincides with the convex hull $\mathrm{conv}(\bigcup_{i=1}^q L_i)$ we observe that

$$A^k(v) \in \mathrm{conv}\{A^{\nu(j)}(a_j) \mid 0 \le j \le (q-1)(d+1)\} \cap A^k(L_1) = A^k(\mathrm{conv}\{a_j \mid \nu(j) = k\}) \,.$$

From here it follows that $v \in \mathrm{conv}\{a_j \mid \nu(j) = k\}$ for all k i.e. $v \in \bigcap_{i=1}^q \Delta_i$ which means that $\{\Delta_i\}_{i=1}^q$ is a desired partition.

2. The guiding principle

The proof of Tverberg theorem given in section 1 raises a hope that other Carathéodory-Bárány type statements might be useful in similar situations. One can argue that, aside from Sarkaria's beautiful idea to use Bárány's theorem, a driving force of this proof is an attempt to realize the simplicial complex

$$\Omega(d,q) := (\{\text{point}\}^{*(\lambda)})^{*(q)}_{\delta},\ \lambda = (q-1)(d+1)+1 \tag{2}$$

which usually arises in the context of Tverberg's theorem (see e.g. [**11**], [**19**]) inside a natural ambient vector space. We will use this as a guiding example to formulate a reasonably general *guiding principle* for finding other Carathéodory-Bárány type results. The relevant ideas can be summarized as follows. A topological proof of a Tverberg type result is typically based on a carefully chosen simplicial complex K which encodes all candidates for a desired partition and which we call a configuration space. This space is always invariant with respect to a symmetry group which is usually a cyclic group. For example in the case of the configuration space $\Omega(d,q)$ described in (2) the symmetry group is Z_q. The second property of the configuration space is its high connectedness, namely if K is equivariantly mapped to a space R^k then K is supposed to be at least $(k-1)$-connected. The importance of this condition is clearly visible from the following well-known statement which is often used in extremal combinatorial geometry.

THEOREM 3. *Let K be a finite simplicial complex equipped with a free action of a cyclic group Z_q. Let R^k be a k-dimensional euclidean space also equipped with a linear action of Z_q which is free on $R^k \setminus \{0\}$. If K is $(k-1)$-connected then for every equivariant map $f : K \to R^k$ the corresponding set of zeros is nonempty, $f^{-1}(0) \neq \emptyset$.*

Example 1. Let $K = [3]^{*(3)} := [3] * [3] * [3]$ be a 2-dimensional complex where Z_3 acts by permuting each of the three element sets [3]. This complex is 1-connected which means that for every continuous map $f : K \to R^2$, $0 \in \text{Image}(f)$ where Z_3 acts on R^2 in the obvious manner.

The significance of this example becomes clear if we observe that it implies the following very special case of Theorem 1. For any three equilateral triangles $\sigma_1, \sigma_2, \sigma_3$ centered at the origin of R^2, there exists a triangle formed by choosing a vertex from each of triangles σ_i which also contains the origin. Indeed, it is enough to define $f : [3]^{*(3)} \to R^2$ as a linear map which maps a copy of $[3] := \{1,2,3\}$ on the vertices of the corresponding equilateral triangle. By preserving the connectedness condition but relaxing the symmetry condition in the spirit of Carathéodory-Bárány theorem, we arrive at the following general principle.

THE GUIDING PRINCIPLE. *Let K be a $(k-1)$-connected simplicial complex equipped with a free, simplicial action of the cyclic group Z_q. Let $\omega : K \to K$ be the generator of this action. Let R^k be a space where the group Z_q acts freely away from the origin $0 \in R^k$.*

If $f : K \to R^k$ is a linear (simplicial) map such that $0 \in \text{conv}\{f(\omega^j(v))\}_{j=0}^{q-1}$ for every vertex $v \in K$, then it is plausible that $0 \in \text{Image}(f)$.

Note that if (K, R^k) is a pair described in the guiding principle then, according to Theorem 3, the desired conclusion $0 \in \text{Image}(f)$ is known to hold for equivariant maps.

Definition 1. A pair of spaces (K, R^k) described in the guiding principle is called Z_q-admissible if $0 \in \mathrm{Image}(f)$ for every linear map which satisfies the condition $0 \in \mathrm{conv}\,\{f(\omega^j(v))\}_{j=0}^{q-1}$.

The following problem is motivated by the fact that if K and L are free Z_q-complexes so that K is $(k-1)$-connected and L is $(l-1)$-connected, then $K * L$ is also a free Z_q-complex which is $(k+l)$-connected.

PROBLEM 1. *Is it true that if both (K, R^k) and (L, R^l) are Z_q-admissible pairs than the pair $(K * L, R^{k+l+1})$ is also Z_q-admissible.*

Of course there is no reason to expect that *the guiding principle* will always lead to correct statements. Nevertheless, we believe that even in those cases where the guiding principle is false, it should lead to other interesting observations. For example it is possible that such examples might shed some light on the mysterious phenomenon that some combinatorial geometric result may be valid for linear maps and generally false if the maps are continuos. The following example shows that the guiding principle does predict correct statements for some configurations of vectors.

PROPOSITION 1. *Let us suppose that $V = \{v_{ij} \mid i \in [3], j \in [5]\}$ is a collection of vectors in the plane so that the origin $0 \in R^2$ is in the convex hull of each of the triangles* $\mathrm{conv}\{v_{1i}, v_{2i}, v_{3i}\}$, $i = 1, \ldots, 5$.

$$\begin{matrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \end{matrix} \tag{3}$$

Then there exist three different indices α, β, γ so that $0 \in \mathrm{conv}\{v_{1\alpha}, v_{2\beta}, v_{3\gamma}\}$.

Proof. Let us assume that the collection $V = \{v_{ij} \mid 1 \le i \le 3, 1 \le j \le 5\} \subset R^2$ is in general position. Hence, for every four element subset $T \subset V$ either exactly two out of four triangles with vertices in T contain the origin or none of these triangles has this property. We call this the "2 or 0"-property of the set T. Let us refer to T as an A-set in case there are exactly two triangles containing the origin and call it a B-set otherwise. Let us assume that the proposition is false. By Theorem 1 applied on the first three columns, with an application of the "2 or 0"-property if necessary and taking the symmetry into account, we can assume that $0 \in \mathrm{conv}\{v_{11}, v_{12}, v_{23}\}$. Hence $\{v_{11}, v_{12}, v_{23}, v_{34}\}$ is an A-set so $0 \in \mathrm{conv}\{v_{11}, v_{12}, v_{34}\}$. So, $\{v_{11}, v_{12}, v_{34}, v_{25}\}$ is also an A-set and $0 \in \mathrm{conv}\{v_{11}, v_{12}, v_{25}\}$. This way, step by step, we conclude that $0 \in \mathrm{conv}\{v_{11}, v_{12}, v_{ij}\}$ for all (i,j) in the region diagonally opposite to $\{(1,1),(1,2)\}$ i.e. for all $2 \le i \le 3$ and $3 \le j \le 5$. A "horizontal edge" $\{v_{i,j}, v_{i,j+1}\}$ will be called a YES-edge if it has an analogous property i.e. if $0 \in \mathrm{conv}\{v_{i,j}, v_{i,j+1}, v_{\mu,\nu}\}$ for all pairs (μ,ν) such that $\mu \ne i$ and $\nu \notin \{j, j+1\}$. Note that because of the "2 or 0"-property no two YES-edges can be diagonally opposite to each other. By applying the Theorem 1 on the last three columns of the configuration and prescribing the vertex v_{35} in advance, we are led to the conclusion that $\{v_{13}, v_{14}\}$ is also a YES-edge. Similarly we conclude that all edges of the form $\{v_{1i}, v_{1j}\}$, are YES-edges where i and j are distinct elements in $\{3,4,5\}$.

This is a contradiction with the "2 or 0"-property since the set $\{v_{21}, v_{13}, v_{14}, v_{15}\}$ has at least three triangles containing the origin.

The set of the form $[m] \times [n]$ will be called a chessboard. The chessboard complex $\Delta_{m,n}$ or the complex of all non-taking rook placements is defined by $\Delta_{m,n} := [n]_\delta^{*(m)} \cong [m]_\delta^{*(n)}$, [**4**], [**15**]. Note that Proposition 1 is just the statement that the pair $(\Delta_{3,5}, R^2)$ is Z_3-admissible. In light of the Problem 1 it would be interesting to know if the pair $((\Delta_{3,5})^{*(3)}, R^8)$ is Z_3-admissible. It is known that chessboard complexes play a special role in topological proofs of colored Tverberg theorems, [**15**], [**20**]. It shouldn't be a big surprise that they implicitly show up in a conjecture which is our first candidate for a colored Carathéodory-Bárány theorem.

CONJECTURE 1. *Let* $V = \{v_{i,j,k} \mid (i,j,k) \in [3] \times [5] \times [3]\} \subset R^8$ *be a configuration of vectors in* R^8. *Let us assume that*

(a) $(\forall i \in [3])(\forall j \in [5])\ 0 \in \mathrm{conv}\{v_{i,j,1}, v_{i,j,2}, v_{i,j,3}\}$,

(b) $(\forall \nu \in [3])\ 0 \notin \mathrm{conv}\{v_{i,j,k} \mid k \neq \nu\}$.

Then there exists a set $A \subset [3] \times [5] \times [3]$ *of size* 9 *so that*

(c) $0 \in \mathrm{conv}\{v_{i,j,k} \mid (i,j,k) \in A\}$ *and*

(d) $(\forall i \in [3])$ If $(i, j_0, k_0) \neq (i, j_1, k_1)$ *are in* A *then both* $j_0 \neq j_1$ *and* $k_0 \neq k_1$.

Note that (a) is the usual assumption typical for a Carathéodory-Bárány type theorem while (b) is an extra hypothesis which narrows the class of configurations for which the conjecture is supposed to hold. Note also that (d) is nothing but the condition that A is a typical simplex in $(\Delta_{3,5})^{*(3)}$.

The following instance of the colored Tverberg theorem, proved in [**15**] by topological methods, can be deduced from Conjecture 1.

THEOREM 4. *A collection of five red, five blue and five white points in* R^3 *always contains three vertex pairwise disjoint triangles formed by points of different color which have a nonempty intersection.*

Proof. Conjecture 1 is supposed to play a role analogous to the role of the Carathéodory-Bárány theorem in the proof of Tverberg's theorem. The details are omitted since a more general proof based on the same construction will be sketched bellow in the proof of Theorem 6. We only make a remark that, given the initial configuration of colored points $S = \{w_{i,j} \mid (i,j) \in [3] \times [5]\}$, the configuration $V = \{v_{i,j,k} \mid (i,j,k) \in [3] \times [5] \times [3]\}$ is defined by $v_{i,j,k} := A^{k-1}(w_{i,j}),\ k = 1,2,3,$ where A is the isometry defined in the proof of Theorem 2.

There are two types of colored Tverberg theorems. Theorem 4 above is an example of a statement of the second type. A characteristic property of these statements, [**15**], is that the dimension of intersecting simplices is strictly less than the dimension of the ambient space. If these two dimensions are equal we have a colored Tverberg theorem of the first type, [**20**], which is exemplified by the following result of Bárány-Larman [**2**] and Jaromczyk-Świątek [**6**].

THEOREM 5. *A collection of three blue, three white and three red points in* R^2 *can always be partitioned into three,* 3*-element sets* S_1, S_2, S_3 *consisting of points* *of different color so that the corresponding triangles* $\mathrm{conv}(S_1), \mathrm{conv}(S_2), \mathrm{conv}(S_3)$ *have a nonempty intersection.*

Let us formulate a general Carathéodory-Bárány type statement which is designed to imply known colored Tverberg theorems of the first type. The reader can easily modify this statement if he is more interested in the colored Tverberg theorems of the second type.

CB(d,t,r) *Let* $S = \{v_{i,j,k} | \ (i,j,k) \in [d+1] \times [t] \times [r]\} \subset R^{(r-1)(d+1)}$ *be a configuration of vectors in a* $(r-1)(d+1)$*-dimensional space. Let us assume that*

(a) $0 \in \mathrm{conv}\{v_{i,j,k} | \, 1 \leq k \leq r\}$ *for every pair* $(i,j) \in [d+1] \times [t]$ *and*

(b) $0 \notin \mathrm{conv}\{v_{i,j,k} | \, k \neq \nu\}$ *for all* $\nu \in [r]$.

Then there exists a subset $\Xi \subset [d+1] \times [t] \times [r]$ *with the following properties:*

(c) $0 \in \mathrm{conv}\{v_{i,j,k} | \ (i,j,k) \in \Xi\}$,

(d) $(\forall i \in [d+1])$ *If* $(i, j_0, k_0) \neq (i, j_1, k_1)$ *are in* Ξ*, then both* $j_0 \neq j_1$ *and* $k_0 \neq k_1$.

Conjecture 2. *For given* d *and* r*,* $CB(d,t,r)$ *is true if* t *is large enough.*

Proposition 2. *The statement* $CB(d,2,2)$ *is true for all* d*.* $CB(d,3,2)$ *is true even if the configuration* S *of vectors in* R^{d+1} *satisfies only the assumption (a) i.e. if* $0 \in \mathrm{conv}\{v_{i,j,1}, v_{i,j,2}\}$ *for every pair* $(i,j) \in [d+1] \times [3]$*. There is a counterexample which shows that the condition (a) alone does not imply the conclusion of* $CB(d,2,2)$.

Proof. The assumption (a) permits us to assume, without loss of generality, that $v_{i,j,2} = -v_{i,j,1}$. By the assumption (b), there exists a hyperplane H such that $v_{i,1,1} \in H^+$ and $v_{i,2,2} \in H^-$ for all i where H^+ and H^- are open halfspaces determined by H. Let $a_i \in H \cap [v_{i,1,1}, v_{i,2,2}]$. Then for some choice of numbers $\epsilon_i \in \{-1,+1\}$, $0 \in \mathrm{conv}\{\epsilon_i a_i\}_{i=1}^{d+1}$ and $C(d,2,2)$ follows.

Let $\{e_i\}_{i=1}^{d+1}$ be a basis of R^{d+1}. Then the configuration $S = \{v_{i,j,k} | \ (i,j,k) \in [d+1] \times [2] \times [2]$ defined by $v_{i,j,k} := (-1)^{j+k} e_i$ shows that $CB(d,2,2)$ is not true if the condition (b) is removed.

The proof of $CB(d,3,2)$ which does not rely on the assumption (b) goes as follows. As before we can assume that $v_{i,j,1} = -v_{i,j,2}$ for all i and j. We observe that the condition of the theorem permits us to define a Z_2-equivariant linear map $f : ([3]^{*(d+1)})^{*(2)}_{\delta} \to R^{d+1}$. Since $([3]^{*(d+1)})^{*(2)}_{\delta} \cong \Delta^{d+1}_{3,2} \simeq (S^1)^{d+1} \cong S^{2d+1}$, **[15]**, **[19]**, **[20]**, the desired conclusion follows from the Borsuk-Ulam theorem.

We show now that $CB(d,t,r)$ implies a general form of Colored Tverberg's theorem.

Theorem 6. *Let* $C_1, \ldots, C_{d+1}$ *be a collection of* $(d+1)$ *disjoint sets in* R^d*, called colors, each of cardinality at least* t*. Both a subset* $S \subset \bigcup_{i=1}^{d+1} C_i$ *of size* $d+1$ *and the possibly degenerated simplex* $\mathrm{conv}(S)$ *are called rainbow sets if* $S \cap C_i \neq \emptyset$ *for all* $1 \leq i \leq d+1$*. Then* $CB(d,t,r)$ *implies that there exist* r *disjoint rainbow* *sets* S_i, $i = 1, \ldots, r$, *such that*

$$\bigcap_{i=1}^{r} \mathrm{conv}(S_i) \neq \emptyset\,.$$

Proof. Let us take the same background picture as in the proof of Theorem 2 above. As before, the ambient space is $R^{r(d+1)}$, $W := \{v \in R^{r(d+1)} \mid (x_1 + \ldots + x_{q(d+1)})(v) = 1\}$, R^d is embedded in W in r different ways and an isometry A permutes these copies $L_1, \ldots, L_r$ of R^d. The starting copy of R^d will be identified with L_1 so the set $C = \bigcup_{i=1}^{d+1} C_i$ is viewed as a subset of L_1. This picture yields a configuration of vectors in $R^{r(d+1)}$ defined by $S' = \bigcup_{i=1}^{d+1} \bigcup_{k=1}^{r} A^{k-1}(C_i)$. This configuration is naturally indexed by three parameters, $(i, j, k) \in [d+1] \times [t] \times [r]$. The projection $\pi : R^{r(d+1)} \to R^{r(d+1)}/D$, where $D := \{v + A(v) + \ldots + A^{r-1}(v) \mid v \in R^{r(d+1)}\}$, yields a configuration $S := \{v_{i,j,k} \mid (i,j,k) \in [d+1] \times [t] \times [r]\}$ where $v_{i,j,k} := \pi(A^{k-1}(v_{i,j}))$ and $C_i = \{v_{i,j}\}_{j=1}^{t}$ is an enumeration of the "color" C_i. By construction this defines a configuration of vectors in $(r-1)(d+1)$-dimensional space which satisfies conditions of the statement $CB(d,t,r)$. Hence, there exists a subset $\Xi \subset [d+1] \times [t] \times [r]$ satisfying conditions (c) and (d) from $CB(d,t,r)$ and it is not difficult to check, following the pattern of proof of Theorem 2, that the sets $S_k := \{(i,j) \mid (i,j,k) \in \Xi\}$, $k = 1, \ldots, r$, form a desired collection of rainbow sets with the property $\bigcap \mathrm{conv}(S_i) \neq \emptyset$. The details are left to the reader.

2. Common transversals

It has been recently shown, [**17**], [**18**], that Tverberg's theorem can be extended to the case where the existence of a common point is replaced by the existence of a common affine transversal. Recall that a common k-dimensional transversal of a family $\mathcal{K}$ of subsets in R^d is a k-dimensional affine subspace L of R^d which meets all elements in $\mathcal{K}$. Specially a common 0-dimensional transversal is just a point in $\bigcap \mathcal{K}$. The proofs of these results require reasonably sophisticated topological methods so it would be interesting to find a more elementary approach. A natural idea would be to formulate the corresponding Carathéodory-Bárány statements. As before in this note, we believe that these statements are interesting in their own right and deserve an independent study.

Let us for the sake of illustration, prove one of these "common transversal" Tverberg type results.

THEOREM 7. *Every set* $C \subset R^3$ *of size* 8 *can be partitioned in four disjoint, nonempty pieces,* $C = \bigcup_{i=1}^{4} C_i$, *so that the corresponding convex hulls* $\mathrm{conv}(C_i)$ *have a common line transversal.*

Proof. The collection of all lines in R^3 is naturally seen as the total space of the canonical plane bundle over the Grassmann manifold $G_2(R^3)$ of all 2-dimensional subspaces of R^3. Indeed each line L is determined by the corresponding orthogonal subspace $L^{\perp}$ and a vector $v \in L^{\perp}$ where $v \in L \cap L^{\perp}$. Let $\pi_P : R^3 \to P$ be the orthogonal projection onto a plane $P \in G_2(R^3)$. Without loss of generality, by an approximation and compactness argument, we can assume that the set C is in general position. If we project the set $C = \{c_i\}_{i=1}^{8}$ on P, the problem of finding a line transversal with desired properties is reduced to the proof of existence of a two dimensional subspace P such that the configuration of points $\pi_P(C)$ admits a partition into four nonempty, disjoint pieces, $\pi_P(C) = A_1 \cup A_2 \cup A_3 \cup A_4$, so that $\bigcap_{i=1}^{4} \mathrm{conv}(A_i) \neq \emptyset$. Note that $\pi_P(C) = \{\pi_P(c_i)\}_{i=1}^{8}$ is a collection of eight continuous cross-sections of the canonical 2-dimensional vector bundle $E \to G_2(R^3)$. Let $D := \{c_1, c_2, c_3, c_4\}$. Then for each $P \in G_2(R^3)$ the set D admits a so called Radon partition $D = D_1^P \cup D_2^P$ so that $\mathrm{conv}(\pi_P(D_1^P)) \cap \mathrm{conv}(\pi_P(D_1^P)) \neq \emptyset$. Since by assumption D is a set of vertices of a nondegenerate tetrahedron $\mathrm{conv}(D)$, the set $\mathrm{conv}(\pi_P(D_1^P)) \cap \mathrm{conv}(\pi_P(D_1^P))$ consists of a single element $s(P)$ which continuously depends on P. Similarly the Radon partitions related to the complementary set $D^c := C \setminus D$ define a continuous cross-section $t : G_2(R^3) \to E$. Let $e : G_2(R^3) \to E$ be the cross-section defined by $e(P) := s(P) - t(P)$. Since the second Stiefel-Whitney class $w_2(E) \in H^2(G_2(R^3); Z_2)$ is nonzero, every continuous cross-section of this bundle vanishes for some $P \in G_2(R^3)$, [**9**]. Hence $s(P) = t(P)$ for some $P \in G_2(R^3)$ and the theorem follows.

Theorem 7 and many other results in [**17**] and [**18**] are not only relatives of the Tverberg theorem but have a clear relationship to "nonembeddability theorems" for simplicial complexes. The earliest and probably the best known results of this type are nonplanarity of graphs K_5 and $K_{3,3}$. The graph K_5 is the 1-skeleton σ_1^4 of a four dimensional simplex and its nonplanarity is rephrased as the fact that for every continuous map $f : \sigma_1^4 \to R^2$ there exist two disjoint simplices $\tau_1, \tau_2 \in \sigma_1^4$ with the property $f(\tau_1) \cap f(\tau_1) \neq \emptyset$. If one asks for a 3-dimensional analogue of this theorem one is led to the following nonlinear version of Theorem 7. For every continuous map $f : \sigma_2^7 \to R^3$, from the 2-skeleton of a 7-dimensional simplex, there exist three disjoint faces $\tau_1, \tau_2, \tau_3 \in \sigma_2^7$ so that $\bigcap_{i=1}^{3} f(\tau_i) \neq \emptyset$. The proof of this theorem follows the same general pattern of the proof above but needs some essentially new ideas to overcome the difficulty of dealing with multivalued sections.

What kind of generalized Carathéodory-Bárány statements correspond to these extensions of Tverberg theorem? From the point of view of [**17**], a correct translation is following. We should replace R^d by a vector bundle $R^d \to E \to B$ which is usually thought of, [9], as a family of vector spaces isomorphic to R^d parametrized by the base space B. Actually the counterpart of R^d is not the bundle itself but the linear space $\Gamma(E)$ of all continuous cross-sections. This space is infinite dimensional over R but as a module over the ring $C(B, R)$ of continuous functions on the base space it has nice properties. A counterpart of a finite configuration of $S \subset R^d$ is then a finite subset $T \subset \Gamma(E)$ i.e. a finite collection of continuous cross-sections of E. One can see T as a family of usual configurations parametrized by the parameter space B and the usual questions asked for a single configuration are translated to a question whether at least one of configurations encoded by T has a given Carathéodory-Bárány property.

References

1. I. Bárány, *A generalizations of Carathéodory's theorem*, Discrete Math. **40** (1982), 141–152.
2. I. Bárány and D.G. Larman, *A colored version of Tverberg's theorem*, J. London Math. Soc. (2) **45** (1992), 314–320.
3. I. Bárány, S.B. Shlosman, A. Szücs, *On a topological generalization of a theorem of Tverberg*, J. London Math. Soc. (2) **23** (1981), 158–164.
4. A. Björner, L. Lovász, S. Vrećica, and R. Živaljević, *Chessboard and matching complexes*, J. London Math. Soc. (2) **49** (1994), 25–39.
5. J. Eckhoff, *Helly, Radon and Carathéodory type theorems*, Handbook of Convex Geometry (P.M. Gruber and J.M. Wills, ed.), vol. A, North-Holland, Amsterdam, 1993, pp. 389–448.
6. J.W. Jaromczyk and G.Światek, *The optimal constant for the colored Tverberg's theorem*, Geom. Dedicata.
7. Ž. Kovijanić, *A proof of Bárány's theorem*, Publ. Inst. Math. Belgrade **55** (1994), 47–50.
8. J. Matoušek, *Topological methods in Combinatorics and Geometry*, Lecture notes, Prague, (preprint).
9. J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton Univ. Press, 1974.
10. J. Pach (Ed.), *New Trends in Discrete and Computational Geometry*, Algorithms and Combinatorics 10, Springer-Verlag, 1993.
11. K. Sarkaria, *A generalized van Kampen-Flores theorem*, Proc. Amer. Math. Soc. **111** (1991), 559–565.
12. ______, *Tverberg's theorem via number fields*, Israel J. Math. **79** (1992), 317–320.
13. H. Tverberg, *A generalization of Radon's theorem*, J. London Math. Soc. **41** (1966), 123–128.
14. H. Tverberg, S. Vrećica, *On generalizations of Radon's theorem and the ham sandwich theorem*, Europ. J. Combinatorics **14** (1993), 259–264.
15. S. Vrećica, R. Živaljević, *New cases of the colored Tverberg theorem*, Jerusalem Combinatorics '93 (H. Barcelo, G. Kalai, eds.), Contemporary mathematics, A.M.S., Providence, 1994.
16. R. Živaljević, *Topological methods*, CRC handbook of discrete and computational geometry (J.E. Goodman, J. O'Rourke, eds.), in preparation.
17. ______, *Tverberg-Vrećica problem and combinatorial geometry on vector bundles*, submitted.
18. ______, *Combinatorial geometry on vector bundles*, in preparation.
19. ______, *User's guide to equivariant methods in Combinatorics*, in preparation.
20. R. Živaljević, S. Vrećica, *The colored Tverberg's problem and complexes of injective functions*, J. Combin. Theory, Ser. A **61 (2)**, 309–318.

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