PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 57 (71), 1995, 135–142
Đuro Kurepa memorial volume

SOME REMARKS ON GENERALIZED MARTIN'S AXIOM

Z. Spasojević

Abstract. Let GMA denote that if $\mathbb{P}$ is well-met, strongly ω_1-closed and ω_1-centered partial order and $\mathcal{D}$ a family of $< 2^{\omega_1}$ dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ which meets every member of $\mathcal{D}$. The consistency of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$ was proved by Baumgartner [**1**] and in [**13**] many of its consequences were considered. In this paper we give a consequence and present an independence result. Namely, we prove that, as a consequence of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$, every $\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some (ω_2, ω_2)-gap and show that the existence of an ω_2-Kurepa tree is consistent with and independent of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$.

1. Introduction. With the discovery of Martin's Axiom [**8**] and its many consequences a number of set-theorists considered the problem of generalizing Martin's Axiom to higher cardinals. Their aim actually was to generalize the consequences of MA to higher cardinals. One of the first generalizations of Martin's Axiom is due to Baumgartner [**1**] and one of the strongest generalizations is due to Shelah [**9**]. We will return to Shelah's version in the last section but now we state Baumgartner's result. A partial order $\mathbb{P}$ is well-met if any two compatible elements in $\mathbb{P}$ have the greatest lower bound. We denote compatibility of $p, q \in \mathbb{P}$ by $p \not\perp q$ and their incompatibility by $p \perp q$. $\mathbb{P}$ is ω_1-closed if any decreasing ω-sequence in $\mathbb{P}$ has a lower bound and it is strongly ω_1-closed if the greatest lower bound exists for any such sequence. $\mathbb{P}$ is centered if any finite sub-collection of $\mathbb{P}$ has a lower bound and it is ω_1-centered if it is a union of ω_1 many centered partial orders. Baumgartner [**1**] constructed a model for

$$(BA) \qquad 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$$

and thus obtained the consistency of one of the first versions of Generalized Martin's Axiom. In fact, Baumgartner considered a somewhat bigger class of partial orders, but in this paper we will only consider partial orders which are well-met, strongly ω_1-closed, ω_1-centered and of size $< 2^{\omega_1}$, where 2^{ω_1} is computed in the final model.

Many consequences of (BA) were considered in [**13**]. The object of this paper is to present one more consequence and an independence result. We show that every

AMS Subject Classification (1991): Primary 03E35

$\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some (ω_2, ω_2)-gap (see §2 for notation and terminology). As usual, this result will be obtained by applying GMA to suitably chosen partial orders. It will be fairly straight-forward to show that this partial order is well-met and strongly ω_1-closed. Somewhat harder will be to show that it is ω_1-centered. For this we first need to recall the notion of a complete embedding.

Definition 1.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. An $i: \mathbb{P} \to \mathbb{Q}$ is a complete embedding if

(*a*) $\forall p, p' \in \mathbb{P}(p' \leq p \to i(p') \leq i(p))$,
(*b*) $\forall p, p' \in \mathbb{P}(p' \perp p \leftrightarrow i(p') \perp i(p))$,
(*c*) $\forall q \in \mathbb{Q} \exists p \in \mathbb{P} \forall p' \in \mathbb{P}(p' \leq p \to i(p') \not\perp i(p))$.

We also recall a result from [**13**].

Proposition 1.2. Assume $2^\omega = \omega_1$. Then any countable support iteration of length $\leq 2^{\omega_1}$ with well-met, strongly ω_1-closed and ω_1-centered partial orders yields an ω_1-centered partial order.

At this stage we also point out that $2^\omega = \omega_1$ is assumed throughout this paper. Now, let $\mathbb{P}$ be a partial order which is well-met and strongly ω_1-closed and suppose that all the conditions in $\mathbb{P}$ are countable. To show that $\mathbb{P}$ is ω_1-centered, it suffices to exhibit a sequence $\langle \mathbb{P}_\xi : \xi \leq \alpha \leq 2^{\omega_1} \rangle$ of sub-orders of $\mathbb{P}$ such that $\mathbb{P}_\alpha = \mathbb{P}$ and each $\mathbb{P}_\xi$, for $\xi < \alpha$, is well-met, strongly ω_1-closed and ω_1-centered, as well as a sequence $\langle i_{\xi\eta} : \xi \leq \eta \leq \alpha \rangle$, with $i_{\xi\eta} : \mathbb{P}_\xi \to \mathbb{P}_\eta$, of complete embeddings such that $\forall \xi, \eta, \theta(\xi \leq \eta \leq \theta \leq \alpha \to i_{\xi\theta} = i_{\eta\theta} \circ i_{\xi\eta})$. Then $\mathbb{P}$ can be viewed as a countable support iteration of length $\alpha \leq 2^{\omega_1}$ with well-met, strongly ω_1-closed and ω_1-centered partial orders so that by Proposition 1.2 $\mathbb{P}$ is also ω_1-centered.

It is well known that $2^\omega = \omega_1$ implies the existence of an ω_2-Aronszajn tree (see [**6**]). The results of Laver and Shelah [**7**] and Shelah and Stanley [**10**] show that the existence of an ω_2-Suslin tree is consistent with and independent of (BA). In the final section we consider the influence of (BA) on the existence of ω_2-Kurepa trees. Our result is that the existence of such trees is consistent with and independent of (BA).

2. Gaps. Let κ^κ be the set of all function from κ to κ. If $f, g \in \kappa^\kappa$ then $f \leq^* g$ if and only if $\exists n < \kappa \forall i < \kappa (i \geq n \to f(i) \leq g(i))$ and $f(i) < g(i)$ on a set of size κ. A (κ^+, κ^+)-pregap in $(\kappa^\kappa, \leq^*)$ is a pair (a, b) where $a = \langle a_\xi : \xi < \kappa^+ \rangle$ and $b = \langle b_\xi : \xi < \kappa^+ \rangle$ are subsets of κ^κ such that $\forall \xi, \eta < \kappa^+ (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \kappa^+ (a_\xi \leq^* a_\eta \wedge b_\eta \leq^* b_\xi)$. If there is a $c \in \kappa^\kappa$ such that $\forall \xi, \eta < \kappa^+ (a_\xi \leq^* c \leq^* b_\eta)$ then c splits the pregap (a, b). If no such c exists then (a, b) is a (κ^+, κ^+)-gap.

Hausdorff [**4**] showed (in ZFC) that $(\omega^\omega, \leq^*)$ contain an (ω_1, ω_1)-gap. Herink [**5**] and independently Blaszczyk and Szymanski [**2**] generalized Hausdorff's result to higher cardinals by proving that if κ is a regular cardinal then $(\kappa^\kappa, \leq^*)$ contains a (κ^+, κ^+)-gap. Hausdorff's result was refined in [**11**] by showing that MA implies that every $\leq^*$-increasing ω_1-sequence in $(\omega^\omega, \leq^*)$ is a lower half of some (ω_1, ω_1)-gap. And this last result was further improved in [**12**] by establishing that $\mathfrak{t} > \omega_1$ is in fact equivalent to the statement that every $\leq^*$-increasing ω_1-sequence in $(\omega^\omega, \leq^*)$ is a lower half of some (ω_1, ω_1)-gap. The goal of this section is to show that (BA) implies that every $\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some (ω_2, ω_2)-gap and thus refine the results of Herink [**5**] and Blaszczyk and Szymanski [**2**].

Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$. A $\leq^*$-decreasing ω_2-sequence $b = \langle b_\xi : \xi < \omega_2 \rangle$ on top of a, such that (a, b) is an (ω_2, ω_2)-gap, will be obtained from an application of GMA to a suitably defined partial order $\mathbb{P}_a$. In order to guarantee that (a, b) is in fact a gap, the elements of the sequences a and b have to satisfy the following condition:

$$(\star) \quad \forall \xi < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\xi(i)) \wedge \forall \xi, \eta < \omega_2 (\xi < \eta \to \exists i < \omega_1 (b_\xi(i) < a_\eta(i))).$$

This condition is a refinement of the following condition due to Kunen for (ω_1, ω_1)-gaps in $(\omega^\omega, \leq^*)$ (unpublished work):

$$\forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i)) \qquad \text{and}$$
$$\forall \xi, \eta < \omega_1 (\xi \neq \eta \to \exists i < \omega (a_\xi(i) \not\leq b_\eta(i) \vee a_\eta(i) \not\leq b_\xi(i))).$$

Now we show that if $2^\omega = \omega_1$ then every (ω_2, ω_2)-pregap in $(\omega_1^{\omega_1}, \leq^*)$ satisfying $(\star)$ is in fact a gap.

LEMMA 2.1. *Assume $2^\omega = \omega_1$ and let $(a, b) = \langle a_\xi, b_\xi : \xi < \omega_2 \rangle$ be an (ω_2, ω_2)-pregap in $(\omega_1^{\omega_1}, \leq^*)$ whose elements satisfy $(\star)$. Then (a, b) is a gap.*

Proof. By way of contradiction, assume (a, b) is split by $c: \omega_1 \to \omega_1$. Then

$$(\circ) \qquad \forall \xi < \omega_2 \exists n_\xi < \omega_1 \forall n \geq n_\xi (a_\xi(n) \leq c(n) \leq b_\xi(n)).$$

By a first thinning process we may assume that $\forall \xi < \omega_2 (n_\xi = m)$, for some fixed $m < \omega_1$. Since $2^\omega = \omega_1$ and m is a countable ordinal, we have $\mid \omega_1^m \mid = \omega_1$. Hence, by another thinning process we may assume that

$$(\bullet) \qquad \forall \xi, \eta < \omega_2 (a_\xi \restriction m = a_\eta \restriction m \wedge b_\xi \restriction m = b_\eta \restriction m).$$

But then $(\circ)$,$(\bullet)$ and the first clause of $(\star)$ imply that $\forall \xi, \eta < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\eta(i))$, which contradicts the second clause of $(\star)$. Hence, (a, b) is a gap and the Lemma is proved. □

Therefore, the definition of $\mathbb{P}_a$ has to incorporate the requirements in $(\star)$.

Definition 2.2. Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$.

$$\begin{aligned}\mathbb{P}_a = \{\langle x, y, n, s \rangle : & f x, y \in [\omega_2]^{<\omega_1} \wedge n < \omega_1 \wedge s : y \to \omega_1^n \wedge \\ & f \forall \xi \in y ((\xi \in x \to \forall i < n (a_\xi(i) \leq s(\xi)(i))) \wedge \\ & f \forall \eta \in x (\eta > \xi \to \exists i < n (s(\xi)(i) < a_\eta(i))))\}\end{aligned}$$

where $\langle x_2, y_2, n_2, s_2 \rangle \leq \langle x_1, y_1, n_1, s_1 \rangle$ if and only if

(1) $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2$,

(2) $\forall \xi \in y_1(s_2(\xi) \restriction n_1 = s_1(\xi))$,

(3) $\forall \xi, \eta \in y_1 \forall i < \omega_1(\xi \leq \eta \wedge n_1 \leq i < n_2 \to s_2(\eta)(i) \leq s_2(\xi)(i))$,

(4) $\forall \xi \in x_1 \forall \eta \in y_1 \forall i < \omega_1(n_1 \leq i < n_2 \to a_\xi(i) \leq s_2(\eta)(i))$.

Clearly $\mathbb{P}_a$ is a partial order and the next step is to show that $\mathbb{P}_a$ is well-met, strongly ω_1-closed and ω_1-centered so that GMA can be applied to it.

So let $\langle x_1, y_1, n_1, s_1\rangle, \langle x_2, y_2, n_2, s_2\rangle \in \mathbb{P}_a$ and suppose $\langle u, v, k, t\rangle \in \mathbb{P}_a$ is their lower bound. We may assume that $u = x_1 \cup x_2$ and $v = y_1 \cup y_2$. Then there is the least m such that $\max(n_1, n_2) \leq m \leq k$ and $\langle u, v, m, t \restriction m\rangle \in \mathbb{P}_a$, where $t \restriction m$ is a function with domain v such that $\forall \xi \in v((t \restriction m)(\xi) = t(\xi) \restriction m)$. Then it is easily seen that $\langle u, v, m, t \restriction m\rangle$ is the greatest lower bound of $\langle x_1, y_1, n_1, s_1\rangle$ and $\langle x_2, y_2, n_2, s_2\rangle$ so that $\mathbb{P}_a$ is well-met.

Now let $\langle x_0, y_0, n_0, s_0\rangle \geq \langle x_1, y_1, n_1, s_1\rangle \geq \cdots$ be a decreasing ω-sequence in $\mathbb{P}_a$. Let $u = \bigcup_{i<\omega} x_i$, $v = \bigcup_{i<\omega} y_i$, $m = \sup_{i<\omega}(n_i)$ and let t be a function with domain v such that $\forall \xi \in v(t(\xi) = \bigcup\{s_i(\xi): \xi \in y_i\})$. Then $\langle u, v, m, t\rangle$ is the greatest lower bound in $\mathbb{P}_a$ of the above sequence so that $\mathbb{P}_a$ is strongly ω_1-closed.

As indicated in §1, to show that $\mathbb{P}_a$ is ω_1-centered it suffices to show that there is a sequence $\langle \mathbb{P}_\alpha: \alpha \leq \omega_2\rangle$ of sub-orders of $\mathbb{P}_a$ such that $\mathbb{P}_{\omega_2} = \mathbb{P}_a$ and a sequence $\langle i_{\alpha\beta}: \alpha \leq \beta \leq \omega_2\rangle$, with $i_{\alpha\beta}: \mathbb{P}_\alpha \to \mathbb{P}_\beta$, of complete embeddings such that $\forall \alpha, \beta, \gamma(\alpha \leq \beta \leq \gamma \leq \omega_2 \to i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$ and such that each $\mathbb{P}_\alpha$, for $\alpha < \omega_2$, is well-met, strongly ω_1-closed and ω_1-centered. Then $\mathbb{P}_a$ can be viewed as a countable support iteration of length ω_2 with well-met, strongly ω_1-closed and ω_1-centered partial orders, since $\mathbb{P}_a$ consists of countable conditions. Then, by Proposition 1.2, $\mathbb{P}_a$ is also ω_1-centered.

For each $\alpha \leq \omega_2$ let $\mathbb{P}_\alpha = \{\langle x, y, n, s\rangle \in \mathbb{P}_a: y \subseteq \alpha\}$ and for each $\alpha \leq \beta \leq \omega_2$ let $i_{\alpha\beta}: \mathbb{P}_\alpha \to \mathbb{P}_\beta$ be the inclusion map $i(p) = p$. Then $\mathbb{P}_a = \mathbb{P}_{\omega_2}$, each $\mathbb{P}_\alpha$ is a sub-order of $\mathbb{P}_a$ with the ordering relation inherited from $\mathbb{P}_a$, and $\forall \alpha \leq \beta \leq \gamma \leq \omega_2(i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$. Analogous proof can be used to show that each $\mathbb{P}_\alpha$, for $\alpha < \omega_2$, is well-met and strongly ω_1-closed as the one used to show that $\mathbb{P}_a$ has these properties.

LEMMA 2.3. *For each $\alpha \leq \beta \leq \omega_2$, $i_{\alpha\beta}$ is a complete embedding.*

Proof. Properties (a) and (b) of Definition 1.1 are satisfied in a trivial way. For (c), let $q = \langle x_q, y_q, n_q, s_q\rangle \in \mathbb{P}_\beta$. Then $p = \langle x_q, y_q \cap \alpha, n_q, s_q \restriction (y_q \cap \alpha)\rangle$ has the required property. □

LEMMA 2.4. *Assume $2^\omega = \omega_1$. Then for each $\alpha < \omega_2$, $\mathbb{P}_\alpha$ is ω_1-centered.*

Proof. Let $\alpha < \omega_2$ and for each $y \in [\alpha]^{<\omega_1}, n < \omega_1, s \in (\omega_1^n)^y$ let $\mathbb{P}_{\alpha yns} = \{\langle x, z, m, t\rangle \in \mathbb{P}_\alpha: z = y \wedge m = n \wedge t = s\}$. Then $\mathbb{P}_\alpha = \bigcup\{\mathbb{P}_{\alpha yns}: y \in [\alpha]^{<\omega_1} \wedge n < \omega_1 \wedge s \in (\omega_1^n)^y\}$ and since $2^\omega = \omega_1$, hence $\omega_1^\omega = \omega_1$, we have that $\mathbb{P}_\alpha$ is a union of ω_1 many sub-orders. Furthermore, if $\langle x_1, y, n, s\rangle, ..., \langle x_k, y, n, s\rangle \in \mathbb{P}_{\alpha yns}$ then $\langle x_1 \cup ... \cup x_k, y, n, s\rangle \in \mathbb{P}_{\alpha yns}$ and $\langle x_1 \cup ... \cup x_k, y, n, s\rangle \leq \langle x_1, y, n, s\rangle, ..., \langle x_k, y, n, s\rangle$. Thus, each $\mathbb{P}_{\alpha yns}$ is centered so that $\mathbb{P}_\alpha$ is ω_1-centered. □

Therefore, Lemmas 2.3 and 2.4 imply that $\mathbb{P}_a$ can be viewed as a countable support iteration of length ω_2 with partial orders which are well-met, strongly ω_1-closed and ω_1-centered. Thus, by Proposition 1.2, $\mathbb{P}_a$ is also ω_1-centered.

Now we come to the main result of this section.

THEOREM 2.5 *Assume* (BA). *Then every* $\leq^*$*-increasing* ω_2*-sequence in* $(\omega_1^{\omega_1}, \leq^*)$ *is a lower half of some* (ω_2, ω_2)*-gap.*

Proof. Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing ω_2-sequence in $(\omega_1^{\omega_1}, \leq^*)$. Then by the previous results, $\mathbb{P}_a$ is well-met, strongly ω_1-closed and ω_1-centered. Let G be a filter in $\mathbb{P}_a$ and for each $\eta < \omega_2$ let

$$b_\eta = \bigcup \{ s(\eta) : \exists p \in G (p = \langle x_p, y_p, n_p, s_p \rangle \wedge s = s_p) \}.$$

Condition (4) of Definition 2.2 together with the requirement that for each $\xi, \eta < \omega_2$ and each $m < \omega_1$ the filter G has a nonempty intersection with the following dense sets

$$D_{\xi\eta m} = \{ \langle x, y, n, s \rangle \in \mathbb{P}_a : \xi \in x \wedge \eta \in y \wedge n \geq m \}$$

will guarantee that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$. In addition, condition (3) of Definition 2.2 together with the requirement that for each $\xi < \eta < \omega_2$ and each $m < \omega_1$ the filter G has a nonempty intersection with the following dense sets

$$E_{\xi\eta m} = \{ \langle x, y, n, s \rangle \in \mathbb{P}_a : \xi, \eta \in y \wedge \mid \{ i : s(\eta)(i) < s(\xi)(i) \} \mid \geq m \}$$

will guarantee that $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$. Then the total number of these dense sets $D_{\xi\eta m}$ and $E_{\xi\eta m}$ is ω_2. Therefore, to satisfy the requirements that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$ the filter G needs to intersect ω_2 dense subsets of $\mathbb{P}_a$ and by (BA) there is one such filter. In addition, the definition of $\mathbb{P}_a$ implies that $\forall \xi < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\xi(i))$ and $\forall \xi, \eta < \omega_2 (\xi < \eta \to \exists i < \omega_1 (b_\xi(i) < a_\eta(i)))$ so that (a, b) is in fact an (ω_2, ω_2)-gap in $(\omega_1^{\omega_1}, \leq^*)$. $\square$

3. Trees. It is well known that $2^\omega = \omega_1$ implies the existence of an ω_2-Aronszajn tree (see [**6**]). Since $2^\omega = \omega_1$ is a part of (BA), it follows that (BA) sattles the existance of an ω_2-Aronszajn tree. By the results of Laver and Shelah [**7**] and Shelah and Stanley [**10**] the existence of an ω_2-Suslin tree is consistent with and independent of (BA). In this section we consider the influence of (BA) on the existence of ω_2-Kurepa trees. We show that the existence of such trees is consistent with and independent of a stronger version of Generalized Martin's Axiom, due to Shelah [**9**], than the one we have considered so far. Recall that an ω_2-Kurepa tree is a tree $\mathbb{T} = (T, \leq_T)$ of hight ω_2 such that any level of $\mathbb{T}$ is of size $< \omega_2$. If $x \in T$ let $\hat{x} = \{ y \in T : y <_T x \}$. We also assume that $T = \omega_2$ and that all our trees have the following properties:

1) $\mid \mathrm{Lev}_0(\mathbb{T}) \mid = 1$,
2) $\forall \alpha < \beta < \mathrm{hight}(\mathbb{T}) \forall x \in \mathrm{Lev}_\alpha(\mathbb{T}) \exists y_1, y_2 \in \mathrm{Lev}_\beta(\mathbb{T}) (y_1 \neq y_2 \wedge x <_T y_1, y_2)$,
3) $\forall \alpha < \mathrm{hight}(\mathbb{T}) \forall x, y \in \mathrm{Lev}_\alpha(\mathbb{T}) (\mathrm{limit}\ \alpha \to (x = y \leftrightarrow \hat{x} = \hat{y}))$.

We begin by formulating Shelah's version of Generalized Martin's Axiom. A partial order $\mathbb{P}$ is ω_2-normal if $\{p_\alpha : \alpha < \omega_2\} \subseteq \mathbb{P}$ then there is a club $C \subseteq \omega_2$ and a regressive function $f: \omega_2 \to \omega_2$ such that for $\alpha, \beta \in C$ if $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \omega_1$ and $f(\alpha) = f(\beta)$ then p_α and p_β are compatible. Note that ω_2-normality is a strengthening of ω_2-Knaster condition, which states that if $\{p_\alpha : \alpha < \omega_2\} \subseteq \mathbb{P}$ then there is an $A \in [\omega_2]^{\omega_2}$ such that any two elements in $\{p_\alpha : \alpha \in A\}$ are compatible. Let GMA^* denote the statement that if $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < 2^{\omega_1}$, it is well-met, strongly ω_1-closed and ω_2-normal and $\mathcal{D}$ is a family of $< 2^{\omega_1}$ dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ meeting all the elements of $\mathcal{D}$. The following Lemma is due to Shelah [**9**].

LEMMA 3.1. *Suppose* $2^\omega = \omega_1$, $2^{<\kappa} = \kappa$ *and* κ *is a regular cardinal. Let* $\langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa\rangle, \langle \mathbb{Q}_\alpha : \alpha < \kappa\rangle\rangle$ *be a countable support iteration such that*

$$\mathbf{1} \Vdash_{\mathbb{P}_\alpha} \text{“ } \mathbb{Q}_\alpha \text{ is well-met, strongly } \omega_1\text{-closed and } \omega_2\text{-normal ”}.$$

Then $\mathbb{P}_\kappa$ *is strongly* ω_1*-closed and* ω_2*-normal.*

This Lemma is essentially all that is needed in Shelah's proof [**9**] of the consistency of

$$(SA) \qquad 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA^*.$$

This Lemma will also play a role in the analysis below.

To obtain a model for (SA) in which there is an ω_2-Kurepa tree, we start with a ground model V for $ZFC + GCH$ in which there is an ω_2-Kurepa tree. For example, the constructible universe L has this property. Then iterate, as in [**9**], to obtain a model for (SA). By Lemma 3.1 cofinalities and hence cardinals are preserved by the iteration so that any ω_2-Kurepa tree in the ground model remains an ω_2-Kurepa tree in the extension. Thus, the existence of an ω_2-Kurepa tree is consistent with (SA).

The construction of a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ in which there are no ω_2-Kurepa trees requires the existence of a strongly inaccessible cardinal and it is analogous to Devlin's construction [**3**] of a model for $2^\omega = \omega_2 + MA$ in which there are no ω_1-Kurepa trees. Therefore we only present an outline of our construction.

The construction will proceed as follows. Start with a model for $ZFC + GCH$ in which κ is a strongly inaccessible cardinal. Then collapse κ to ω_3 by the Levy collapse $\mathbb{L}_\kappa$ (see below). In the extension, there are no ω_2-Kurepa trees. Then iterate, as in [**9**], to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. We use Lemma 3.1 to show that in the final model there are no ω_2-Kurepa trees.

Now, we define the Levy collapse $\mathbb{L}_\kappa$ and present some of its properties whose proofs are standard.

Definition 3.2. $\mathbb{L}_\kappa = \{p : |p| \leq \omega_1 \wedge p \text{ is a function } \wedge \mathrm{dom}(p) \subseteq \kappa \times \omega_2 \wedge \forall(\alpha, \xi) \in \mathrm{dom}(p)(p(\alpha, \xi) \in \alpha)\}$, where $p \leq q$ if and only if $p \supseteq q$.

For $\lambda < \kappa$ let $\mathbb{L}_\lambda = \{p \in \mathbb{L}_\kappa : \mathrm{dom}(p) \subseteq \lambda \times \omega_2\}$ and $\mathbb{L}^\lambda = \{p \in \mathbb{L}_\kappa : \mathrm{dom}(p) \subseteq (\kappa \setminus \lambda) \times \omega_2\}$. then $\mathbb{L}_\lambda \times \mathbb{L}^\lambda$ is isomorphic to $\mathbb{L}_\kappa$.

LEMMA 3.3. *$\mathbb{L}_\kappa$ is ω_2-closed. If κ is strongly inaccessible, then $\mathbb{L}_\kappa$ has the κ-cc.*

LEMMA 3.4. *Let M be a countable transitive model (c.t.m.) for $ZFC+GCH$ and suppose κ is strongly inaccessible in M and G is $\mathbb{L}_\kappa$-generic over M. Then $\omega_1^{M[G]} = \omega_1^M$, $\omega_2^{M[G]} = \omega_2^M$, $\omega_3^{M[G]} = \kappa$ and, in $M[G]$, there are no ω_2-Kurepa trees.*

So, by extending with $\mathbb{L}_\kappa$, ω_1 and ω_2 remain unchanged and κ gets collapsed to ω_3 and if GCH holds in M it also holds in $M[G]$.

The idea now is to start with a model $M[G]$, as above, and iterate, as in [**9**], ω_3 times to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. But we need to know that the iteration does not introduce any new ω_2-Kurepa trees. The next two Lemmas are toward this end. We omit the proofs as the Lemmas and their proofs are the analogues of the corresponding Lemmas for ω_1-trees. The first one is the analogue of Lemma 3.6 in [**3**] and the second one is the analogue of Theorem 8.5 in [**1**].

LEMMA 3.5. *Let M be a c.t.m. for ZFC and suppose that, in M, $\mathbb{P}$ and $\mathbb{Q}$ are partial orders such that $\mathbb{P}$ is strongly ω_1-closed and ω_2-normal and $\mathbb{Q}$ is ω_2-closed. Let G be $\mathbb{P}\times\mathbb{Q}$-generic over M. Let $G_\mathbb{P} = \{p \in \mathbb{P}\colon (p,1) \in G\}$ and $G_\mathbb{Q} = \{q \in \mathbb{Q}\colon (1,q) \in G\}$. Then if $\mathbb{T}$ is an ω_2-tree in $M[G_\mathbb{P}]$ and b is an ω_2-branch of $\mathbb{T}$ in $M[G]$, then $b \in M[G_\mathbb{P}]$. In addition $\omega_1^{M[G]} = \omega_1^M$ and $\omega_2^{M[G]} = \omega_2^M$.*

LEMMA 3.6. *Suppose $\mathbb{T}$ is an ω_2-tree and $\mathbb{P}$ is strongly ω_1-closed and ω_2-normal partial order. Then forcing with $\mathbb{P}$ adds no new ω_2-branches through $\mathbb{T}$.*

Now we state and prove the main result of this section.

THEOREM 3.7. *Let M be a c.t.m. for $ZFC + GCH$ and κ strongly inaccessible in M. Then there is an extension of M which is a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ in which there are no ω_2-Kurepa trees.*

Proof. Let M be as above and G $\mathbb{L}_\kappa$-generic over M. Then, by Lemma 3.4, in $N = M[G]$ there are no ω_2-Kurepa trees and GCH still holds. In N, we perform a countable support iteration of length ω_3, as in [**9**], to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. Let $\langle\langle \mathbb{P}_\alpha\colon \alpha \le \omega_3\rangle, \langle \mathbb{Q}_\alpha\colon \alpha < \omega_3\rangle\rangle$ be such iteration and H $\mathbb{P}_{\omega_3}$-generic over N. Then $N[H]$ is a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ and we now show that there are no ω_2-Kurepa trees in $N[H]$. In N, let τ be a nice $\mathbb{P}_{\omega_3}$-name for an ω_2-tree in $N[H]$ (see [**6**] for the definition of a nice name). Since, by Lemma 3.1, $\mathbb{P}_{\omega_3}$ has ω_2-cc there is an $\alpha < \omega_3$ such that τ is in fact a $\mathbb{P}_\alpha$-name. Then H_α, the restriction of H to $\mathbb{P}_\alpha$, is $\mathbb{P}_\alpha$ generic over N. Since $\alpha < \omega_3$, the iteration is with countable supports, we are considering only partial orders of size $< \omega_3$ (i.e. $\mathbf{1} \Vdash_{\mathbb{P}_\alpha}$ " $\mid \mathbb{Q}_\alpha \mid < \check{\omega}_3 \mid$ "), GCH holds in $M[G]$, hence the density of $\mathbb{P}_\alpha$ is $< \omega_3$, we may assume that $\mid \mathbb{P}_\alpha \mid < \omega_3$. Now, in M, $\mathbb{L}_\kappa$ has the κ-cc (by Lemma 3.3), so there is some $\lambda < \kappa$ such that if G_λ is the restriction of G to $\mathbb{L}_\lambda$ then $\mathbb{P}_\alpha \in M[G_\lambda]$ and H_α is $\mathbb{P}_\alpha$-generic over $M[G_\lambda]$. Now $\mathbb{T} = \tau[G] \in M[G_\lambda][H_\alpha]$ and, by Lemma 3.5, any ω_2-branch of $\mathbb{T}$ which is in $M[G_\lambda][H_\alpha][G^\lambda]$ is already in

$M[G_\lambda][H_\alpha]$. So, in $M[G_\lambda][H_\alpha]$, $\mathbb{T}$ has at most $2^{\omega_2} = \theta$ such branches and since κ is still strongly inaccessible we have that $\theta < \kappa$. But, in $M[G_\lambda][H_\alpha][G^\lambda]$, κ is collapsed to ω_3. So $\mathbb{T}$ can have at most $\aleph_2$ many ω_2-branches in $M[G_\lambda][H_\alpha][G^\lambda]$. But $M[G_\lambda][H_\alpha][G^\lambda] = M[G_\lambda][G^\lambda][H_\alpha] = M[G][H_\alpha] = N[H_\alpha]$. So $\mathbb{T}$ has at most $\aleph_2$ many ω_2-branches in $N[H_\alpha]$. However, by Lemma 3.1, $\mathbb{P}^\alpha$ is ω_2-normal so that, by Lemma 3.6, $\mathbb{T}$ does not obtain any new ω_2-branches in the extension $N[H_\alpha][H^\alpha]$. But $N[H_\alpha][H^\alpha] = N[H]$. So $\mathbb{T}$ can not be an ω_2-Kurepa tree in $N[H]$ which proves that in the model $N[H]$ there are no ω_2-Kurepa trees. This finishes the proof of the Theorem. $\square$

Therefore, the existence of an ω_2-Kurepa tree is consistent with and independent of (SA) and hence consistent with and independent of (BA).

REFERENCES

[1] J. Baumgartner, *Iterated Forcing, in: A.R.D. Mathias, ed., Surveys in Set Theory,* London Mathematical Society Lecture Note Series **87**, 1983, 1–60.

[2] A. Blaszczyk and A. Szymanski, *Hausdorff's gaps versus normality*, Bull. Acad. Polon. Sci. **30** (1982), 371–378.

[3] K. Devlin, *$\aleph_1$-Trees*, Ann. Math. Logic **13** (1978), 267–330.

[4] F. Hausdorff, Die Graduierung nach dem Endverlauf, Abh. König. Sächs. Ges. Wiss., Math.-Physis. Kl. **31** (1909), 296–334.

[5] C. Herink, *Some Applications of Iterated Forcing*, Ph.D. Thesis, University of Wisconsin-Madison, 1977.

[6] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.

[7] R. Laver and S. Shelah, *The $\aleph_2$-Suslin hypothesis*, Trans. Amer. Math. Soc. **264** (1981), 411–418.

[8] D. Martin and R. Solovay, *Internal Cohen Extensions*, Ann. Math. Logic **2** (1970), 143–178.

[9] S. Shelah, *A weak generalization of MA to higher cardinals*, Israel J. Math. **36** (1978), 297–304.

[10] S. Shelah and L. Stanley, *Generalized Martin's axiom and Suslin's hypothesis for higher cardinals*, Israel J. Math. **43** (1982), 225–236.

[11] Z. Spasojević, *Some results on gaps*, Topology Appl. **56** (1994), 129–139.

[12] Z. Spasojević, *Gaps in $(\mathcal{P}(\omega), \subset^*)$ and $(\omega^\omega, \leq^*)$*, Proc. American Math. Soc., to appear.

[13] F. Tall, *Some applications of generalized Martin's axiom*, Topology Appl. **57** (1994), 215–248.

[14] S. Todorčević, *Trees, Subtrees and Order Types*, Ann. Math. Logic **20** (1980), 233–268.

Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel

(Received 01 07 1995)
(Revised 26 08 1995)
(Revised 28 11 1995)