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TAUBERIAN CONDITIONS AND STRUCTURES OF TAYLOR AND FOURIER COEFFICIENTS

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Dedicated to Slobodan Aljančić

Abstract. From equivalent forms of Tauberian conditions of Hardy–Littlewood [**1**] and Č.V. Stanojević [**2**] new structural information is obtained for the Taylor, respectively Fourier coefficient. Consequently new Tauberian theorems are established.

1. Introduction. Various Tauberian conditions such as Hardy–Littlewood [**1**], Schmidt [**2**], Č.V. Stanojević [**3**] and D. Grow and Č.V. Stanojević [**4**], determine the structure of Taylor coefficients of power series and Fourier coefficients of series in L^1. Of particular interest is Hardy–Littlewood Tauberian condition

$$V_n(|a|,p) = \frac{1}{n}\sum_{k=1}^{n} k^p |a_k|^p = O(1), \quad n \to \infty, \quad p > 1, \tag{1.1}$$

and its analogue in [**4**]

$$V_n(|\Delta \widehat{f}|,p) = \frac{1}{n}\sum_{|k|\le n} k^p |\Delta \widehat{f}(k)|^p = O(1), \quad n \to \infty, \quad p \in (1,2]. \tag{1.2}$$

The condition (1.1) implies the existence of a function $h \in H^r$, $r \ge 2$, such that the Taylor coefficients $\{a_n\}$ of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1, \tag{1.3}$$

are equal to the Taylor coefficients $\{b_n\}$ of h. The condition (1.2) implies the existence of a function $h \in L^r$, $r \ge 2$, such that $\Delta \widehat{f}(n) = \widehat{h}(n)$, $n \in Z$. The

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purpose of this paper is to show that there are equivalent forms of (1.1) and (1.2) which provide more structural information about Taylor and Fourier coefficients.

2. Structure of Taylore Coefficients. Consider (1.3) and denote $S_n(a) = \sum_{k=0}^{n} a_k$. A positive sequence $\{R(n)\}$ is O-regularly varying [**5,6**] if $\overline{\lim_n} \frac{R([\lambda n])}{R(n)} < \infty$, $\lambda > 1$. If $\{R(n)\}$ is O-regularly varying sequence $\{\log R(n)\}$ is slowly varying i.e. $\overline{\lim_n} \frac{R([\lambda n])}{\log R(n)} = 1$.

Theorem 2.1. *For $p \in (1,2]$ let (1.1) hold, and let*

$$\frac{1}{n}\sum_{k=0}^{n} |S_k(a)|^p = O(1), \quad n \to \infty. \tag{2.1}$$

Then: (i) *there exists $h \in H^r$, $r \geq 2$, such that*

$$S_n(a) = \frac{1}{n}\sum_{k=1}^{n} kb_k$$

where $\{b_n\}$ is the sequence of Taylor coefficients of h; and

(ii) *the sequence $\{S_n(a)/n\}$ is of bounded variation.*

Proof. As shown in [**7**], the condition (1.1) is equivalent to each of the following statements

$$\lim_{\lambda \to 1+0} \overline{\lim} \sum_{k=n+1}^{[\lambda_n]} k^{p-1}|a_k|^P < \infty, \tag{2.2}$$

$$\left\{\exp\left(\sum_{k=0}^{n} k^{p-1}|a_k|^p\right)\right\} \text{ is } O\text{-regularly varying.} \tag{2.3}$$

From the inequalities below

$$\begin{aligned} C_1 V_n(|a|,p) + C_2\frac{1}{n}\sum_{k=1}^{n}|S_k(a)|^p &\leq \frac{1}{n}\sum_{k=1}^{n}|\Delta(kS_k(a))|^p \\ &\leq C_3 V_n(|a|,p) + C_2\frac{1}{n}\sum_{k=1}^{n}|S_k(a)|^p \end{aligned}$$

and from the condition (2.1) it folows that (1.1) is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n}|\Delta(kS_k(a))|^p = O(1), \quad n \to \infty.$$

Hence, for some O-regularly varying sequence $\{R(n)\}$,

$$\sum_{k=1}^{n} \frac{|\Delta(kS_k(a))|^p}{k} = \log R(n).$$

Therefore,

$$\sum_{k=1}^{n} |\Delta(kS_k(a))|^p < \infty.$$

For $p \in (1,2]$, from F. Riesz's Theorem [**8**] it folows that there exists $h \in H^q$, $1/p + 1/q = 1$, such that $\Delta(nS_n(a)) = b_n$ where $\{b_n\}$ is the sequence of Taylor coefficients of h. Finally

$$S_n(a) = \frac{1}{n}\sum_{k=1}^{n} kb_k.$$

To prove (ii) notice that

$$\left|\frac{S_n(a)}{n} - \frac{S_{n+1}(a)}{n+1}\right| \le \frac{|b_n|}{n} + \frac{1}{n(n+1)}\sum_{k=1}^{n}|b_k|$$

where both series $\sum\limits_{n=1}^{\infty} \frac{|b_n|}{n}$ and $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\sum\limits_{k=1}^{n}|b_k|\right)$ are convergent. Thus the sequence $\{S_n(a)/n\}$ is of bounded variation.

3. Structure of Fourier Coefficients of Functions in $\mathbf{L}^1$ and Convergence of Fourier Series in $\mathbf{L}^1$-norm. For $f \in L^1(T)$, $T = \mathbb{R}/2\pi\mathbb{Z}$, let

$$\lim_{\lambda\to 1+0}\overline{\lim_{n}}\sum_{|k|\le n+1}^{[\lambda_n]} |k|^{p-1}|\Delta\widehat{f}(k)|^p < \infty, \quad p \in (1,2].$$

It was shown in [**3**] that there exsists $h \in L^q$, $1/p + 1/q = 1$, such that $\Delta\widehat{f}(n) = \widehat{h}(n)$, $n \in \mathbb{Z}$, and that the Fourier series of f converges to $f \in L^1$-norm if and only if

$$\widehat{f}(n)\log|n| = o(1), \quad |n| \to \infty.$$

The following theorem is an analogue to Theorem 2.1.

THEOREM 3.1. *Let $f \in L^1(T)$ and let* (1.2) *hold. Then:*

(i) *there exists $h \in L^q$, $1/p + 1/q = 1$, such that*

$$\widehat{f}(n) = \frac{1}{n}\sum_{|k|\le n} k\widehat{h}(k) + C_4,$$

where C_4 is an absolute constant;

(ii) *the Fourier series of f converges to f in L^1-norm if and only if*

$$\widehat{f}(n)\log|n| = o(1), \quad |n| \to \infty.$$

Proof. After the following inequalities

$$C_5V_n(|\Delta\widehat{f}|,p) + C_6\frac{1}{n}\sum_{|k|\le n}|\widehat{f}(k)|^p \le \frac{1}{n}\sum_{|k|\le n}|\Delta(k\widehat{f}(k))|^p$$
$$\le C_7V_n(|\Delta\widehat{f}|,p) + C_6\frac{1}{n}\sum_{|k|\le n}|\widehat{f}(k)|^p$$

are established, the proof of (i) follows along the lines of the proof of (i) in Theorem 2.1. The proof of (ii) requires appropriate slight modifications of the proof given in [**4**].

4. Additional Remarks. From the conclusion of Theorem 2.1 we have that

$$a_n = b_n + O\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right), \quad n \to \infty,$$

and from the conclusion of Theorem 3.1

$$\Delta\widehat{f}(n) = \widehat{h}(n) + O\left(\frac{1}{n}\sum_{k=1}^{n} \widehat{h}(k)\right), \quad n \to \infty.$$

In the above asymptotic relations we see the measure of improvements of the representations of sequences $\{a_n\}$ and $\{\Delta\widehat{f}(n)\}$ in terms of corresponding Taylor and Fourier coefficients of functions in h^q and L^q respectively.

In [**7**] the modulo of regularity of a complex sequence $\{Q(n)\}$ is defined as

$$\rho(n) = n\left(\frac{Q(n)}{Q(n+1)} - 1\right),$$

and the sequence $\{Q(n)/n\}$ is defined to be regularly varying in $(C,1)$-mean if for some $p > 1$

$$\frac{1}{n}\sum_{|k|\le n} |\rho(k)|^p = O(1), \quad n \to \infty.$$

Consequently, Theorem 2.1 and Theorem 3.1 can be reformulated in the following way.

THEOREM 4.1. *For some* $p \in (1,2]$ *let* (2.1) *hold. If*

$$n\left(\frac{nS_n(a)}{(n+1)S_{n+1}(a)} - 1\right) = O(1), \quad n \to \infty,$$

then the conclusions of Theorem 2.1 hold.

THEOREM 4.2. *For some* $p \in (1,2]$ *let* $\{\widehat{f}(n)\}$ *be* $(C,1)$*-regularly varying in mean, and let*

$$\frac{1}{n}\sum_{|k|\le n} |k||\Delta\widehat{f}(k)| = O(1), \quad n \to \infty.$$

Then the conclusions of Theorem 3.1 hold.

REFERENCES

[1] G.H. Hardy and T.E. Littlewood, *Tauberian theorems concerning power series and Dirichlelet's series whose coefficients are positive*, Proc. London Math. Soc. (2) **13** (1913/14), 174–191.

[2] R. Schmidt, *Überdivergente Folgen un lineare Mittelbildungen*, Math. Z. **22** (1924), 89–152.

[3] Č.V. Stanojević, *Structure of Fourier and Fourier–Stiltjes coefficients of series with slowly varying convergence moduli*, Bull. Amer. Math. Soc. **19**(1) (1988), 283–286.

[4] D. Grow, Č.V. Stanojević, *Convergence and the Fourier character of trigonometric transforms with slowly varying convergence moduli*, Math. Ann. **302** (1995), 433–472

[5] J. Karamata, *Remarks on the paper by V.G. Avakumović* (in Serbian), Rad Jugoslav. Akad. Znan. Umjet. **254** (1936), 187–200.

[6] V.G. Avakumović, *Sur une extensions de la condition de conergence des theorems inverses de sommabilite*, C. R. Acad. Sci. Paris **200** (1935), 1515–1517.

[7] V.B. Stanojević, *Fourier and Trigonometric Transforms With Complex Coefficients Regularly Varying in Mean, Fourier Analysis*, Lecture Notes in Pure and Applied Mathematics 157, Marcel Dekker, New York, 1994.

[8] F. Riesz, *Über eine Verrallgemeinerung des Parsevalschen Formel*, Math. Z. **18** (1923), 117–124.

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