PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 59 (73), 1996, 49–76

ESTIMATES FOR DERIVATIVES AND INTEGRALS OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF NONSELFADJOINT STURM–LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENTS (I)

Nebojša L. Lažetić

Communicated by Miroljub Jevtić

Abstract. We consider derivatives of the eigenfunctions and associated functions of the formal Sturm–Liouville operator

$$\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x)$$

defined on a finite or infinite interval $G \subseteq \mathbb{R}$. We suppose that the complex-valued potential $q = q(x)$ belongs to the class $L_1^{\rm loc}(G)$ and that piecewise continuously differentiable coefficient $p = p(x)$ has a finite number of the discontinuity points in G.

Order-sharp upper estimates are obtained for the suprema of the moduli of the first derivative of the eigenfunctions and associated functions of the operator $\mathcal{L}$ in terms of their norms in metric L_2 on compact subsetes of G (on the entire interval G).

Introduction

1. Definitions. Consider the formal Sturm–Liouville operator

$$\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x), \tag{1}$$

which is defined on an arbitrary interval $G = (a,b)$ of the real axis $\mathbb{R}$. Let $x_0 \in G$ be a point of discontinuity of the coefficient p. If we suppose that

$$p(x) = \begin{cases} p_1(x), & x \in (a, x_0), \\ p_2(x), & x \in (x_0, b), \end{cases}$$

then the following conditions are imposed on the coefficients:

AMS Subject Classification (1991): Primary 34B25
Key Words and Phrases: formal differential operator, eigenfunction, associated function

1) $p_1(x) \in \mathcal{C}^{(1)}(a, x_0]$, and $p_2(x) \in \mathcal{C}^{(1)}[x_0, b)$.

2) $p_1(x) \geq \alpha_1 > 0$ everywhere on $(a, x_0]$, and $p_2(x) \geq \alpha_2 > 0$ everywhere on $[x_0, b)$.

3) $q(x) \in L_1^{\text{loc}}(G)$ is a complex-valued function.

Definition 1. A complex-valued function $\overset{\circ}{u}_\lambda(x) \not\equiv 0$ is called an *eigenfunction of the operator* (1) *corresponding to the* (*complex*) *eigenvalue* λ ($\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$) if it satisfies the following conditions:

(a) $\overset{\circ}{u}_\lambda(x)$ is absolutely continuous on any finite closed subinterval of G.

(b) $\overset{\circ}{u}{}'_\lambda(x)$ is absolutely continuous on any finite closed subinterval of the half-open intervals $(a, x_0]$ and $[x_0, b)$.

(c) $\overset{\circ}{u}_\lambda(x)$ satisfies the differential equation

$$- \big(p_1(x)\, \overset{\circ}{u}{}'_\lambda(x)\big)' + q(x)\, \overset{\circ}{u}_\lambda(x) = \lambda\, \overset{\circ}{u}_\lambda(x) \tag{2}$$

almost everywhere on (a, x_0), and the differential equation

$$- \big(p_2(x)\, \overset{\circ}{u}{}'_\lambda(x)\big)' + q(x)\, \overset{\circ}{u}_\lambda(x) = \lambda\, \overset{\circ}{u}_\lambda(x) \tag{3}$$

almost everywhere on (x_0, b).

(d) $\overset{\circ}{u}_\lambda(x)$ satisfies the junction condition

$$p_1(x_0)\, \overset{\circ}{u}{}'_\lambda(x_0 - 0) = p_2(x_0)\, \overset{\circ}{u}{}'_\lambda(x_0 + 0).$$

Definition 2. A complex-valued function $\overset{i}{u}_\lambda(x) \not\equiv 0$ ($i = 1, 2, \ldots$) is called an *associated function* (*of the* i*–th order*) *of the operator* (1) *corresponding to the eigenfunction* $\overset{\circ}{u}_\lambda(x)$ *and the eigenvalue* λ if it satisfies the following conditions:

($a^\star$) Conditions (a), (b) and (d) of Definition 1 hold for $\overset{i}{u}_\lambda(x)$.

($b^\star$) $\overset{i}{u}_\lambda(x)$ satisfies the differential equation

$$- \big(p_1(x)\, \overset{i}{u}{}'_\lambda(x)\big)' + q(x)\, \overset{i}{u}_\lambda(x) = \lambda\, \overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{4}$$

almost everywhere on (a, x_0), and the differential equation

$$- \big(p_2(x)\, \overset{i}{u}{}'_\lambda(x)\big)' + q(x)\, \overset{i}{u}_\lambda(x) = \lambda\, \overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{5}$$

almost everywhere on (x_0, b).

1.1. Let K be any compact set of positive measure lying strictly within G. We will use the notation

$$K_R \overset{\text{def}}{=} \{\, x \in G \mid \rho(x, \overline{K}\,) \leq R \,\},$$

where $R \in (0, \rho(K, \partial G))$, and $\overline{K}$ is the intersection of all closed intervals containing K. (By $\rho(A, B)$ we denote the distance of a set $A \subset \mathbb{R}$ from a set $B \subset \mathbb{R}$.)

If $\lambda = r\, e^{i\varphi}$, then $\sqrt{\lambda} \overset{\text{def}}{=} \sqrt{r}\, e^{i\varphi/2}$, where $\varphi \in (-\pi/2, 3\pi/2]$.

2. Main theorem. We present the following results.

THEOREM 1. *(a) If* $q(x) \in L_1^{\mathrm{loc}}(G)$, *then for any compact subset* K *of the interval* G *there exist a number* $R \in (0, \rho(K, \partial G))$ *and constants* $r(K_R, \operatorname{Im}\sqrt{\lambda})$, $C_{i1}(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ *such that*

$$\sup_{x \in K} |\overset{i}{u}{}'_\lambda(x)| \leq C_{i1}(K_R, p, q, \operatorname{Im}\sqrt{\lambda}) \, \| \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{6}$$

for $0 \leq |\operatorname{Re}\sqrt{\lambda}| \leq r(K_R, \operatorname{Im}\sqrt{\lambda})$, *and*

$$\sup_{x \in K} |\overset{i}{u}{}'_\lambda(x)| \leq C_{i1}(K_R, p, q, \operatorname{Im}\sqrt{\lambda}) \, |\sqrt{\lambda}| \, \| \overset{i}{u}_\lambda \|_{L_2(K_R)} \tag{7}$$

for $|\operatorname{Re}\sqrt{\lambda}| > r(K_R, \operatorname{Im}\sqrt{\lambda})$.

(b) Let $q(x) \in L_1(G)$ *and (when the interval* G *is infinite)* $\overset{i}{u}_\lambda(x) \in L_2(G)$. *If the functions* $p_1(x)$ *and* $p_2(x)$ *are bounded together wiht their first derivatives, then there exist constants* $r(G, \operatorname{Im}\sqrt{\lambda})$ *and* $C_{i1}(G, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ *such that*

$$\sup_{x \in G} |\overset{i}{u}{}'_\lambda(x)| \leq C_{i1}(G, p, q, \operatorname{Im}\sqrt{\lambda}) \, \| \overset{i}{u}_\lambda \|_{L_2(G)} \tag{8}$$

for $0 \leq |\operatorname{Re}\sqrt{\lambda}| \leq r(G, \operatorname{Im}\sqrt{\lambda})$, *and*

$$\sup_{x \in G} |\overset{i}{u}{}'_\lambda(x)| \leq C_{i1}(G, p, q, \operatorname{Im}\sqrt{\lambda}) \, |\sqrt{\lambda}| \, \| \overset{i}{u}_\lambda \|_{L_2(G)} \tag{9}$$

for $|\operatorname{Re}\sqrt{\lambda}| > r(G, \operatorname{Im}\sqrt{\lambda})$.

Let us note that by $\overset{i}{u}{}'_\lambda(x_0)$ we mean $\overset{i}{u}{}'_\lambda(x_0 - 0)$ or/and $\overset{i}{u}{}'_\lambda(x_0 + 0)$.

Remark 1. If G is a finite interval, then condition imposed in the propositon (b) on the functions $p_1'(x)$ and $p_2'(x)$ can be replaced by the following condition: $p_1'(x) \in L_1(a, x_0)$, $p_2'(x) \in L_1(x_0, b)$.

Remark 2. It will be shown in the proof of Theorem 1 that actually "better" estimates than the ones formulated above are valid. Namely, it is possible to replace $\| \overset{i}{u}_\lambda \|_{L_2(K_R)}$ in the estimates (6)–(7) by $\max_{x \in K_{R_0}} |\overset{i}{u}_\lambda(x)|$, for some $R_0 \in (0, R)$. Moreover, if G is a finite interval, then there exists a closed interval $\tilde{K} \subset G$ such that we can replace $\| \overset{i}{u}_\lambda \|_{L_2(G)}$ and $C_{i1}(G, \cdot)$ in the estimates (8)–(9) by $\max_{x \in \tilde{K}_{R_0}} |\overset{i}{u}_\lambda(x)|$ and $C_{i1}(\tilde{K}_{R_0}, \cdot)$ respectively.

Remark 3. Let $\sigma(\mathcal{L})$ be some set of eigenvalues of the operator (1). If there exists a constant A not depending on the numbers $\lambda \in \sigma(\mathcal{L})$ and such that

$$|\operatorname{Im}\sqrt{\lambda}| \leq A, \qquad \lambda \in \sigma(\mathcal{L}), \tag{10}$$

then the constants $C_{01}(\cdot)$ and $r(\cdot)$ do not depend on the numbers λ, which means that it is possible to define them uniformly with respect to the parameter $\lambda \in \sigma(\mathcal{L})$.

If the numbers $\lambda \in \sigma(\mathcal{L})$ satisfy (10) and zero is not a limit point of the set $\{\, |\operatorname{Re}\sqrt{\lambda}| \mid \lambda \in \sigma(\mathcal{L})\,\}$, then the constants $C_{i1}(\cdot)$ $(i \in \mathbb{N})$ do not depend on these numbers, too.

Remark 4. The constants $C_{i1}(\cdot)$ $(i = 1, 2, \dots)$ actually do not depend on the order i of the associated function, which means that they can be the same for all associated functions corresponding to the specific eigenfunction.

Remark 5. Theorem 1 is an extension of known results for the formal Schrödinger operator

$$\mathcal{L}(u)(x) \;=\; -\,u''(x) \;+\; q(x)\,u(x). \tag{11}$$

Namely, in this case the estimates (6)–(9) were announced in [**8**] and proved in [**5**]. The corresponding estimates for derivatives of eigenfunctions of an arbitrary nonnegative selfadjoint extension of the operator (13) were first derived in [**3**]–[**4**].

If $G = (a, b)$ is a finite interval, then for the operator

$$\mathcal{L}(u)(x) \;=\; p(x)\,u''(x) + r(x)\,u'(x) + q(x)\,u(x),$$

with coefficients $p(x) \in W_1^2(a, x_0) \cap W_1^2(x_0, b)$, $r(x) \in W_1^1(a, x_0) \cap W_1^1(x_0, b)$ and $q(x) \in L_1(a, b)$, the global estimate (9) was announced in [**10**]. There some perturbed junction condition (on the first derivatives) at the discontinuity point $x_0 \in G$ was imposed.

Remark 6. The following example shows that the estimates (6)–(9) are best possible with respect to the order of the parameter λ. Let the operator

$$\mathcal{L}(u)(x) \;=\; -\,u''(x)$$

be defined on the interval $G = (0, 1)$, and let the eigenfunctions and associated functions of $\mathcal{L}$ satisfy the boundary conditions $u_\lambda(0) = 0$, $u'_\lambda(0) = u'_\lambda(1)$. Then $\sigma(\mathcal{L}) = \{\, \lambda_n = (2n\pi)^2 \mid n = 0, 1, 2, \dots\}$ is the set of all eigenvalues, the eigenfunctions have the form $\overset{\circ}{u}_0(x) = x$, $\overset{\circ}{u}_n(x) = \sin 2n\pi x$ $(n \in \mathbb{N})$; the associated functions corresponding to the eigenfunction $\overset{\circ}{u}_0$ do not exist, and the others have the form

$$\overset{1}{u}_n(x) \;=\; -\,x\,\frac{\cos 2n\pi x}{4n\pi}, \qquad n \in \mathbb{N}$$

(see [**1**]). It is not difficult to verify that in this case the order of parameter λ in the corresponding estimates (6)–(9) can not be improved.

Remark 7. For the sake of simplicity we have supposed that the coefficient $p(x)$ has only one point of discontinuity. But all stated results remain valid when this function has an arbitrary finite number of such points. In that case definitions 1 and 2 should be formulated in the corresponding way.

3. Estimates of eigenfunctions and associated functions. In the proof of Theorem 1 we will essentially use the following estimates for eigenfunctions and associated functions of the operator (1), which were announced in [**7**] and proved in [**9**].

LEMMA 1. *(a) If $q(x) \in L_1^{\rm loc}(G)$, then for any compact set $K \subset G$ there exist a number $R \in (0, \rho(K, \partial G))$ and constants $C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ such that*

$$\max_{x \in K} |\overset{i}{u}_\lambda(x)| \ \le\ C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\ \|\overset{i}{u}_\lambda\|_{L_2(K_R)}\,. \tag{12}$$

(b) Suppose that $q(x) \in L_1(G)$, and that $\overset{i}{u}_\lambda(x) \in L_2(G)$ if G is an infinite interval. If $p_1(x)$ and $p_2(x)$ are bounded along with their first derivatives, then there exist constants $C_i(G, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ such that

$$\sup_{x \in G} |\overset{i}{u}_\lambda(x)| \ \le\ C_i(G, p, q, \operatorname{Im}\sqrt{\lambda})\ \|\overset{i}{u}_\lambda\|_{L_2(G)}\,. \tag{13}$$

LEMMA 2. *(a) If $q(x) \in L_1^{\rm loc}(G)$, then for any compact set $K \subset G$ there exist a number $R \in (0, \rho(K, \partial G))$ and constants $A_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$, $A_i(K_R, p, q)$ $(i = 1, 2, \dots)$ such that*

$$\begin{aligned} \max_{x \in K} |\overset{i-1}{u}_\lambda(x)| \ &\le\ A_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\,|\sqrt{\lambda}| \cdot \max_{x \in K_R} |\overset{i}{u}_\lambda(x)| \quad \textit{for} \quad \lambda \ne 0\,, \\ \max_{x \in K} |\overset{i-1}{u}_\lambda(x)| \ &\le\ A_i(K_R, p, q) \cdot \max_{x \in K_R} |\overset{i}{u}_\lambda(x)| \quad \textit{for} \quad \lambda = 0\,. \end{aligned} \tag{14}$$

(b) Suppose that $q(x) \in L_1(G)$, and that $\overset{i}{u}_\lambda(x) \in L_2(G)$ if G is an infinite interval. If $p_1(x)$ and $p_2(x)$ are bounded along with their first derivatives, then there exist constants $A_i(G, p, q, \operatorname{Im}\sqrt{\lambda})$, $A_i(G, p, q)$ $(i = 1, 2, \dots)$ such that

$$\begin{aligned} \sup_{x \in G} |\overset{i-1}{u}_\lambda(x)| \ &\le\ A_i(G, p, q, \operatorname{Im}\sqrt{\lambda})\ |\sqrt{\lambda}| \cdot \sup_{x \in G} |\overset{i}{u}_\lambda(x)| \quad \textit{for}\ \lambda \ne 0\,, \\ \sup_{x \in G} |\overset{i-1}{u}_\lambda(x)| \ &\le\ A_i(G, p, q) \cdot \sup_{x \in G} |\overset{i}{u}_\lambda(x)| \quad \textit{for} \quad \lambda = 0\,. \end{aligned} \tag{15}$$

3.1. If G is a finite interval, then condition imposed on the functions $p_1'(x)$ and $p_2'(x)$ in the propositions (b) of the previous lemmas can be replaced by the following one: $p_1' \in L_1(a, x_0)$, $p_2'(x) \in L_1(x_0, b)$.

Also, the global estimate (13) may be sharpened in the following sense: If G is a finite interval, then for any closed interval $K \subset G$ there exist constants $C_i(K, p, q, \operatorname{Im}\sqrt{\lambda})$ such that

$$\sup_{x \in G} |\overset{i}{u}_\lambda(x)| \ \le\ C_i(K, p, q, \operatorname{Im}\sqrt{\lambda}) \cdot \max_{x \in K} |\overset{i}{u}_\lambda(x)|\,.$$

3.2. Having in mind the specific applications of estimates (12)–(15), we note that the constants appearing in these estimates have the following properties of independence of the parameters λ and i:

1) If condition (10) is satisfied, then it is possible to make the constants $C_0(\cdot)$ independent of the numbers $\lambda \in \sigma(\mathcal{L})$.

2) If the numbers $\lambda \in \sigma(\mathcal{L})$ satisfy (10) and zero is not a limit point of the set $\{|\operatorname{Re}\sqrt{\lambda}|\mid \lambda \in \sigma(\mathcal{L})\}$, then the constants $C_i(\cdot)$ and $A_i(\cdot)$ $(i \in \mathbb{N})$ do not depend on those numbers.

3) The constants $C_i(\cdot)$, $A_i(\cdot)$ $(i \in \mathbb{N})$ are independent of the parameter i.

As will be shown in the proof of Theorems 1, the statements from Remark 3 are actually consequences of 1)–2).

Remark 8. The constants from estimates (12) and (14) have an important property concerning the dependence on the "variable" K_R. Namely, a careful analysis of proofs of the corresponding theorems in papers [**9**] may show the following fact: Under assumptions from the proposition (b) of Lemma 1 (or Lemma 2) it is possible to define the mentioned constants in such way, that they contain only the length of the closed interval $\overline{K}$.

4. Methods and applications. The estimates (6)–(9) are obtained by a method based only on the mean-value formulas for the first derivatives of solutions of the differential equations (2)–(5) and on the mean-value formulas for these solutions. This method is a subsequent development of the method worked out in [**6**].

4.1. The estimates formulated in Theorem 1 are results of independent interest. They also play a basic role in study of the following problems concerning expansions in eigenfunctions and associated functions of the operators (1) and (11):

1) Uniform convergence on G of the first derivative of partial sum of spectral expansion (for any absolutely continuous function) generated by an arbitrary complete and minimal system of eigenfunctions and associated functions of the mentioned operators.

2) Uniform equiconvergence on compact subsets of G of the first derivative of partial sums of spectral expansions (for any absolutely continuous function) corresponding to two nonselfadjoint Sturm–Liouville (or Schrödinger) operators.

4.2. The present paper is the first one in a series of three papers devoted to derivatives and integrals of the eigenfunctions and associated functions of the operator (1). It contains three sections. In § 1 the necessary mean-value formulas for the first derivative of the eigenfunctions and associated functions are derived. In § 2 the proof of estimates (6)–(7) is given, and in § 3 the estimates (8)–(9) are proved.

In the second paper we will establish order–sharp upper estimates for integrals (over arbitrary closed intervals $[y_1, y_2] \subseteq \overline{G}$) of the eigenfunctions and associated functions in terms of their L_2-norms when G is a finite interval.

Finally, in the third subsequent paper we intend to discuss the problem of estimates for the higher derivatives and some "double" integrals of eigenfunctions and associated functions of the operator (1). There the corresponding theorems will be formulated and proved.

§1. Mean-value formulas for the first derivative

1. Auxiliary functions. In this section we will establish so–called *mean-value formulas* for the first derivative of eigenfunctions and associated functions of the operator (1).

1.1. Throughout this paper we will constantly use functions $h = \rho_1(x,t)$ and $h = \rho_2(x,t)$ defined by

$$\int\limits_{x-\rho_1(x,t)}^{x} \frac{d\tau}{\sqrt{p(\tau)}} = t\,, \qquad \int\limits_{x}^{x+\rho_2(x,t)} \frac{d\tau}{\sqrt{p(\tau)}} = t\,, \tag{16}$$

where $x \in (a,b)$, $t \in [0, t_x]$, and t_x is a sufficiently small positive number. These functions are continuous with respect to the variable x on every closed interval $K \subset G$, and for any $x \neq x_0$ they have the first derivative. Moreover, for a fixed $x \in G$ they are continuous and (strictly) increasing with respect to the variable t on the corresponding closed interval $[0, t_x]$. Hence, there exist the inverse functions $t = \overline{\rho}_1(x,h)$ and $t = \overline{\rho}_2(x,h)$, which will be used in the following form:

$$\overline{\rho}_1(x, x-\xi) = \int\limits_{\xi}^{x} \frac{d\tau}{\sqrt{p(\tau)}}\,, \qquad \overline{\rho}_2(x, \xi - x) = \int\limits_{x}^{\xi} \frac{d\tau}{\sqrt{p(\tau)}}\,. \tag{17}$$

The functions (16) and (17) were first introduced by V.A. Il'in in [**2**] (case $x = x_0$).

1.2. Let $x \in G$ and $t \in (0, t_x]$ be arbitrary fixed numbers. In order to establish the mentioned mean-value formulas for the first derivative we need the function

$$w(x,\xi) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\sqrt{\lambda}} \cos\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\,, & x - \rho_1(x,t) \le \xi \le x\,, \\ \frac{1}{\sqrt{\lambda}} \cos\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\,, & x \le \xi \le x + \rho_2(x,t)\,, \end{cases} \tag{18}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ is an arbitrary (complex) number. It is not difficult to see that for $\xi \neq x_0$ the following holds:

$$\begin{aligned} \big(p(\xi)\, w'_\xi(x,\xi)\big)' &= -\,\lambda\, w(x,\xi) \,+ \\ &+ \begin{cases} \frac{p'(\xi)}{2\sqrt{p(\xi)}} \sin\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\,, & x - \rho_1(x,t) < \xi < x\,, \\ \frac{-p'(\xi)}{2\sqrt{p(\xi)}} \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\,, & x < \xi < x + \rho_2(x,t)\,. \end{cases} \end{aligned} \tag{19}$$

2. Mean-value formulas for the first derivative of an eigenfunction. Let $\overset{\circ}{u}_\lambda(\xi)$ be an eigenfunction of the operator (1) corresponding to the eigenvalue $\lambda \neq 0$. Fix $x \in G$ and $t \in [0, t_x]$, where $t_x > 0$ is a number such that $[x - \rho_1(x, t_x), x + \rho_2(x, t_x)] \subset G$.

2.1. We first suppose that $x_0 \in (x - \rho_1(x,t), x)$ and start from the integral

$$\tag{20} \int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \big(p(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi \;=\; \int\limits_{x-\rho_1(x,t)}^{x_0} \big(p_1(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi \;+ \\ +\int\limits_{x_0}^{x} \big(p_2(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi \;+\; \int\limits_{x}^{x+\rho_2(x,t)} \big(p_2(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi$$

Using equalities (19), we obtain

$$\tag{21} \int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \big(p(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi = -\,\lambda \int\limits_{x-\rho_1(x,t)}^{x_0} \overset{\circ}{u}_\lambda(\xi)\, w(x,\xi)\, d\xi \;+ \\ +\int\limits_{x-\rho_1(x,t)}^{x_0} \frac{p_1'(\xi)}{2\sqrt{p_1(\xi)}}\, \overset{\circ}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\, d\xi - \\ -\,\lambda \int\limits_{x_0}^{x} \overset{\circ}{u}_\lambda(\xi)\, w(x,\xi)\, d\xi \;+\; \int\limits_{x_0}^{x} \frac{p_2'(\xi)}{2\sqrt{p_2(\xi)}}\, \overset{\circ}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\, d\xi \;- \\ -\lambda \int\limits_{x}^{x+\rho_2(x,t)} \overset{\circ}{u}_\lambda(\xi)\, w(x,\xi)\, d\xi - \int\limits_{x}^{x+\rho_2(x,t)} \frac{p_2'(\xi)}{2\sqrt{p_2(\xi)}}\, \overset{\circ}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi-x) - t)\, d\xi.$$

On the other hand, applying twice the partial integration to the integrals on the right-hand side of (20), and using then the junction condition, the continuity of function (18), the equations (2)–(3) and the differentiability properties of function (18), we conclude that

$$\tag{22} \int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \big(p(\xi)\, w'_\xi(x,\xi)\big)'\, \overset{\circ}{u}_\lambda(\xi)\, d\xi \;= \\ = \frac{1}{\sqrt{\lambda}} \big[\, p_1(x-\rho_1(x,t))\, \overset{\circ}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_2(x+\rho_2(x,t))\, \overset{\circ}{u}{}'_\lambda(x+\rho_2(x,t)) \big] \;- \\ -2\sqrt{p_2(x)}\, \overset{\circ}{u}_\lambda(x)\, \sin\sqrt{\lambda}\, t + \big(\sqrt{p_1(x_0)} - \sqrt{p_2(x_0)}\ \big)\, \overset{\circ}{u}_\lambda(x_0)\, \sin\sqrt{\lambda}\,(\overline{\rho}_1(x, x-x_0) - t) + \\ + \int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} q(\xi)\, \overset{\circ}{u}_\lambda(\xi)\, w(x,\xi)\, d\xi \;-\; \lambda \int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{\circ}{u}_\lambda(\xi)\, w(x,\xi)\, d\xi\,.$$

Finally, the equalities (21) and (22) yield the following *mean-value formula for the first derivative* of the function $\overset{\circ}{u}_\lambda(\xi)$:

$$
\begin{aligned}
&p_1(x-\rho_1(x,t))\,\overset{\circ}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_2(x+\rho_2(x,t))\,\overset{\circ}{u}{}'_\lambda(x+\rho_2(x,t)) \;=\\
&\quad= 2\sqrt{p_2(x)}\,\overset{\circ}{u}_\lambda(x)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t\;-\\
&\quad-\big(\sqrt{p_1(x_0)}-\sqrt{p_2(x_0)}\,\big)\,\overset{\circ}{u}_\lambda(x_0)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(x,x-x_0)-t)\;+\\
&\quad+\sqrt{\lambda}\cdot\int\limits_{x-\rho_1(x,t)}^{x}\frac{p_j'(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi\;-\\
&\quad-\sqrt{\lambda}\cdot\int\limits_{x}^{x+\rho_2(x,t)}\frac{p_2'(\xi)}{2\sqrt{p_2(\xi)}}\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\;-\\
&\quad-\int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} q(\xi)\,\overset{\circ}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\,d\xi\,,
\end{aligned}
\tag{23}
$$

where $\overline{\rho}_k(x,|\,x-\xi\,|)=\overline{\rho}_1(x,x-\xi)$ if $\xi<x$, and $\overline{\rho}_k(x,|\,x-\xi\,|)=\overline{\rho}_2(x,\xi-x)$ if $x<\xi$; also, $j=1$ if $\xi\le x_0$, and $j=2$ if $x_0\le\xi$. Denote by $I_{(23)}(x;p_j,p_2,q;\overset{\circ}{u}_\lambda)$ the sum of integrals (with corresponding signs $+$ or $-$) appearing in (23).

2.2. Suppose now that $x_0\in(x,x{+}\rho_2(x,t))$. Then analogously to the previous case one can obtain the following *mean-value formula*:

$$
\begin{aligned}
&p_1(x-\rho_1(x,t))\,\overset{\circ}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_2(x+\rho_2(x,t))\,\overset{\circ}{u}{}'_\lambda(x+\rho_2(x,t)) \;=\\
&\quad= 2\sqrt{p_1(x)}\,\overset{\circ}{u}_\lambda(x)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t\;+\;I_{(23)}(x;p_1,p_j,q;\overset{\circ}{u}_\lambda)\;-\\
&\quad-\big(\sqrt{p_2(x_0)}-\sqrt{p_1(x_0)}\,\big)\,\overset{\circ}{u}_\lambda(x_0)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(x,x_0-x)-t)\,.
\end{aligned}
\tag{24}
$$

2.3. The case when $x_0\notin(x-\rho_1(x,t),x+\rho_2(x,t))$ is much simpler. Then the corresponding *mean-value formulas* have the form

$$
\begin{aligned}
&p_j(x-\rho_1(x,t))\,\overset{\circ}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_j(x+\rho_2(x,t))\,\overset{\circ}{u}{}'_\lambda(x+\rho_2(x,t)) \;=\\
&\quad= 2\sqrt{p_j(x)}\,\overset{\circ}{u}_\lambda(x)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t\;+\;I_{(23)}(x;p_j,p_j,q;\overset{\circ}{u}_\lambda)\,,
\end{aligned}
\tag{25}
$$

where $j=1$ if $x+\rho_2(x,t)<x_0$, and $j=2$ if $x_0<x-\rho_1(x,t)$.

3. Mean-value formulas for the first derivative of an associated function. Let $\overset{i}{u}_\lambda(\xi)$ be an associated function of the operator (1) corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(\xi)$ and the eigenvalue $\lambda\neq 0$. Let $x\in G$ and $t\in[0,t_x]$ be arbitrary fixed numbers.

3.1. If $x_0 \in (x-\rho_1(x,t), x+\rho_2(x,t)) \setminus \{x\}$, then the following *mean-value formulas for the first derivative* of the function $\overset{i}{u}_\lambda(\xi)$ hold:

$$
\begin{aligned}
&p_1(x-\rho_1(x,t))\,\overset{i}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_2(x+\rho_2(x,t))\,\overset{i}{u}{}'_\lambda(x+\rho_2(x,t)) \;=\\
&\qquad = 2\sqrt{p_{j_1}(x)}\,\overset{i}{u}_\lambda(x)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t \;-\\
&\qquad -\Big(\sqrt{p_{j_2}(x_0)}-\sqrt{p_{j_1}(x_0)}\,\Big)\,\overset{i}{u}_\lambda(x_0)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,(\overline{\rho}_{j_2}(x,x-x_0)-t)\;+\\
&\qquad + I_{(23)}(x;p_j,p_j,q;\overset{i}{u}_\lambda) \;-\int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\,d\xi\,,
\end{aligned}
\tag{26}
$$

where $j_1=2,\ j_2=1$ if $x_0<x$, and $j_1=1,\ j_2=2$ if $x<x_0$; also, $j=1$ if $\xi\le x_0$, and $j=2$ if $x_0\le\xi$.

The proof of these formulas is (almost) the same as the one of the formula (23). The only difference is that instead of (2)–(3) we use now the equations (4)–(5).

3.2. If $x_0 \notin (x-\rho_1(x,t), x+\rho_2(x,t))$, then the corresponding *mean-value formulas* have the form

$$
\begin{aligned}
&p_j(x-\rho_1(x,t))\,\overset{i}{u}{}'_\lambda(x-\rho_1(x,t)) \;-\; p_j(x+\rho_2(x,t))\,\overset{i}{u}{}'_\lambda(x+\rho_2(x,t)) \;=\\
&\qquad = 2\sqrt{p_j(x)}\,\overset{i}{u}_\lambda(x)\,\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t \;+\; I_{(23)}(x;p_j,p_j,q;\overset{i}{u}_\lambda)\;-\\
&\qquad -\int\limits_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\,d\xi\,,
\end{aligned}
\tag{27}
$$

where $j=1$ if $x+\rho_2(x,t)<x_0$, and $j=2$ if $x_0<x-\rho_1(x,t)$.

§2. Local estimates of the first derivative

1. Local estimate (7). In this section the proof of the propositon (a) of Theorem 1 will be given. We will consider in detail the case $i>1$ only, the case $i=0$ being more simple.

1.1. Let $K\subset G$ be an arbitrary compact set, with $x_0\in K$. There are points $c,d\in K$ such that $c\le x\le d$ for every $x\in K$. Let $\delta\in(0,1)$ be a fixed number. We introduce the notations

$$\alpha \;=\; \min\{\sqrt{\alpha_1},\,\sqrt{\alpha_2}\}\,, \qquad R_0 \;=\; \delta\,\rho(K,\partial G)\,,$$

$$\gamma(K_{R_0},p) \;=\; \max\left\{ \max_{x\in[c-R_0,x_0]} \sqrt{p_1(x)}\,,\; \max_{[x_0,d+R_0]} \sqrt{p_2(x)} \right\},$$

$$\gamma'(K_{R_0},p) \;=\; \max\left\{ \max_{x\in[c-R_0,x_0]} |\,p_1'(x)\,|\,,\; \max_{x\in[x_0,d+R_0]} |\,p_2'(x)\,| \right\},$$

$$\tau(K_{R_0},p) \;=\; \min\left\{ \int\limits_{c-R_0}^{c} \frac{d\tau}{\sqrt{p_1(\tau)}}\,,\; \int\limits_{d}^{d+R_0} \frac{d\tau}{\sqrt{p_2(\tau)}} \right\},$$

fix a number $\tau_0 \in (0,\tau(K_{R_0},p))$, and for every $x \in K$ define numbers $\rho_1(x,\tau_0)$ and $\rho_2(x,\tau_0)$ by equalities (18). Then $(x-\rho_1(x,\tau_0), x+\rho_2(x,\tau_0)) \subset K_{R_0}$, and the following estimate holds:

$$\max\{\,\rho_1(x,\tau_0)\,,\,\rho_2(x,\tau_0)\,\} \;\le\; \gamma(K_{R_0},p)\,\tau_0\,. \tag{28}$$

We will use the function

(29)

$$\omega(x,y;\mu) \stackrel{\text{def}}{=} \begin{cases} \cos\mu\,\overline{\rho}_1(x,x-y); & x-\rho_1(x,\tau_0)\le y\le x;\\ \cos\mu\,\overline{\rho}_2(x,y-x); & x\le y\le x+\rho_2(x,\tau_0);\\ 0; & y\in(a,x-\rho_1(x,\tau_0))\cup(x+\rho_2(x,\tau_0),b); \end{cases}$$

where $x \in K$, and $\mu > 0$ is an arbitrary number. Consider the integral

$$\overset{i}{\omega}(x;\mu) \;\stackrel{\text{def}}{=}\; \int\limits_a^b \omega(x,y;\mu)\,\overset{i}{u}_\lambda(y)\,dy \;= \tag{30}$$

$$= \int\limits_{x-\rho_1(x,\tau_0)}^{x} \overset{i}{u}_\lambda(y)\,\cos\mu\,\overline{\rho}_1(x,x-y)\,dy \;+\; \int\limits_{x}^{x+\rho_2(x,\tau_0)} \overset{i}{u}_\lambda(y)\,\cos\mu\,\overline{\rho}_2(x,y-x)\,dy\,,$$

where $\overset{i}{u}_\lambda(\xi)$ is an arbitrary associated function corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(\xi)$ and to the eigenvalue $\lambda \neq 0$.

1.2. At first we suppose *that* $x \in K_- \stackrel{\text{def}}{=} \{\,x\in K \mid x\le x_0\,\}$ *is a fixed point and* $x_0 < x+\rho_2(x,\tau_0)$. Putting $t=\overline{\rho}_1(x,x-y)$ in the first integral on the right–hand side of (30), and $t=\overline{\rho}_2(x,y-x)$ in the second one, we obtain the equality

$$(31)\qquad \overset{i}{\omega}(x;\mu) = \int_0^{\overline{\rho}_2(x,x_0-x)} \Big[\sqrt{p_1(x-\rho_1(x,t))}\, \overset{i}{u}_\lambda(x-\rho_1(x,t)) + \sqrt{p_1(x+\rho_2(x,t))}\, \overset{i}{u}_\lambda(x+\rho_2(x,t)) \Big] \cos\mu t\, dt + \int_{\overline{\rho}_2(x,x_0-x)}^{\tau_0} \Big[\sqrt{p_1(x-\rho_1(x,t))}\, \overset{i}{u}_\lambda(x-\rho_1(x,t)) + \sqrt{p_2(x+\rho_2(x,t))}\, \overset{i}{u}_\lambda(x+\rho_2(x,t)) \Big] \cos\mu t\, dt\,.$$

In order to "evaluate" the first (the second) integral (31), we will use the mean-value formula (16) (the mean-value formula (18)) from the first paper [**9**]. Hence we conclude that the following equality holds:

$$(32)\qquad \overset{i}{\omega}(x;\mu) = 2\sqrt{p_1(x)}\,\overset{i}{u}_\lambda(x) \int_0^{\tau_0} \cos\mu t\, \cos\sqrt{\lambda}\, t\, dt + \big(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\,\big)\, \overset{i}{u}_\lambda(x_0) \cdot \int_{\overline{\rho}_2(x,x_0-x)}^{\tau_0} \cos\mu t\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x,x_0-x)-t)\, dt -$$

$$- \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_{x-\rho_1(x,t)}^{x} \frac{p_1'(\xi)}{2\sqrt{p_1(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\, d\xi \Bigg) dt + \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_x^{x+\rho_2(x,t)} \frac{p_j'(\xi)}{2\sqrt{p_j(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\, d\xi \Bigg) dt -$$

$$- \frac{1}{\sqrt{\lambda}} \cdot \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\, d\xi \Bigg) dt - \frac{1}{\sqrt{\lambda}} \cdot \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{i-1}{u_\lambda}(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\, d\xi \Bigg) dt\,.$$

It follows from (30) and the differentiability properties of functions (16) (with respect to the variable x) that for $x \neq x_0$ we have

$$(33)\qquad \frac{d}{dx}\overset{i}{\omega}(x;\mu) = \frac{\mu}{\sqrt{p_1(x)}} \Bigg(\int_x^{x+\rho_2(x,\tau_0)} \overset{i}{u}_\lambda(y)\, \sin\mu\,\overline{\rho}_2(x,y-x)\, dy - \int_{x-\rho_1(x,\tau_0)}^{x} \overset{i}{u}_\lambda(y)\, \sin\mu\,\overline{\rho}_1(x,x-y)\, dy \Bigg) + \frac{\cos\mu\,\tau_0}{\sqrt{p_1(x)}} \big[\sqrt{p_2(x+\rho_2(x,\tau_0))}\, \overset{i}{u}_\lambda(x+\rho_2(x,\tau_0)) - \sqrt{p_1(x-\rho_1(x,\tau_0))}\, \overset{i}{u}_\lambda(x-\rho_1(x,\tau_0)) \big].$$

On the other hand, by applying the well-known rules for differentiation under the integral sign to equality (32), it is possible to obtain still another expression for $\frac{d}{dx}\overset{\circ}{\omega}(x;\mu)$. In that way we get from (32)–(33) that for $x \neq x_0$ the following equality holds:

$$
\begin{aligned}
(34)\quad & \overset{i}{u}{}'_\lambda(x) \int_0^{\tau_0} \cos\mu t \, \cos\sqrt{\lambda}\, t \, dt = \\
&= \frac{\mu}{2\,p_1(x)} \Bigg(\int_x^{x+\rho_2(x,\tau_0)} \overset{i}{u}_\lambda(y) \sin\mu\overline{\rho}_2(x, y-x)\, dy \\
&\qquad - \int_{x-\rho_1(x,\tau_0)}^{x} \overset{i}{u}_\lambda(y) \sin\mu\overline{\rho}_1(x, x-y)\, dy \Bigg) \\
&+ \frac{\cos\mu\,\tau_0}{2\,p_1(x)} \Big[\sqrt{p_2(x+\rho_2(x,\tau_0))}\; \overset{i}{u}_\lambda(x+\rho_2(x,\tau_0)) - \\
&\qquad - \sqrt{p_1(x-\rho_1(x,\tau_0))}\; \overset{i}{u}_\lambda(x-\rho_1(x,\tau_0)) \Big] - \\
&- \sqrt{\lambda}\, \overset{i}{u}_\lambda(x_0)\, \frac{\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}}{2\,p_1(x)} \cdot \int_{\overline{\rho}_2(x,x_0-x)}^{\tau_0} \cos\mu t\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, x_0-x) - t)\, dt - \\
&\qquad - \overset{i}{u}_\lambda(x_0)\, \frac{\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}}{2\,p_1(x)} \cos\mu\,\overline{\rho}_2(x, x_0-x) - \\
&- \frac{\sqrt{\lambda}}{4\,p_1(x)} \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_{x-\rho_1(x,t)}^{x} \frac{p_1'(\xi)}{\sqrt{p_1(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\, d\xi + \\
&\qquad + \int_x^{x+\rho_2(x,t)} \frac{p_j'(\xi)}{\sqrt{p_j(\xi)}}\, \overset{i}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi-x) - t)\, d\xi \Bigg) dt + \\
&+ \frac{1}{4\,p_1(x)} \int_0^{\tau_0} \cos\mu t \cdot \Big[p_1'(x-\rho_1(x,t))\, \overset{i}{u}_\lambda(x-\rho_1(x,t)) - \\
&\qquad - p_j'(x+\rho_2(x,t))\, \overset{i}{u}_\lambda(x+\rho_2(x,t)) \Big]\, dt + \\
&+ \frac{1}{2\,p_1(x)} \int_0^{\tau_0} \cos\mu t \cdot \Bigg(\int_{x-\rho_1(x,t)}^{x} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_1(x, x-\xi) - t)\, d\xi - \\
&\qquad - \int_x^{x+\rho_2(x,t)} q(\xi)\, \overset{i}{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi-x) - t)\, d\xi \Bigg) dt +
\end{aligned}
$$

$$+\ \frac{1}{2\,p_1(x)}\int\limits_0^{\tau_0}\cos\mu\,t\cdot\Bigg(\int\limits_{x-\rho_1(x,t)}^{x}\overset{i-1}{u_\lambda}(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi\ -$$

$$-\int\limits_{x}^{x+\rho_2(x,t)}\overset{i-1}{u_\lambda}(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\Bigg)dt\,.$$

1.3. The equality (34) will serve as a starting point in our proof of the local estimate (7). We will also need a lower–bound estimate for the integral

$$\int\limits_0^{\tau_0}\cos\mu\,t\,\cos\sqrt{\lambda}\,t\,dt\ =\ \frac{\tau_0}{2}\left(\frac{\sin\tau_0\,(\mu+\sqrt{\lambda}\,)}{\tau_0\,(\mu+\sqrt{\lambda}\,)}\ +\ \frac{\sin\tau_0\,(\mu-\sqrt{\lambda}\,)}{\tau_0\,(\mu-\sqrt{\lambda}\,)}\right). \tag{35}$$

It follows from $\lim\limits_{z\to 0}\frac{\sin z}{z}=1$ that there exists a number $\delta_1>0$ such that for every $z\in\mathbb{C}$ we have

$$|\,z\,|\ <\ \delta_1\ \Longrightarrow\ \left|\frac{\sin z}{z}\right|\ >\ \frac{2}{3}\,. \tag{36}$$

From now on we will suppose that τ_0 satisfies the additional condition

$$\tau_0\ <\ \min\left\{1,\ \frac{\delta_1}{1+\big(\mathrm{Im}\,\sqrt{\lambda}\,\big)^2}\right\}. \tag{37}$$

Introduce number $\mu_0\overset{\mathrm{def}}{=}(2/\tau_0)\cdot\sqrt{1+\mathrm{sh}^2(\tau_0\ \mathrm{Im}\,\sqrt{\lambda}\,)}$. Then for every $\mu>\mu_0$ we have

$$\left|\frac{\sin\tau_0\,(\mu+\sqrt{\lambda}\,)}{\tau_0\,(\mu+\sqrt{\lambda}\,)}\right|\ \le\ \frac{\sqrt{1+\mathrm{sh}^2(\tau_0\ \mathrm{Im}\,\sqrt{\lambda}\,)}}{\tau_0\,\mu}\ <\ \frac{1}{2}\,.$$

On the other–hand, if μ satisfies $|\,\mu-\mathrm{Re}\,\sqrt{\lambda}\,|\le 1$, then $|\,\mu-\sqrt{\lambda}\,|\le 1+\big(\mathrm{Im}\,\sqrt{\lambda}\,\big)^2$. Therefore, for such numbers μ it holds

$$|\,\tau_0\,(\mu-\sqrt{\lambda}\,)\,|\ <\ \frac{\delta_1}{1+\big(\mathrm{Im}\,\sqrt{\lambda}\,\big)^2}\,\Big(1+\big(\mathrm{Im}\,\sqrt{\lambda}\,\big)^2\Big)\ =\ \delta_1\,,$$

wherefrom we conclude, by (36), that $|\,\tau_0\,(\mu-\sqrt{\lambda}\,)\,|^{-1}\,|\,\sin\tau_0\,(\mu-\sqrt{\lambda}\,|>2/3$.

Now, we can state the mentioned estimate for the integral (35) in the form of the following assertion: There exists a number $\mu_0=\mu_0(\tau_0,\mathrm{Im}\,\sqrt{\lambda}\,)$ such that for every number $\mu>\mu_0$, satisfying condition $|\,\mu-\mathrm{Re}\,\sqrt{\lambda}\,|\le 1$, we have the estimate

$$\left|\int\limits_0^{\tau_0}\cos\mu\,t\,\cos\sqrt{\lambda}\,t\,dt\right|\ >\ \frac{\tau_0}{6}\,. \tag{38}$$

1.4. Let we define number $r(K_{R_0}, \mathrm{Im}\sqrt{\lambda}) > 1$ by

$$r(K_{R_0}, \mathrm{Im}\sqrt{\lambda}) \;=\; \sqrt{\left(\mathrm{Im}\sqrt{\lambda}\right)^2 + \left(\mu_0(\tau_0, \mathrm{Im}\sqrt{\lambda})\right)^2}\,,$$

suppose that $|\,\mathrm{Re}\sqrt{\lambda}\,| > r(K_{R_0}, \mathrm{Im}\sqrt{\lambda})$, and put $\mu \stackrel{\mathrm{def}}{=} |\,\mathrm{Re}\sqrt{\lambda}\,|$. For the number μ equality (34) and estimate (38) hold. Then estimates (28) and equality (34) imply the following inequality:

$$\begin{aligned}
|\,\overset{i}{u}{}'_\lambda(x)\,| \;\le\; & |\sqrt{\lambda}\,| \cdot \max_{\xi \in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,| \cdot \Bigg(\frac{6}{\alpha^2}\,\gamma(K_{R_0}, p) \;+ \\
& + \frac{6}{\alpha^2\,\tau_0}\,\gamma(K_{R_0}, p) \;+\; \frac{6}{\alpha^2}\,\gamma(K_{R_0}, p)\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})} \;+\; \frac{6}{\alpha^2\,\tau_0}\,\gamma(K_{R_0}, p) \;+ \\
& + \frac{3\,\tau_0}{\alpha^3}\,\gamma(K_{R_0}, p)\,\gamma'(K_{R_0}, p)\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})} \;+\; \frac{3}{\alpha^2}\,\gamma'(K_{R_0}, p) \;+ \\
& + \frac{6}{\alpha^2}\,\|\,q\,\|_{L_1(K_{R_0})}\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})}\,\Bigg) \;+ \\
& + \frac{6\,\tau_0}{\alpha^2}\,\gamma(K_{R_0}, p)\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})} \cdot \max_{\xi \in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,|\,, \qquad \text{or}
\end{aligned}$$

$$\begin{aligned}
(39) \qquad |\,\overset{i}{u}{}'_\lambda(x)\,| \;\le\; & |\sqrt{\lambda}\,|\,\tilde{C}_{i1}(K_{R_0}, p, q, \mathrm{Im}\sqrt{\lambda}) \cdot \max_{\xi \in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,| \;+ \\
& + \frac{6\,\tau_0}{\alpha^2}\,\gamma(K_{R_0}, p)\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})} \cdot \max_{\xi \in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,|\,.
\end{aligned}$$

At this place we have to use the "anti–apriori" estimate (14) : According to the proposition (a) of Lemma 2 , there exist a number $R_1 \in (0, \rho(K_{R_0}, \partial G))$ and a constant $A_i(K_R, p, q, \mathrm{Im}\sqrt{\lambda})$ such that

$$(40) \qquad \max_{\xi \in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,| \;\le\; A_i(K_R, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}\,| \cdot \max_{\xi \in K_R} |\,\overset{i}{u}_\lambda(\xi)\,|\,,$$

where $K_R \stackrel{\mathrm{def}}{=} K_{R_0+R_1}$. But we need a more convenient form of this inequality. Namely, if the compact K (from Lemma 2) is a closed interval, then instead of $\max_{\xi \in K_R} |\,\overset{i}{u}_\lambda(\xi)\,|$ on the right–hand side of (14) it is possible to write $\max_{\xi \in K} |\,\overset{i}{u}_\lambda(\xi)\,|$, with a constant $\tilde{A}_i(\cdot)$ depending on K (see Remark 2 in the introductory part of the first paper [**9**]). Hence, instead of (40) we have the stronger estimate

$$(41) \qquad \max_{\xi \in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,| \;\le\; \tilde{A}_i(K_{R_0}, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}\,| \cdot \max_{\xi \in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,|\,.$$

Using this estimate, we obtain from (39) the following inequality:

$$\begin{aligned}
(42) \qquad |\,\overset{i}{u}{}'_\lambda(\xi)\,| \;\le\; & |\sqrt{\lambda}\,| \Bigg(\tilde{C}_{i1}(K_{R_0}, p, q, \mathrm{Im}\sqrt{\lambda}) \;+ \\
& + \frac{6\,\tau_0}{\alpha^2}\,\gamma(K_{R_0}, p)\,\tilde{A}_i(K_{R_0}, p, q, \mathrm{Im}\sqrt{\lambda})\sqrt{1 + \mathrm{sh}^2(\tau_0\ \mathrm{Im}\sqrt{\lambda})}\,\Bigg) \cdot \max_{\xi \in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,|\,.
\end{aligned}$$

According to the proposition (a) of Lemma 1, there exist a number R_1 in the interval $(0, \rho(K_{R_0}, \partial G))$ and a constant $C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ such that

$$\max_{\xi \in K_{R_0}} |\overset{i}{u}_\lambda(\xi)| \le C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda}) \|\overset{i}{u}_\lambda\|_{L_2(K_R)}, \tag{43}$$

where $K_R \overset{\text{def}}{=} K_{R_0+R_1}$.

It follows from (42) and (43) that *for the points* $x \in K^-$, *satisfying condition* $x < x_0 < x + \rho_2(x, \tau_0)$, the estimate

$$|\overset{i}{u}{}'_\lambda(x)| \le C'_{i1}(K_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})\, C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\, |\sqrt{\lambda}|\, \|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{44}$$

holds, where $C'_{i1}(K_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})$ denotes the constant from estimate (42).

1.5. Suppose now *that* $x \in K^-$ *is a fixed point and* $x + \rho_2(x, \tau_0) < x_0$. In that case, instead of (31) we obtain the following equality:

$$\overset{i}{\omega}(x; \mu) = \int_0^{\tau_0} \Big[\sqrt{p_1(x - \rho_1(x,t))}\, \overset{i}{u}_\lambda(x - \rho_1(x,t)) + \\ + \sqrt{p_1(x + \rho_2(x,t))}\, \overset{i}{u}_\lambda(x + \rho_2(x,t)) \Big] \cos\mu t\, dt.$$

Using here the corresponding mean-value formula (16) from the first paper [**9**], instead of (32) we get the equality

$$\overset{i}{\omega}(x; \mu) = R_{(32)}(x; \mu; \overset{i}{u}_\lambda; \overset{i-1}{u}_\lambda), \tag{45}$$

where $R_{(32)}(\cdot)$ denotes the right–hand side of equality (32), in which $p_j(\cdot)$ is replaced by $p_1(\cdot)$ and the term containing x_0 is omitted.

It follows then from equality (45), by the same "differentiability procedure" as in the previous case, that

$$\overset{i}{u}{}'_\lambda(x) \int_0^{\tau_0} \cos\mu t\, \cos\sqrt{\lambda} t\, dt = R_{(34)}(x; \mu; \overset{i}{u}_\lambda; \overset{i-1}{u}_\lambda), \tag{46}$$

where $R_{(34)}(\cdot)$ denotes the right–hand side of equality (34), with $p_j(\cdot)$ replaced by $p_1(\cdot)$ and with all three terms containing x_0 omitted.

If we suppose that $|\operatorname{Re}\sqrt{\lambda}| > r(K_{R_0}, \operatorname{Im}\sqrt{\lambda})$ and put $\mu \overset{\text{def}}{=} |\operatorname{Re}\sqrt{\lambda}|$, then, comparing (34) with (46), we see that for $|\overset{i}{u}{}'_\lambda(x)|$ the estimate (42), and therefore the estimate (44), also holds.

1.6. Finally, it remains to consider *the case when* $x = x_0$. From our assumptions on the function $\overset{i}{u}_\lambda(\xi)$ it follows that $\overset{i}{u}{}'_\lambda(x_0 - 0) = \lim_{x \to x_0 - 0} \overset{i}{u}{}'_\lambda(x)$.

Hence, using the well-known theorems on continuity of the parameter Riemann or Lebesgue integrals, we obtain from (34) that

$$\overset{i}{u}{}'_\lambda(x_0-0)\int_0^{\tau_0}\cos\mu t\,\cos\sqrt{\lambda}\,t\,dt \;=\; R_{(34)}(x_0;\mu;\overset{i}{u}_\lambda;\overset{i-1}{u}_\lambda)\,, \tag{47}$$

where $R_{(34)}(\cdot)$ denotes the right–hand side of equality (34), with x replaced by x_0. Comparing (47) with (34), we see that for $|\overset{\circ}{u}{}'_\lambda(x_0-0)|$ the estimate (42), and therefore the estimate (44), also holds.

1.7. By considerations in 1.2–1.6 we may conclude that the estimate

$$\begin{aligned}&\sup_{x\in K^-}|\overset{i}{u}{}'_\lambda(x)| \;\le\\ &\le\; C'_{i1}(K_{R_0},p,q,\operatorname{Im}\sqrt{\lambda})\,C_i(K_R,p,q,\operatorname{Im}\sqrt{\lambda})\,|\sqrt{\lambda}\,|\,\|\overset{i}{u}_\lambda\|_{L_2(K_R)}\end{aligned} \tag{48}$$

holds if λ satisfies $|\operatorname{Re}\sqrt{\lambda}\,|>r(K_{R_0},\operatorname{Im}\sqrt{\lambda})$.

1.8. The procedure of estimation the function $\overset{i}{u}{}'_\lambda(\xi)$ *on the compact set* $K^+\overset{\text{def}}{=}\{x\in K\,|\,x_0\le x\}$ is completely analogous to the derivation of estimate (48). Thus, we first consider the points $x\in K^+\backslash\{x_0\}$ such that $x-\rho_1(x,\tau_0)<x_0$. For these points we obtain, using the corresponding mean-value formulas (16) and (18) from [**9**], that equality corresponding to the equality (32) have the form

$$\begin{aligned}\overset{i}{\omega}(x;\mu) \;=\; & 2\sqrt{p_2(x)}\;\overset{i}{u}_\lambda(x)\int_0^{\tau_0}\cos\mu t\,\cos\sqrt{\lambda}\,t\,dt\;+\\ &+\;\Big(\sqrt{p_1(x_0)}-\sqrt{p_2(x_0)}\,\Big)\,\overset{i}{u}_\lambda(x_0)\cdot\int_{\overline{\rho}_1(x,x-x_0)}^{\tau_0}\cos\mu t\,\cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-x_0)-t)\,dt\;-\\ &-\int_0^{\tau_0}\cos\mu t\cdot\Bigg(\int_{x-\rho_1(x,t)}^{x}\frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{i}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi\;+\\ &\qquad+\int_x^{x+\rho_2(x,t)}\frac{p'_2(\xi)}{2\sqrt{p_2(\xi)}}\,\overset{i}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\Bigg)dt\;-\\ &-\frac{1}{\sqrt{\lambda}}\int_0^{\tau_0}\cos\mu t\cdot\Bigg(\int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)}q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\,d\xi\Bigg)\,dt\;-\\ &-\frac{1}{\sqrt{\lambda}}\cdot\int_0^{\tau_0}\cos\mu t\cdot\Bigg(\int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)}\overset{i-1}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_k(x,|\,x-\xi\,|)-t)\,d\xi\Bigg)dt\,.\end{aligned} \tag{49}$$

Starting from this equality, one may verify that the estmate (42) holds (with the *same* constant $C'_{i1}(K_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda}))$.

When $x \in K^+$ is a point such that $x_0 < x - \rho_1(x, \tau_0)$, then using the corresponding mean-value formula (16) from the first paper **[9]**, instead of (45) we obtain the following equality:

$$\overset{i}{\omega}(x;\mu) \;=\; R_{(49)}(x;\mu;\overset{i}{u}_\lambda;\overset{i-1}{u}_\lambda)\,,$$

where $R_{(49)}(\cdot)$ denotes the right–hand side of equality (49), with $p_j(\cdot)$ replaced by $p_1(\cdot)$ and the term containing x_0 omitted. By this equality we can get an equality corresponding to (46), and then prove the estimate (42).

The estimates (42) and (44) hold true for $|\overset{i}{u}{}'_\lambda(x_0+0)|$, too. This fact results from $\overset{i}{u}{}'_\lambda(x_0+0) = \lim\limits_{x\to x_0+0} \overset{i}{u}{}'_\lambda(x)$ and from an equality which is analogous to the equality (47).

Finally, by the previous consideration we may conclude that the estimate

$$\begin{aligned}\sup_{x\in K^+} |\overset{i}{u}{}'_\lambda(x)| \;\le& \\ \le\; C'_{i1}(K_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})\, C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\, &|\sqrt{\lambda}|\, \|\overset{i}{u}_\lambda\|_{L_2(K_R)}\end{aligned} \tag{50}$$

holds if λ is an eigenvalue such that $|\operatorname{Re}\sqrt{\lambda}| > r(K_{R_0}, \operatorname{Im}\sqrt{\lambda})$.

1.9. The estimates (48) and (50) show that **the local estimate** (7) **is valid** if we define

$$C_{i1}(K_R, p, q, \operatorname{Im}\sqrt{\lambda}) \;\overset{\text{def}}{=}\; C'_{i1}(K_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})\, C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\,, \tag{51}$$

and put $r(K_R, \operatorname{Im}\sqrt{\lambda}) \overset{\text{def}}{=} r(K_{R_0}, \operatorname{Im}\sqrt{\lambda})$.

2. Local estimate (6). In the second part of the present section we will prove the estimate (6). Note that already introduced symbols for constants and sets keep their meaning.

2.1. Let $\lambda \ne 0$ be an eigenvalue such that $0 \le |\operatorname{Re}\sqrt{\lambda}| \le r(K_R, \operatorname{Im}\sqrt{\lambda})$. This time we will start from the function

$$\omega(x,y) \;\overset{\text{def}}{=}\; \begin{cases} 1/\tau\,, & y \in (x-\rho_1(x,\tau), x+\rho_2(x,\tau))\,, \\ 0\,, & y \in G \setminus (x-\rho_1(x,\tau), x+\rho_2(x,\tau))\,, \end{cases}$$

where $x \in K$, and $\tau \in (0, \tau(K_{R_0}, p))$ is a fixed number. Introduce the integral

$$\overset{i}{\omega}(x) \;\overset{\text{def}}{=}\; \int_a^b \omega(x,y)\,\overset{i}{u}_\lambda(y)\,dy \;=\; \frac{1}{\tau}\left(\;\int\limits_{x-\rho_1(x,\tau)}^{x} \overset{i}{u}_\lambda(y)\,dy \;+\; \int\limits_{x}^{x+\rho_2(x,\tau)} \overset{i}{u}_\lambda(y)\,dy\right).$$

Let first $x \in K^-$ *be a fixed point satisfying* $x_0 < x + \rho_2(x,\tau)$. (We will expose in some detail this case only.) Then proceeding as in derivation of equality (34), we obtain that for $x \neq x_0$ the following equality is valid:

$$\overset{i}{u}{}'_\lambda(x)\,\frac{2}{\tau}\int_0^\tau \cos\sqrt{\lambda}\,t\,dt \;= \tag{52}$$

$$= \frac{1}{p_1(x)}\Big[\sqrt{p_2(x+\rho_2(x,\tau))}\;\overset{i}{u}_\lambda(x+\rho_2(x,\tau)) \;-$$

$$-\;\sqrt{p_1(x-\rho_1(x,\tau))}\;\overset{i}{u}_\lambda(x-\rho_1(x,\tau))\Big] \;-$$

$$-\;\frac{\sqrt{p_2(x_0)}-\sqrt{p_1(x_0)}}{\tau\,p_1(x)}\,\overset{i}{u}_\lambda(x_0)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(x,x_0-x)-\tau) \;-$$

$$-\;\frac{\sqrt{\lambda}}{2\,\tau\,p_1(x)}\int_0^\tau\Bigg(\int_{x-\rho_1(x,t)}^{x}\frac{p_1'(\xi)}{\sqrt{p_1(\xi)}}\,\overset{i}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi \;+$$

$$+\int_x^{x+\rho_2(x,t)}\frac{p_j'(\xi)}{\sqrt{p_j(\xi)}}\,\overset{i}{u}_\lambda(\xi)\sin\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\Bigg)dt \;+$$

$$+\;\frac{1}{2\,\tau\,p_1(x)}\int_0^\tau\Big[p_1'(x-\rho_1(x,t))\,\overset{i}{u}_\lambda(x-\rho_1(x,t)) \;-$$

$$-\;p_j'(x+\rho_2(x,t))\,\overset{i}{u}_\lambda(x+\rho_2(x,t))\Big]\,dt \;+$$

$$+\;\frac{1}{\tau\,p_1(x)}\int_0^\tau\Bigg(\int_{x-\rho_1(x,t)}^{x} q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi \;-$$

$$-\int_x^{x+\rho_2(x,t)} q(\xi)\,\overset{i}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\Bigg)dt \;+$$

$$+\;\frac{1}{\tau\,p_1(x)}\int_0^\tau\Bigg(\int_{x-\rho_1(x,t)}^{x}\overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_1(x,x-\xi)-t)\,d\xi \;-$$

$$-\int_x^{x+\rho_2(x,t)}\overset{i-1}{u}_\lambda(\xi)\,\cos\sqrt{\lambda}\,(\overline{\rho}_2(x,\xi-x)-t)\,d\xi\Bigg)dt\,.$$

2.2. Suppose additionally that the number τ satisfies the condition

$$\tau \;<\; \frac{\delta_1}{\Big[\big(r(K_R,\operatorname{Im}\sqrt{\lambda})\big)^2+\big(\operatorname{Im}\sqrt{\lambda}\big)^2\Big]^{1/2}}\,. \tag{53}$$

Then we have the estimate

$$\left| \frac{2}{\tau} \int\limits_0^\tau \cos\sqrt{\lambda}\, t\, dt \right| \;>\; \frac{4}{3}\,,$$

and for every $x \in K^-$ the numbers $\rho_1(x,\tau),\, \rho_2(x,\tau)$ satisfy the estimate (28):

$$\max\{\, \rho_1(x,\tau)\,,\ \rho_2(x,\tau)\,\} \;\le\; \tau\,\gamma(K_{R_0},p)\,.$$

By virtue of these estimates, we obtain from (52) that

$$\begin{aligned} |\, \overset{i}{u}{}'_\lambda(x)\,| \;\le\; \max_{\xi\in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,| \cdot \Bigg(&\frac{3}{2\,\alpha^2}\,\gamma(K_{R_0},p) \;+ \\ &+\; \frac{3}{2\,\tau\,\alpha^2}\,\gamma(K_{R_0},p)\,\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\;+ \\ +\; \frac{3\,\tau}{4\,\alpha^3}\,\gamma(K_{R_0},p)\,\gamma'(K_{R_0},p)\,&\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\;\cdot \\ &\cdot\sqrt{\big(r(K_R,\mathrm{Im}\sqrt{\lambda}\,)\big)^2+\big(\mathrm{Im}\sqrt{\lambda}\,\big)^2}\;+ \\ +\; \frac{3}{4\,\alpha^2}\,\gamma'(K_{R_0},p) \;+\; \frac{3}{2\,\alpha^2}\,\|\,q\,\|_{L_1(K_{R_0})}\,&\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\;\Bigg)\;+ \\ &+\; \frac{3}{2\,\alpha^2}\,\gamma(K_{R_0},p)\,\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\cdot \max_{\xi\in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,|\,, \qquad \text{or} \end{aligned}$$

$$\begin{aligned} (54)\qquad |\, \overset{i}{u}{}'_\lambda(x)\,| \;\le\; &\tilde{C}'_{i1}(K_{R_0},p,q,\mathrm{Im}\sqrt{\lambda}\,)\cdot \max_{\xi\in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,| \;+ \\ &+\; \frac{3}{2\,\alpha^2}\,\gamma(K_{R_0},p)\,\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\cdot \max_{\xi\in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,|\,. \end{aligned}$$

Applying here the estimate (41) to $\max\limits_{\xi\in K_{R_0}} |\,\overset{i-1}{u}_\lambda(\xi)\,|$, and then the estimate (43) to $\max\limits_{\xi\in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,|$, we get the estimate

$$\begin{aligned} (55)\qquad |\, \overset{i}{u}{}'_\lambda(x)\,| \;\le\; |\sqrt{\lambda}\,|\,\Bigg(&\tilde{C}'_{i1}(K_{R_0},p,q,\mathrm{Im}\sqrt{\lambda}\,) \;+ \\ +\; \frac{3}{2\,\alpha^2}\,\gamma(K_{R_0},p)\,\tilde{A}_i(K_{R_0},p,q,\mathrm{Im}\sqrt{\lambda}\,)\,&\sqrt{1+\mathrm{sh}^2(\tau\ \mathrm{Im}\sqrt{\lambda}\,)}\;\Bigg)\cdot \max_{\xi\in K_{R_0}} |\,\overset{i}{u}_\lambda(\xi)\,|\,, \end{aligned}$$

and then the final estimate

$$(56)\qquad |\, \overset{i}{u}{}'_\lambda(x)\,| \;\le\; C'_{i1}(K_{R_0},p,q,\mathrm{Im}\sqrt{\lambda}\,)\,C_i(K_R,p,q,\mathrm{Im}\sqrt{\lambda}\,)\cdot \|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}\,,$$

where $C'_{i1}(K_{R_0},p,q,\operatorname{Im}\sqrt{\lambda})$ is the constant from (55). Note again that these estimates are valid for the points $x \in K^-$ such that $x < x_0 < x + \rho_2(x,\tau)$.

2.3. By the corresponding arguments one may verify that the estimates (55) and (56) are valid *in the other cases of points* $x \in K$, i.e., when $x \in K^-$ is such that $x + \rho_2(x,\tau) < x_0$, or $x = x_0$, or $x \in K^+$. Therefore, if $\lambda \neq 0$ is an eigenvalue satisfying $0 \leq |\operatorname{Re}\sqrt{\lambda}| \leq r(K_R, \operatorname{Im}\sqrt{\lambda})$, then **the estimate** (6) **holds**:

$$\sup_{x\in K} |\overset{i}{u}{}'_{\lambda}(x)| \leq C'_{i1}(K_R,p,q,\operatorname{Im}\sqrt{\lambda})\, \|\overset{i}{u}_{\lambda}\|_{L_2(K_R)}, \quad \text{where}$$

$$C_{i1}(K_R,p,q,\operatorname{Im}\sqrt{\lambda}) \overset{\text{def}}{=} C'_{i1}(K_{R_0},p,q,\operatorname{Im}\sqrt{\lambda})\, C_i(K_R,p,q,\operatorname{Im}\sqrt{\lambda}). \tag{57}$$

Note that using max if necessary, we may obtain *the same constant* in both estimates (6) and (7), as it is stated in the proposition (*a*) of Theorem 1.

2.4. It remains to consider *the case* $\lambda = 0$. The corresponding mean-value formulas for the associated function $\overset{i}{u}_0(\xi)$ are much simpler then in the case $\lambda \neq 0$:

$$\begin{aligned}
&\sqrt{p_j(x-\rho_1(x,t))}\,\overset{i}{u}_0(x-\rho_1(x,t)) + \sqrt{p_j(x+\rho_2(x,t))}\,\overset{i}{u}_0(x+\rho_2(x,t)) = \\
&= \left(\sqrt{p_j(x-0)}+\sqrt{p_j(x+0)}\right)\overset{i}{u}_0(x) - \int_{x-\rho_1(x,t)}^{x} \frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{i}{u}_0(\xi)\,d\xi + \qquad (58)\\
&+ \int_{x}^{x+\rho_2(x,t)} \frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{i}{u}_0(\xi)\,d\xi - \int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} q(\xi)\,\overset{i}{u}_0(\xi)\,(\overline{\rho}_k(x,|x-\xi|)-t)\,d\xi - \\
&- \int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{i-1}{u}_0(\xi)\,(\overline{\rho}_k(x,|x-\xi|)-t)\,d\xi,
\end{aligned}$$

where it is supposed that $x_0 \notin (x-\rho_1(x,t), x+\rho_2(x,t))$ if $x \neq x_0$, and $p_j(\cdot)$, $\overline{\rho}_k(\cdot)$ have the corresponding indices;

$$\begin{aligned}
&\sqrt{p_1(x-\rho_1(x,t))}\,\overset{i}{u}_0(x-\rho_1(x,t)) + \sqrt{p_2(x+\rho_2(x,t))}\,\overset{i}{u}_0(x+\rho_2(x,t)) = \\
&= 2\sqrt{p_{j_1}(x)}\,\overset{i}{u}_0(x) + \left(\sqrt{p_{j_2}(x_0)} - \sqrt{p_{j_1}(x_0)}\right)\overset{i}{u}_0(x_0) + I_{(58)}(x;p_j,p_j,q;\overset{i}{u}_0;\overset{i-1}{u}_0),
\end{aligned}$$

where $j_1 = 2$, $j_2 = 1$ if $x_0 \in (x-\rho_1(x,t), x)$, and $j_1 = 1$, $j_2 = 2$ if x_0 belongs to the interval $(x, x+\rho_2(x,t))$.

Using these formulas, one can prove that **the estimate** (6) **is valid** in the considered case, too.

2.5. So far we have been assuming that $x_0 \in K$. If $x_0 \notin K$, then the estimates (6) and (7) can be proved by application of the corresponding formulas (16) from the first paper [**9**] and (58), and by arguments that are analogous to the previous ones. Except simplicity, the only difference is that the number R_0 should be defined now by

$$R_0 \;=\; \delta \min \{ \rho(K, \partial G) \, , \, \rho(x_0, K) \} \, .$$

2.6. Proof of estimates (6)–(7) in *the case of the eigenfunction* $\overset{\circ}{u}_\lambda(\xi)$ is based on the following remark. The necessary mean-value formulas for this function can be obtained from the mean-value formulas for $\overset{i}{u}_\lambda(\xi)$ by omitting first the integral which contains $\overset{i-1}{u}_\lambda(\xi)$ and then using replacement $\overset{i}{u}_\lambda \longmapsto \overset{\circ}{u}_\lambda$ (see formulas (13)–(15) in the first paper [**9**]). That is why the content of 1.1–2.5 give us also the proof of estimates (6)–(7) in the case $i = 0$.

3. On Remarks 2–4. We end this section by consideration of assertions from Remarks 2, 3 and 4 concerning the estimates (6) and (7).

3.1. The estimates (39) and (55) show that the first part of Remark 2 holds true.

3.2. Let the set $\sigma(\mathcal{L})$ satisfies the conditions descibed in 3.2 of Introduction. Then the constants $C_i(K_R, p, q, \cdot)$ do not depend on the numbers $\lambda \in \sigma(\mathcal{L})$, i.e., they have some upper bound $C_0(K_R, p, q, A)$.

Let us replace $\operatorname{Im}\sqrt{\lambda}$ by A in (37), in the definition of numbers $r(K_{R_0}, \cdot)$ and in the constants from (39) (see 1.3–1.4). Replace also $\operatorname{Im}\sqrt{\lambda}$ by A in (53), and in the constants from (55) (see 2.2). Then we get, by (51) and (57), that the estimates (6)–(7) are valid, with the constants (and the number $r(K_R, A)$) not depending on the numbers $\lambda \in \sigma(\mathcal{L})$.

3.3. By virtue of statement 3) in 3.2 of Introduction it follows from (51) and (57) that the constants $C_{i1}(\cdot)$ $(i \ge 1)$ actually do not depend on parameter i.

§3. Global estimates of the first derivative

1. Case of the finite interval. We begin the proof of the proposition (b) with consideration of the case when G is a finite interval.

1.1. We will first prove the estimate (9) in the case $i \ge 1$. Using compactness of the closed interval $\overline{G} = [a, b]$ and continuity of functions $\rho_1 = \rho_1(x, t)$, $\rho_2 = \rho_2(x, t)$ (with respect to the variable t), it is possible to find points $c, d \in G$ and positive numbers t_c^1, t_c^2, t_d^1, t_d^2 such that

$$c - \rho_1(c, t_c^1) < a < c < c + \rho_2(c, t_c^2) < x_0 < d - \rho_1(d, t_d^1) < d < b < d + \rho_2(d, t_d^2) \, .$$

Then there exist numbers $\tau_c \in (0, \min\{t_c^1, t_c^2\})$ and $\tau_d \in (0, \min\{t_d^1, t_d^2\})$ for which the following holds: $\rho_1(c, [0, \tau_c]) = [0, c - a]$, $\rho_2(d, [0, \tau_d]) = [0, b - d]$.

Introduce the closed interval $\tilde{K} \overset{\text{def}}{=} [c,d]$, put $x = c$ and $j = 1$ in the mean-value formula (27), and assume that $t \in [0, \tau_c)$. Also, put $x = d$ and $j = 2$ in the same mean-value formula, and assume that $t \in [0, \tau_d]$. Therefore we obtain two equalities from which it results the inequality

$$
\begin{aligned}
(59) \quad \max\Big\{ \sup_{x\in(a,c]} |\overset{i}{u}{}'_\lambda(x)|, \sup_{x\in[d,b)} |\overset{i}{u}{}'_\lambda(x)| \Big\} \le \frac{1}{\alpha^2}\gamma(\tilde{K},p)\cdot \sup_{x\in\tilde{K}} |\overset{i}{u}{}'_\lambda(x)| + \\
+ \Big[|\sqrt{\lambda}| \Big(\frac{2\gamma(\tilde{K},p)}{\alpha^2} + \frac{(b-a)\,\gamma'(G,p)}{\alpha^3} \Big) + \\
+ \|q\|_{L_1(G)} \Big] \cdot \sqrt{1 + \mathrm{sh}^2(\max\{\tau_c,\tau_d\}\ \mathrm{Im}\sqrt{\lambda})} \cdot \sup_{x\in G} |\overset{i}{u}_\lambda(x)| + \\
+ \frac{b-a}{\alpha^2}\sqrt{1+\mathrm{sh}^2(\max\{\tau_c,\tau_d\}\ \mathrm{Im}\sqrt{\lambda})} \cdot \sup_{x\in G} |\overset{i-1}{u}_\lambda(x)|,
\end{aligned}
$$

where $\gamma(\tilde{K},p)$ has the obvious meaning (see 1.1 § 2), and

$$
(60) \qquad \gamma'(G,p) \overset{\text{def}}{=} \max\Big\{ \sup_{x\in(a,x_0]} |p_1'(x)|, \sup_{x\in[x_0,b)} |p_2'(x)| \Big\}.
$$

1.2. Let us define the number $r(G, \mathrm{Im}\sqrt{\lambda}) \overset{\text{def}}{=} r(\tilde{K}_{R_0}, \mathrm{Im}\sqrt{\lambda})$, where the number $r(\tilde{K}_{R_0}, \mathrm{Im}\sqrt{\lambda})$ is introduced in 1.4 § 2 (and generated by our closed interval $\tilde{K}$).

Suppose first that $|\mathrm{Re}\sqrt{\lambda}| > r(G, \mathrm{Im}\sqrt{\lambda})$. Then we may use estimate (7) and obtain that the following holds:

$$
(61) \qquad
\begin{aligned}
\sup_{x\in\tilde{K}} |\overset{i}{u}{}'_\lambda(x)| &\le C_{i1}(\tilde{K}_R, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(\tilde{K}_R)} \\
&\le C_{i1}(\tilde{K}_R, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(G)}.
\end{aligned}
$$

Also, by virtue of estimates (15) and (13) it holds

$$
\begin{aligned}
(62) \quad \sup_{x\in G} |\overset{i-1}{u}_\lambda(x)| &\le A_i(G,p,q,\mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}| \cdot \sup_{x\in G} |\overset{i}{u}_\lambda(x)| \\
&\le A_i(G,p,q,\mathrm{Im}\sqrt{\lambda})\,C_i(G,p,q,\mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(G)}.
\end{aligned}
$$

By inequalities (61)–(62) we obtain from (59) that

$$\max\Big\{ \sup_{x\in(a,c]} |\overset{i}{u}{}'_\lambda(x)|,\ \sup_{x\in[d,b)} |\overset{i}{u}{}'_\lambda(x)| \Big\} \le \tag{63}$$

$$\le \Big[\frac{1}{\alpha^2}\gamma(\tilde K,p)\, C_{i1}(\tilde K_R,p,q,\operatorname{Im}\sqrt\lambda) + $$

$$+ \Big(\frac{2\gamma(\tilde K,p)}{\alpha^2} + \frac{(b-a)\,\gamma'(G,p)}{\alpha^3} + \|q\|_{L_1(G)} \Big)\cdot$$

$$\cdot\sqrt{1+\operatorname{sh}^2(\max\{\tau_c,\tau_d\}\operatorname{Im}\sqrt\lambda)}\cdot C_i(G,p,q,\operatorname{Im}\sqrt\lambda) + $$

$$+ \frac{b-a}{\alpha^2} A_i(\cdot)\, C_i(\cdot)\sqrt{1+\operatorname{sh}^2(\max\{\tau_c,\tau_d\}\operatorname{Im}\sqrt\lambda)}\Big]\, |\sqrt\lambda|\, \|\overset{i}{u}_\lambda\|_{L_2(G)}\,.$$

Now, we can conclude from (61) and (63) that $\sup_{x\in G}|\overset{i}{u}{}'_\lambda(x)|$ exists, and that **the estimate** (9) **is valid**:

$$\sup_{x\in G}|\overset{i}{u}{}'_\lambda(x)| \le C_{i1}(G,p,q,\operatorname{Im}\sqrt\lambda)\,|\sqrt\lambda|\,\|\overset{\circ}{u}_\lambda\|_{L_2(G)}\,,$$

where $C_{i1}(G,p,q,\operatorname{Im}\sqrt\lambda)$ is the maximum of the constants appearing in the inequalities (61) and (63).

1.3. Suppose now that $0 \le |\operatorname{Re}\sqrt\lambda| \le r(G,\operatorname{Im}\sqrt\lambda)$ and $\lambda \ne 0$. This case can be treated analogously to the previous one. Namely, using the mean-value formulas (27), we can get the estimate (59) in which instead of $|\sqrt\lambda|$ the number

$$\sqrt{\big(r(G,\operatorname{Im}\sqrt\lambda)\big)^2 + \big(\operatorname{Im}\sqrt\lambda\big)^2}$$

stands. The rest of the proof of estimate (8) is the same; instaed of estimate (7) it is necessary to apply estimate (6).

1.4. Finally, if $\lambda = 0$, then it is possible to verify that the mean-value formulas

$$p_j(x-\rho_1(x,t))\,\overset{i}{u}{}'_0(x-\rho_1(x,t)) - p_j(x+\rho_2(x,t))\,\overset{i}{u}{}'_0(x+\rho_2(x,t)) =$$

$$= -\int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} q(\xi)\,\overset{i}{u}_0(\xi)\,d\xi - \int_{x-\rho_1(x,t)}^{x+\rho_2(x,t)} \overset{i-1}{u}_0(\xi)\,d\xi$$

hold if $x_0 \notin (x-\rho_1(x,t), x+\rho_2(x,t))$.

By these formulas and the procedure used in 1.2, one may easily prove the corresponding estimate (8).

1.5. Proof of estimates (8)–(9) in *the case of an eigenfunction* is based also on the remark given in 2.6 § 2.

2. Case of the infinite interval. Our general assumption in this case is that $\overset{\circ}{u}_\lambda(\xi) \in L_2(G)$ $(i \geq 0)$.

2.1. Suppose first that $G = (-\infty, +\infty)$, and analyse the dependence of the constants $C_{i1}(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ on the "variable" K_R. We want to show that in this case, under assumptions from the proposition (b) of Theorem 1, constants $C_{i1}(\cdot, p, q, \operatorname{Im}\sqrt{\lambda})$ can be chosen "almost independently" of the compact K_R.

Let $R_0 \in (0,1)$ be an arbitrary fixed number, and let

$$\gamma(G,p) \overset{\text{def}}{=} \max\left\{ \sup_{x\in(-\infty,x_0]} \sqrt{p_1(x)}\,,\ \sup_{x\in[x_0,+\infty)} \sqrt{p_2(x)} \right\},\ \tau(G,p) \overset{\text{def}}{=} \frac{R_0}{\gamma(G,p)}\,,$$

$$\gamma'(G,p) \overset{\text{def}}{=} \max\left\{ \sup_{x\in(-\infty,x_0]} |\,p_1'(x)\,|\,,\ \sup_{x\in[x_0,+\infty)} |\,p_2'(x)\,| \right\}. \tag{64}$$

Fix a number $\tau_0 \in (0, \tau(G,p))$ and define functions $\rho_1(\cdot,\tau_0)$, $\rho_2(\cdot,\tau_0)$ on the compact K by equalities (16).

Proceed further like in 1.1–1.3 § 2. Then define $\mu_0(\tau_0, \operatorname{Im}\sqrt{\lambda})$ as in 1.3 § 2, and the number $r(G, \operatorname{Im}\sqrt{\lambda}) \overset{\text{def}}{=} \sqrt{\left(\operatorname{Im}\sqrt{\lambda}\right)^2 + \left(\mu_0(\tau_0, \operatorname{Im}\sqrt{\lambda})\right)^2}$. Therefore, if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G, \operatorname{Im}\sqrt{\lambda})$, then we get the estimate (39), with a constant $\tilde{C}_{i1}(G,p,q,\operatorname{Im}\sqrt{\lambda})$ obtained from the constant $\tilde{C}_{i1}(K_R,p,q,\operatorname{Im}\sqrt{\lambda})$ by replacement $K_R \longmapsto G$.

According to Remark 8, the constant $C_i(K_{R_0+R_1}, p, q, \operatorname{Im}\sqrt{\lambda})$ from estimate (43) can be replaced by a constant $C_i(\overline{K}, p, q, \operatorname{Im}\sqrt{\lambda})$ depending only on the length of the closed interval $\overline{K} \supset K$.

Analysing further the content of 1.4–1.9 § 2, we obtain the following conclusion: There are constants $r(G, \operatorname{Im}\sqrt{\lambda})$ and $C_{i1}(G_s, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i \geq 0)$ such that for every closed interval $K \subset G$ (of the fixed length equal to some $s > 0$) the estimate

$$\sup_{x\in K} |\,\overset{i}{u}{}'_\lambda(x)\,| \ \leq\ C_{i1}(G_s,p,q,\operatorname{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\ \|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}$$

holds if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G, \operatorname{Im}\sqrt{\lambda})$, where $R > 0$ is a fixed number.

Analogous analysis of the proof of estimate (6) (see 2.1–2.4 § 2) shows that there is a constant $C'_{i1}(G_s, p, q, \operatorname{Im}\sqrt{\lambda})$ such that every closed interval $K \subset G$ (of the fixed length equal to some $s > 0$) the estimate

$$\sup_{x\in K} |\,\overset{i}{u}{}'_\lambda(x)\,| \ \leq\ C'_{i1}(G_s,p,q,\operatorname{Im}\sqrt{\lambda})\ \|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}$$

holds if $0 \leq |\operatorname{Re}\sqrt{\lambda}\,| \leq r(G, \operatorname{Im}\sqrt{\lambda})$, where $R > 0$ is a fixed number.

2.2. Prove now the estimates (8)–(9). Define constant $r(G, \operatorname{Im}\sqrt{\lambda})$ as in 2.1. Then, for arbitrary fixed number $s > 0$, define constants $C_{i1}(G_s, p, q, \operatorname{Im}\sqrt{\lambda})$

and $C'_{i1}(G_s, p, q, \mathrm{Im}\sqrt{\lambda})$ as it was already explained. If $x \in G$ is an arbitrary point and $K(x)$ an arbitrary closed interval of the length s, such that $x \in K(x)$, then

$$\begin{aligned} |\overset{i}{u}{}'_\lambda(x)| &\le C_{i1}(G_s, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(K(x)_R)} \\ &\le C_{i1}(G_s, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(G)} \end{aligned}$$

for $|\,\mathrm{Re}\sqrt{\lambda}\,| > r(G, \mathrm{Im}\sqrt{\lambda})$, and

$$\begin{aligned} |\overset{i}{u}{}'_\lambda(x)| &\le C'_{i1}(G_s, p, q, \mathrm{Im}\sqrt{\lambda})\,\|\overset{i}{u}_\lambda\|_{L_2(K(x)_R)} \\ &\le C'_{i1}(G_s, p, q, \mathrm{Im}\sqrt{\lambda})\,\|\overset{i}{u}_\lambda\|_{L_2(G)} \end{aligned}$$

if $0 \le |\sqrt{\lambda}| \le r(G, \mathrm{Im}\sqrt{\lambda})$.

Therefore, it results that $\sup_{x\in\mathbb{R}} |\overset{i}{u}{}'_\lambda(x)|$ exists and **the estimates** (8)–(9) **hold**.

2.3. Consider *the case when* $G = (a, +\infty)$, $a \in \mathbb{R}$. Define first numbers $\gamma(G,p)$ and $\gamma'(G,p)$ as in 2.1. Let $c \in (a, x_0)$ be an arbitrary fixed number and $\tau_c > 0$ such that $\rho_1(c, [0, \tau_c)) = [0, c-a)$. Then $\rho_2(c, \tau_c) \le \gamma(G,p)\,\tau_c$. Choose a number $d \in G$ such that $c + \gamma(G,p)\,\tau_c < d$, $c - a < (d-c)/4$, and denote by K the closed interval $[c,d]$. Define the number $r(G, \mathrm{Im}\sqrt{\lambda})$ by $r(G, \mathrm{Im}\sqrt{\lambda}) = r(K_{R_0}, \mathrm{Im}\sqrt{\lambda})$, with $r(K_{R_0}, \mathrm{Im}\sqrt{\lambda})$ introduced in 1.4 § 2.

Return after that to the mean-value formulas (27). Putting there $x = c$ and $j = 1$, and assuming that $t \in [0, \tau_c)$, we obtain an inequality for $\sup_{x\in(a,c]} |\overset{i}{u}{}'_\lambda(x)|$ having the same forme as inequality (59), with $\tilde{K}$ and $\max\{\tau_c, \tau_d\}$ replaced by K and τ_c respectively. By that inequality and Lemma 1 we obtain the estimate

$$\sup_{x\in(a,c]} |\overset{i}{u}{}'_\lambda(x)| \le \tilde{C}_{i1}(G, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(G)}$$

if $|\,\mathrm{Re}\sqrt{\lambda}\,| > r(G, \mathrm{Im}\sqrt{\lambda})$.

On the compact set K the estimate (7) is valid, wherefrom we get

$$\sup_{x\in K} |\overset{i}{u}{}'_\lambda(x)| \le C_{i1}(K_R, p, q, \mathrm{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\,\|\overset{i}{u}_\lambda\|_{L_2(G)}$$

if $|\,\mathrm{Re}\sqrt{\lambda}\,| > r(G, \mathrm{Im}\sqrt{\lambda})$.

Finally, it remains to show the existence of $\sup_{x\in(d,+\infty)} |\overset{i}{u}{}'_\lambda(x)|$ and the validity of estimate (9) for this suprema. This can be done like in 2.1; the only difference is that we now choose $R_0 \in (0, \min\{1, c-a\})$ and cover each point $x \in (d, +\infty)$

by a closed interval $K(x)$, where intervals $K(x)$ have the same fixed length $s \in (0, d-c)$.

As far as the estimate (8) concerned, it can be proved by corresponding arguments; we omitt the details.

2.4. The verification of the estimates (8)–(9) in *the case* $G = (-\infty, b)$ $(b \in \mathbb{R})$ is completely analogous to the procedure presented in 2.3.

2.5. If we use the maen–value formulas (25) instaed of (27), then the considerations from 2.3–2.4 are valid in the case of eigenfunction $\overset{\circ}{u}_\lambda(\xi)$, too.

Proof of Theorem 1 is completed.

3. On Remarks 1–4. In order to verify that Remark 1 holds true, we should change definitions (60), (64) of $\gamma'(G,p)$ replacing $\sup\limits_{x\in(a,x_0]} |\, p_1'(x)\,|$, $\sup\limits_{x\in[x_0,b)} |\, p_2'(x)\,|$ by integrals $\int\limits_a^{x_0} |\, p_1'(x)\,|\, dx$ and $\int\limits_{x_0}^{b} |\, p_2'(x)\,|\, dx$ respectively. Also, it is necessary to use the first statement from 3.1 in Introduction.

3.1. Analysing the content of 3.1 in Introduction, 3.1 § 2 and 1.1–1.3, we see that the global estimates (8)–(9) may be sharpened in the following way:

$$\sup_{x\in G} |\, \overset{i}{u}{}_\lambda'(x)\,| \;\le\; C_{i1}(\tilde{K}_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda}\,)\,|\sqrt{\lambda}\,| \cdot \max_{x\in\tilde{K}_{R_0}} |\, \overset{i}{u}_\lambda(x)\,|$$

if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G, \operatorname{Im}\sqrt{\lambda}\,)$, and

$$\sup_{x\in G} |\, \overset{i}{u}{}_\lambda'(x)\,| \;\le\; C_{i1}(\tilde{K}_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda}\,) \cdot \max_{x\in\tilde{K}_{R_0}} |\, \overset{i}{u}_\lambda(x)\,|$$

if $0 \le |\operatorname{Re}\sqrt{\lambda}\,| \le r(G, \operatorname{Im}\sqrt{\lambda}\,)$, where G is a finite interval, and $\tilde{K} \subset G$ is the closed interval defined in 1.1.

3.2. The constants $C_{i1}(G,p,q,\cdot)$ $(i \ge 0)$ and $r(G,\cdot)$ from the proposition (b) of Theorem 1 have the property of independence of the numbers $\lambda \in \sigma(\mathcal{L})$, satisfying conditions described in Remark 3. This assertion is a consequence of the definition of numbers $r(G,\cdot)$ (see 1.2, 2.1 and 2.3), and of the structure of constants $C_{i1}(G,p,q,\cdot)$ in different cases (see 3.2 § 2, 1.1–1.4, 2.1 and 2.3). In the proof of the mentioned property one should also use the content of 3.2 in Introduction.

3.3. Analysing the structure of "global" constants $C_{i1}(\cdot)$ $(i \ge 1)$ and having in mind statement 3) in 3.2 of Introduction, we see that these constants do not depend on the parameter i.

References

1. В.А. Ильин, *Необходимые и достаточные условия базисности подсистемы собственных и присоединенных функций пучка М.В. Келдыша обыкновенных дифференциальных операторов*, ДАН СССР **227**(4) (1976), 796–799.

2. В.А. Ильин, *О сходимости разложений по собственным функциям в точках разрыва коэффициентов дифференциального оператора*, Матем. заметки **22**(5) (1977), 679–698
3. Н. Лажетич, *Равномерные оценки первых производных собственных функций оператора Штурма–Лиувилля с потенциалом из класса* L_1, ДАН СССР **248**(6) (1979), 1304–1305
4. И. Йо, Н. Лажетич, *Оценка разности производных частичных сумм разложений, отвечающих двум произвольным неотрицательным самосопряженным расширениям двух операторов типа Штурма–Лиувилля, для абсолютно непрерывной функции*, Дифф. уравнения **16**(4) (1980), 598–619
5. N. Lažetić, *On the convergence of spectral decompositions corresponding to ordinary differential operators of second order*, Dissertation, University of Belgrade, 1980 (in Serbian)
6. Н. Лажетич, *Равномерные оценки для производных собственных функций оператора Штурма–Лиувилля*, Дифф. уравнения **17**(11) (1981), 1978–1983
7. Н. Лажетич, *Оценки собственных и присоединенных функций оператора Штурма–Лиувилля с разрывными коэфициентами*, ДАН СССР **258**(3) (1981), 541–544
8. Н. Лажетич, *О производных частичных сумм разложений по собственным и присоединенным функциям несамосопряженных операторов типа Штурма–Лиувилля*, ДАН СССР **260**(1) (1981), 22–26
9. Н. Лажетич, *Оценки собственных и присоединенных функций оператора Штурма–Лиувилля с разрывными коэффициентами* (И)–(ИИ), Bull. Acad. Serbe Sci. Arts **84** Cl. Sci. Math. Natur. **13** (1984), 75–101; **88**, Cl. Sci. Math. Natur. **14** (1985), 21–47 .
10. В.Д. Будаев, *О безусловной базисности на замкнутом интервале систем собственных и присоединенных функций оператора второго порядка с разрывными коэффициентами*, Дифф. уравнения **23**(6) (1987), 941–952

Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 29 02 1995)
(Revised 15 03 1996)