PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 60 (74), 1997, 5–14

ON GENERALIZATION OF FUNCTIONS $n!$ AND $!n$

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Communicated by Aleksandar Ivić

Abstract. A sequence y_n is defined, and its relation to Đuro Kurepa's left factorial hypothesis is discussed. Also, a generalization of functions $n!$, $!n$ and y_n, a sequence $u_{n,m}$ is defined, and a number of its properties is proved.

1. Introduction. Kurepa in [**6**] defined $!n$ (left factorial) by:

$$!n = 0! + 1! + 2! + \cdots + (n-1)!, \qquad n \in N \tag{1.1}$$

and stated the hypothesis that

$$(!n, n!) = 2, \qquad \text{for} \qquad n > 1, \tag{KH}$$

where (a, b) denotes the greatest common divisor of integers a and b. In [**6**] was proved that (KH) is equivalent to assertion that

$$!p \not\equiv 0 \pmod p, \quad \text{for all primes} \quad p > 2 \tag{1.2}$$

and this is the usual form of KH.

In [**8**], [**9**], [**11**], [**12**] and [**13**] there are several statements equivalent to KH, which are all exposed in [**5**]. Here we cite, for example, the assertion proved in [**14**], that KH is equivalent to

$$\sum_{k=1}^{p-2} (k+1)^{p-k} k^{k-1} \not\equiv 0 \pmod p, \quad \text{for all primes } p > 2 \tag{1.3}$$

KH is verified in [**2**] for $n < 10^6$. In this paper we will try to open some new possibilities for considering KH.

AMS Subject Classification (1991): Primary 11A05

2. The sequence y_n. Let $f(x)=\dfrac{e^{-x}}{1-x}$ and $n\in N\cup\{0\}$. We define a sequence y_n by:

$$y_n=f^{(n)}(0). \tag{2.1}$$

It is easy to see that

$$y_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!. \tag{2.2}$$

The first few members, are: $y_0=1$, $y_1=0$, $y_2=1$, $y_3=2$, $y_4=9$, $y_5=44,\ldots$. Let us notice that the sequence y_n has a combinatorial meaning to. Namely, y_n, $n>0$, is the number of derangements of the set of n elements, i.e. number of permutations of an n-element set, in which no element is fixed. Let us establish some properties of the sequence y_n.

PROPOSITION 2.1. *For every $n\in N$ we have*

$$y_n=ny_{n-1}+(-1)^n, \quad (2.3) \qquad \sum_{k=0}^{n}\binom{n}{k}y_k=n!, \quad (2.4)$$

$$\sum_{k=1}^{n}\binom{n}{k}y_{k-1}=!n, \quad (2.5) \qquad \sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}(!k)=y_{n-1}. \quad (2.6)$$

Proof. Since $(1-x)f(x)=e^{-x}$, it follows that

$$(-1)^n=[(1-x)f(x)]^{(n)}_{x=0}=f^{(n)}(0)-nf^{(n-1)}(0)=y_n-ny_{n-1},$$

so the equality (2.3) is correct. Further we have:

$$\frac{1}{1-x}=e^xf(x)\Rightarrow n!=[e^xf(x)]^{(n)}_{x=0}=\sum_{k=0}^{n}\binom{n}{k}y_k,$$

$$!n=\sum_{k=0}^{n-1}k!=\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}y_i=\sum_{i=0}^{n-1}y_i\sum_{k=0}^{n-1}\binom{k}{i}=\sum_{i=0}^{n-1}\binom{n}{i+1}y_i,$$

i.e. the equalities (2.4) and (2.5) are correct. From (2.5) it follows that:

$$!n=u_1^{(n)}(0), \quad u_1(x)=e^x\int_0^x f(t)dt, \quad (!0=0) \tag{2.7}$$

and further $u_1(x)e^{-x}=\int_0^x f(t)dt\Rightarrow f^{(n-1)}(0)=y_{n-1}=[u_1(x)e^{-x}]^{(n)}_{x=0}$, i.e. the equality (2.6) also holds.

PROPOSITION 2.2. *For every $n \in N$, and every $m \in N \cup \{0\}$ the following holds:*

$$\sum_{k=0}^{n} \binom{n}{k} y_{k+m} = n! \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{n+i}{i} (i!). \tag{2.8}$$

The proof is easy and can be omitted.

Let us notice, that from proven equalities, one can obtain a number of congruences by module p, when $p \in P$, and P denotes the same set as above. From (2.3) it follows

$$y_p \equiv -1 \pmod p, \tag{2.9}$$
$$y_{p-1} + y_{p-2} \equiv 1 \pmod p, \tag{2.10}$$

Also, the substitution $n = p - 1$ in (2.4) give

$$\sum_{k=0}^{p-1} (-1)^k y_k \equiv -1 \quad \pmod p, \tag{2.11}$$

Finally, by substitution $n = p$ in (2.6) and (2.8) we obtain

$$y_{p-1} \equiv !p \pmod p \tag{2.12}$$
$$y_m + y_{m+p} \equiv 0 \pmod p, \quad m \in N \cup \{0\} \tag{2.13}$$

Bearing in mind (1.2), one can, without great effort, formulate a number of assertions equivalent to KH. Really, according to congruences (2.9)-(2.13), KH is equivalent to every one of the following statements:

$$y_{p-1} \not\equiv 0 \pmod p, \quad \text{for all primes } p \geq 3, \tag{2.14}$$
$$y_{p-2} \not\equiv 1 \pmod p, \quad \text{for all primes } p \geq 3, \tag{2.15}$$
$$y_p \not\equiv -1 \pmod{p^2}, \quad \text{for all primes } p \geq 3, \tag{2.16}$$
$$\sum_{k=0}^{p-2} (-1)^k y_k \not\equiv -1 \pmod p, \quad \text{for all primes } p \geq 3. \tag{2.17}$$

Obviously, one can formulate a number of similar statements.

The sequence y_n can be represented in another way.

PROPOSITION 2.3. *For every $n \in N$*

$$y_n = \left[\frac{n!}{e}\right] + \frac{1 + (-1)^n}{2}, \tag{2.18}$$

where $[x]$ denotes integer part of x, i.e. $[x] \in Z$ and $[x] \leq x < [x] + 1$.

Proof. From (2.2) it follows

$$\begin{aligned} y_n &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\Gamma(k+1) = \int_0^{+\infty} e^{-x}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^k dx \\ &= \int_0^{+\infty}(x-1)^n e^{-x}dx = \int_0^{1}(x-1)^n e^{-x}dx + \frac{1}{e}\int_0^{+\infty} t^n e^{-t}dt \\ &= \frac{n!}{e} + \int_0^{1}(x-1)^n e^{-x}dx. \end{aligned}$$

Since we have

$$\Big|\int_0^1 (x-1)^n e^{-x}dx\Big| \le \int_0^1 |x-1|e^{-x}dx = \frac{1}{e}.$$

It follows that $y_n = \left[\frac{n!}{e}\right] + 1$, for n even and $y_n = \left[\frac{n!}{e}\right]$, for n odd and thus, the equality (2.18) holds.

Bearing in mind properties of the sequence y_n from (2.18) one immediately obtains

$$\left[\frac{n!}{e}\right] = \left[\frac{e^{-x}}{1-x} - \frac{e^x + e^{-x}}{2}\right]^{(n)}_{x=0} \tag{2.19}$$

Also, according to Proposition 2.1, it follows that for every $n \in N$, the following equalities hold:

$$\left[\frac{n!}{e}\right] = n\left[\frac{(n-1)!}{e}\right] + \frac{1-(-1)^n}{2}(n-1), \tag{2.20}$$

$$\sum_{k=0}^{n}\binom{n}{k}\left[\frac{k!}{e}\right] = n! - 2^{n-1}, \tag{2.21}$$

$$\sum_{k=1}^{n}\binom{n}{k}\left[\frac{(k-1)!}{e}\right] = !n - 2^{n-1}. \tag{2.22}$$

Bearing in mind (2.9)–(2.13), it follows that for every prime $p \ge 3$, the following congruences hold:

$$\left[\frac{p!}{e}\right] \equiv -1 \pmod{p}, \tag{2.23}$$

$$\left[\frac{(p-1)!}{e}\right] + \left[\frac{(p-2)!}{e}\right] \equiv 0 \pmod{p}, \tag{2.24}$$

$$\sum_{k=0}^{p-1}(-1)^k\left[\frac{k!}{e}\right] \equiv \frac{p-3}{2} \pmod{p}, \tag{2.25}$$

$$\left[\frac{(p-1)!}{e}\right] \equiv !p - 1 \pmod{p}, \tag{2.26}$$

$$\left[\frac{m!}{e}\right] + \left[\frac{(m+p)!}{e}\right] \equiv -1 \pmod{p},\;\; m \in N \cup \{0\}. \tag{2.27}$$

The statements (2.14)–(2.17) all equivalent to KH, could be reformulated similarly. From the discussion above, it is clear, that the sequence y_n is closely related to the functions $n!$ and $!n$.

3. The sequence $u_{n,m}$. Let $f(x) = \dfrac{e^{-x}}{1-x}$ and let $u_m(x)$, $m \in Z$, be the sequence of functions defined by:

$$u_m(x) = \begin{cases} e^x \int\limits_0^x dt_1 \int\limits_0^{t_1} dt_2 \cdots \int\limits_0^{t_{m-1}} f(t_m)dt_m, & m > 0 \\ e^x f^{(-m)}(x), & m \le 0 \end{cases}. \tag{3.1}$$

The sequence of numbers $u_{n,m}$, $n \in N \cup \{0\}$, is defined by

$$u_{n,m} = u_m^{(n)}(0). \tag{3.2}$$

Let us first notice, that the sequence $u_{n,m}$, in some special cases represents the functions $n!$, $!n$ and y_n.

PROPOSITION 3.1. *For every $n \in N \cup \{0\}$ following equalities hold:*

$$u_{n,0} = n!, \tag{3.3}$$

$$u_{n,1} = !n, \quad (!0 = 0), \tag{3.4}$$

$$u_{0,-n} = y_n. \tag{3.5}$$

Proof. Referring to (2.1) and (3.1) and bearing in mind (2.7), it follows immediately that

$$u_{n,0} = \left[\frac{1}{1-x}\right]^{(n)}_{x=0} = n!,$$

$$u_{n,1} = \left[e^x \int\limits_0^x f(t)dt\right]^{(n)}_{x=0} = !n,$$

$$u_{0,-n} = \left[e^x f^{(n)}(x)\right]_{x=0} = f^{(n)}(0) = y_n,$$

which proves the assertion.

Considering further properties of the sequence $u_{n,m}$, let us show, that the sequence $u_{n,m}$ has some properties similar to those of binomial coefficients.

PROPOSITION 3.2. *For every $n \in N \cup \{0\}$ we have:*

$$u_{n,m} + u_{n,m+1} = u_{n+1,m+1}, \quad m \in Z, \tag{3.6}$$
$$m > n \Rightarrow u_{n,m} = 0, \quad m \in N, \tag{3.7}$$
$$u_{n,n} = 1, \tag{3.8}$$
$$u_{n,n-1} = n. \tag{3.9}$$

Proof. Referring to (3.1) we have

$$u_m(x) = e^x(u_{m+1}(x)e^{-x})' = u'_{m+1}(x) - u_{m+1}(x) \Rightarrow$$
$$u_{n,m} = u_m^{(n)}(0) = [u'_{m+1}(x) - u_{m+1}(x)]_{x=0}^{(n)} = u_{n+1,m+1} - u_{n,m+1},$$

and thus, the equality (3.6) holds.

Let $m > n$, then

$$u_m^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} \Big[\int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} f(t_m)dt_m \Big]^{(k)}$$
$$= e^x \sum_{k=0}^{n} \binom{n}{k} \int_0^x dt_{k+1} \cdots \int_0^{t_{m-1}} f(t_m)dt_m,$$

and after substituting $x = 0$, we obtain (3.7).

Further, from (3.6) and (3.7) it follows

$$u_{n,n} = u_{n-1,n-1} + u_{n-1,n} = u_{n-1,n-1} = \cdots = u_{0,0} = 1$$
$$u_{n,n-1} = u_{n-1,n-2} + u_{n-1,n-1} = u_{n-1,n-2} + 1 = \cdots = u_{1,0} + (n-1) = n$$

and thus, the assertion is proved.

We can easily calculate members of the sequence $u_{n,m}$. For example, for $|m| < 5$ and $n < 6$, we have:

	⋯−4	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**	**4**	⋯
0	9	2	1	0	1	0	0	0	0	
1	53	11	3	1	1	1	0	0	0	
2	362	64	14	4	2	2	1	0	0	
3	2790	426	78	18	6	4	3	1	0	
4	24024	3216	504	96	24	10	7	4	1	
5	229080	27240	3720	600	120	34	17	11	5	

PROPOSITION 3.3. *For every $n \in N$, and every $m \in Z$, the following equalities hold:*

$$\sum_{k=m}^{n} (-1)^{k-m} u_{n,k} = u_{n-1,m-1}, \tag{3.10}$$

$$\sum_{k=0}^{n-1} u_{k,m} = u_{n,m+1} - u_{0,m+1}. \tag{3.11}$$

Proof. Referring to (3.6) and (3.7) we obtain

$$\sum_{k=m}^{n} (-1)^{k-m} u_{n,k} = \sum_{k=m}^{n} (-1)^{k-m} (u_{n-1,k-1} - u_{n-1,k})$$
$$= u_{n-1,m-1} + \sum_{k=m+1}^{n} (-1)^{k-m} u_{n-1,k-1} + (-1)^{n-m} u_{n-1,n} + \sum_{k=m}^{n-1} (-1)^{k-m} u_{n-1,k}$$
$$= u_{n-1,m-1} + \sum_{k=m}^{n-1} (-1)^{k-m+1} u_{n-1,k} + \sum_{k=m}^{n-1} (-1)^{k-m} u_{n-1,k} = u_{n-1,m-1}.$$

Also

$$\sum_{k=0}^{n-1} u_{k,m} = \sum_{k=0}^{n-1} (u_{k+1,m+1} - u_{k,m+1}) = \sum_{k=1}^{n} u_{k,m+1} - \sum_{k=0}^{n-1} u_{k,m+1} = u_{n,m+1} - u_{0,m+1}.$$

and that proves the proposition.

PROPOSITION 3.4. *For every $k, n \in N \cup \{0\}$ and every $m \in Z$, the following equalities hold:*

$$\sum_{i=0}^{n} (-1)^{n-1} \binom{n}{i} u_{k+i,m} = u_{k,m-n}, \tag{3.12}$$

$$\sum_{i=0}^{n} \binom{n}{i} u_{k,m-i} = u_{n+k,m}. \tag{3.13}$$

Proof. According to (3.1), we have

$$u_{m-n}(x) = e^x (u_m(x) e^{-x})^{(n)} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} u_m^{(i)}(x).$$

Differentiating the last equality k times and substituting $x = 0$ we obtain (3.12). Further we have

$$u_m^{(n)}(x) = \sum_{i=0}^{n} \binom{n}{i} u_{m-i}(x) \Rightarrow u_m^{(n+k)}(x) = \sum_{i=0}^{n} \binom{n}{i} u_{m-i}^{(k)}(x),$$

and by substituting $x = 0$ we obtain (3.13).

Let us notice, that by substituting $k=0$ in (3.13) and considering (3.5) and (3.7) we obtain

$$u_{n,m} = \sum_{i=0}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=0}^{n} \binom{n}{i} y_{i-m} \quad (m \le 0), \text{ i.e.}$$
$$u_{n,m} = \sum_{i=0}^{m-1} \binom{n}{i} u_{0,m-i} + \sum_{i=m}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=m}^{n} \binom{n}{i} y_{i-m}, \quad (0 < m \le n).$$

We can conclude that, for every $n \in N \cup \{0\}$ and every $m \in Z$, $m \le n$ holds

$$u_{n,m} = \sum_{i=s}^{n} \binom{n}{i} y_{i-m}, \qquad s = \max(0,m). \tag{3.14}$$

Referring to (2.8), substitution $m=-k$, $k \in N \cup \{0\}$ in (3.14), gives

$$u_{n,-k} = \sum_{i=0}^{n} \binom{n}{i} y_{i+k} = n! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{n+i}{i} i!. \tag{3.15}$$

Especially, for $k=1$ we obtain

$$u_{n,-1} = n \cdot n!. \tag{3.16}$$

It is obvious that for $m \le 0$

$$u_{n,m} \equiv 0 \pmod{n!}. \tag{3.17}$$

Substitution $n=p$, $p \in P$ in (3.13), gives us

$$u_{k,m} + u_{k,m-p} \equiv u_{p+k,m} \pmod{p}. \tag{3.18}$$

According to (3.17) and (3.18), for $m \le p \le k$, we obtain

$$u_{k,m} \equiv u_{p+k,m} \pmod{p}. \tag{3.19}$$

Let us notice, that for $n=p+1$, $p \in P$, $p>2$, in (3.14), referring to (2.10) we obtain:

$$u_{p+1,2} \equiv 1 \pmod{p}. \tag{3.20}$$

At last, let us prove two recurrent formulas for the sequence $u_{n,m}$.

PROPOSITION 3.5. *For every $n \in N$ and every $m \in Z$ following equality holds*

$$(m-1)u_{n,m} = (n-m+1)u_{n,m-1} - u_{n,m-2} + a(n,m), \tag{3.21}$$

where

$$a(n,m) = \begin{cases} \binom{n}{m-2}, & m \geq 2 \\ 0, & m < 2 \end{cases}.$$

Proof. For $m = 1$, the equality holds, since (3.21) reduces to (3.16). For $m < 1$, according to (2.3), (3.6) and (3.14), we obtain

$$\begin{aligned} u_{n,m-1} + u_{n,m-2} = u_{n+1,m-1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} y_{i-m+1} \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} (i-m+1) y_{i-m} + (-1)^{1-m} \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i \\ &= (n+1) \sum_{i=1}^{n+1} \binom{n}{i-1} y_{i-m} - (m-1) \sum_{i=0}^{n+1} \binom{n+1}{i} y_{i-m} \\ &= (n+1) u_{n,m-1} - (m-1)(u_{n,m} + u_{n,m-1}). \end{aligned}$$

and, after putting this in order, we obtain (3.21).

For $m > 1$, we have

$$\begin{aligned} u_{n,m-1} + u_{n,m-2} = u_{n+1,m-1} &= \sum_{i=m-1}^{n+1} \binom{n+1}{i} y_{i-m+1} \\ = \binom{n+1}{m-1} &+ \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-m}(i-m+1) + (-1)^{1-m} \sum_{i=m}^{n+1} \binom{n+1}{i} (-1)^i \\ = \binom{n+1}{m-1} &+ (n+1) \sum_{i=m}^{n+1} \binom{n}{i-1} y_{i-m} - (m-1) \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-m} \\ &+ (-1)^m \sum_{i=0}^{m-1} \binom{n+1}{i} (-1)^i \\ = \binom{n+1}{m-1} &+ (n+1) \sum_{i=m-1}^{n} \binom{n}{i} y_{i-m+1} - (m-1) u_{n+1,m} - \binom{n}{m-1} \\ = \binom{n}{m-2} &+ (n+1) u_{n,m-1} - (m-1)(u_{n,m} + u_{n,m-1}), \end{aligned}$$

and, after putting this in order, we obtain (3.21).

PROPOSITION 3.6. *For every $n \in N \cup \{0\}$ and every $m \in Z$, the following equality holds:*

$$u_{n+2,m} = (n-m+3) u_{n+1,m} - (n+1) u_{n,m} + a(n,m). \tag{3.22}$$

Proof. According to (3.6), (3.12) and (3.21), we obtain

$$\begin{aligned} u_{n+2,m} - 2u_{n+1,m} + u_{n,m} &= u_{n,m-2} \\ &= (n-m+1)u_{n,m-1} - (m-1)u_{n,m} + a(n,m) \\ &= (n-m+1)(u_{n+1,m} - u_{n,m}) - (m-1)u_{n,m} + a(n,m), \end{aligned}$$

and, after putting this in order, we get (3.22).

Notice that, according to (3.4) and (3.6), after substituting $m = 2$ in (3.22), we obtain

$$u_{n+2,2} = (n+1)(!n) + 1. \tag{3.23}$$

From properties of the sequence $u_{n,m}$, it is clear that one can state number of assertions equivalent to KH. For example KH is equivalent to all following assertions:

$$(\exists k)(k \geq p \wedge u_{k,2} \not\equiv 1 \pmod{p}), \quad \text{for all primes } p > 2, \tag{3.24}$$

$$u_{p-1,2} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2, \tag{3.25}$$

$$u_{p-2,2} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2, \tag{3.26}$$

$$u_{p+1,2} \not\equiv p + 1 \pmod{p^2}, \quad \text{for all primes } p > 2. \tag{3.27}$$

Really, (3.24) and (3.27) are direct corollaries of (1.2) and (3.23), while (3.6) implies (3.25) and (3.26).

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(Received 23 05 1996)