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ESTIMATES FOR DERIVATIVES AND INTEGRALS OF EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF NONSELFADJOINT STURM–LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENTS (II)

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Abstract. We study integrals of the eigenfunctions and associated functions of the formal Sturm–Liouville operator $\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x)$ defined on a finite interval $G \subset \mathbb{R}$. We suppose that the complex-valued potential $q = q(x)$ belongs to the class $L_1(G)$ and that piecewise continuously differentiable coefficient $p = p(x)$ has a finite number of the discontinuity points in G. Order-sharp upper estimates are established for integrals (over arbitrary closed intervals $[y_1, y_2] \subseteq \overline{G}$) of the eigenfunctions and associated functions in terms of their L_2-norms when G is finite.

Introduction

1. Definitions. Consider the formal Sturm–Liouville operator

$$\mathcal{L}(u)(x) = -\big(p(x)\,u'(x)\big)' + q(x)\,u(x), \tag{1}$$

which is defined on an arbitrary interval $G = (a, b)$ of the real axis $\mathbb{R}$. Let $x_0 \in G$ be a point of discontinuity of the coefficient p. If we suppose that

$$p(x) = \begin{cases} p_1(x), & x \in (a, x_0), \\ p_2(x), & x \in (x_0, b), \end{cases}$$

then the following conditions are imposed on the coefficients:

1) $p_1(x) \in \mathcal{C}^{(1)}(a, x_0]$, and $p_2(x) \in \mathcal{C}^{(1)}[x_0, b)$.

2) $p_1(x) \geq \alpha_1 > 0$ everywhere on $(a, x_0]$, and $p_2(x) \geq \alpha_2 > 0$ everywhere on $[x_0, b)$.

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3) $q(x) \in L_1^{loc}(G)$ is a complex-valued function.

Definition 1. A complex-valued function $\overset{\circ}{u}_\lambda(x) \not\equiv 0$ is called an *eigenfunction of the operator* (1) *corresponding to the* (*complex*) *eigenvalue* $\lambda(\,\lambda = \operatorname{Re}\lambda + i\ \operatorname{Im}\lambda\,)$ if it satisfies the following conditions:

$(a)\overset{\circ}{u}_\lambda(x)$ is absolutely continuous on any finite closed subinterval of G.

$(b)\overset{\circ}{u}{}'_\lambda(x)$ is absolutely continuous on any finite closed subinterval of the half–open intervals $(a, x_0]$ and $[x_0, b)$.

$(c)\overset{\circ}{u}_\lambda(x)$ satisfies the differential equation

$$-\big(p_1(x)\,\overset{\circ}{u}{}'_\lambda(x)\big)' + q(x)\,\overset{\circ}{u}_\lambda(x) = \lambda\,\overset{\circ}{u}_\lambda(x) \tag{2}$$

almost everywhere on (a, x_0), and the differential equation

$$-\big(p_2(x)\,\overset{\circ}{u}{}'_\lambda(x)\big)' + q(x)\,\overset{\circ}{u}_\lambda(x) = \lambda\,\overset{\circ}{u}_\lambda(x) \tag{3}$$

almost everywhere on (x_0, b).

$(d)\overset{\circ}{u}_\lambda(x)$ satisfies the junction condition

$$p_1(x_0)\,\overset{\circ}{u}{}'_\lambda(x_0 - 0) = p_2(x_0)\,\overset{\circ}{u}{}'_\lambda(x_0 + 0).$$

Definition 2. A complex-valued function $\overset{i}{u}_\lambda(x) \not\equiv 0$ $(i = 1, 2, \dots)$ is called an *associated function* (*of the* i*-th order*) *of the operator* (1) *corresponding to the eigenfunction* $\overset{\circ}{u}_\lambda(x)$ *and the eigenvalue* λ if it satisfies the following conditions:

(a*) Conditions (a), (b) and (d) of Definition 1 hold for $\overset{i}{u}_\lambda(x)$.

(b*) $\overset{i}{u}_\lambda(x)$ satisfies the differential equation

$$-\big(p_1(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + q(x)\,\overset{i}{u}_\lambda(x) = \lambda\,\overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{4}$$

almost everywhere on (a, x_0), and the differential equation

$$-\big(p_2(x)\,\overset{i}{u}{}'_\lambda(x)\big)' + q(x)\,\overset{i}{u}_\lambda(x) = \lambda\,\overset{i}{u}_\lambda(x) - \overset{i-1}{u}_\lambda(x) \tag{5}$$

almost everywhere on (x_0, b).

1.1. Let K be any compact set of positive measure lying strictly within G. We will use the notation

$$K_R \overset{\text{def}}{=} \{\, x \in G \mid \rho(x, \overline{K}\,) \le R \,\},$$

where $R \in (0, \rho(K, \partial G))$, and $\overline{K}$ is the intersection of all closed intervals containing K. (By $\rho(A, B)$ we denote the distance of a set $A \subset \mathbb{R}$ from a set $B \subset \mathbb{R}$.)

If $\lambda = r\,e^{i\varphi}$, then $\sqrt{\lambda} \stackrel{\text{def}}{=} \sqrt{r}\,e^{i\varphi/2}$, where $\varphi \in (-\pi/2, 3\pi/2]$.

2. Main theorem. We present the following results.

THEOREM 1. *Let $q(x) \in L_1(G)$, where G is a finite interval. If the functions $p_1(x)$ and $p_2(x)$ are bounded together with their first derivatives, then there exist a closed interval $K \subset G$ and constants $r(G, \operatorname{Im}\sqrt{\lambda})$, $D_{i1}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ ($i = 0, 1, 2, \dots$) such that*

$$\left| \int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy \right| \le D_{i1}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda})\ \|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{6}$$

for every eigenvalue λ, and

$$\left| \int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy \right| \le D_{i1}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda})\ \frac{1}{|\sqrt{\lambda}|}\ \|\overset{i}{u}_\lambda\|_{L_2(K_R)} \tag{7}$$

if $|\operatorname{Re}\sqrt{\lambda}| > r(G, \operatorname{Im}\sqrt{\lambda})$, where $R \in (0, \rho(K, \partial G))$ is some fixed number. The estimates (6) and (7) hold uniformly with respect to numbers $a \le y_1 < y_2 \le b$.

Remark 1. The condition imposed on $p_1'(x)$ and $p_2'(x)$ in Theorem 1 can be replaced by the following one: $p_1'(x) \in L_1(a, x_0), p_2'(x) \in L_1(x_0, b)$.

Remark 2. It is possible to replace $\|\overset{i}{u}_\lambda\|_{L_2(K_R)}$ in estimates (6)–(7) by $\max_{x \in K} |\overset{i}{u}_\lambda(x)|$, with constants $D_{i1}(\cdot)$ changed correspondingly. As a consequence we obtain, by virtue of estimate (10) and the proposition stated in 3.1, that the estimates (6)–(7) are valid for an *arbitrary* closed interval $K \subseteq G$ (with corresponding constants $D_{i1}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda})$).

Remark 3. Let $\sigma(\mathcal{L})$ be some set of eigenvalues of the operator (1). If there exists a constant A not depending on the numbers $\lambda \in \sigma(\mathcal{L})$ and such that

$$|\operatorname{Im}\sqrt{\lambda}| \le A, \qquad \lambda \in \sigma(\mathcal{L}), \tag{8}$$

then the constants $D_{01}(\cdot)$ and $r(\cdot)$ do not depend on the numbers λ, which means that it is possible to define them uniformly with respect to the parameter $\lambda \in \sigma(\mathcal{L})$.

If the numbers $\lambda \in \sigma(\mathcal{L})$ satisfy (8) and zero is not a limit point of the set $\{\,|\operatorname{Re}\sqrt{\lambda}|\,|\,\lambda \in \sigma(\mathcal{L})\,\}$, then the constants $D_{i1}(\cdot)(i \in \mathbb{N})$ do not depend on these numbers, too.

Remark 4. The constants $D_{i1}(\cdot)$ ($i = 1, 2, \dots$) actually do not depend on the order i of the associated function, which means that they can be the same for all associated functions corresponding to the specific eigenfunction.

Remark 5. Theorem 1 includes the case when the function $p(x)$ is continuous at the point x_0 (and has the required differentiability properties at that point).

Especially, if $p_1(x) = p_2(x) = 1$ for $x \in G$, then the operator (1) reduces to the formal Schrödinger operator

$$\mathcal{L}(u)(x) = -u''(x) + q(x)\,u(x). \tag{9}$$

In that case the estimates (6)–(7) were established in [**2**]. The corresponding estimates for integrals of eigenfunctions of an arbitrary nonnegative selfadjoint extension of the operator (9) were first derived in [**1**].

Remark 6. The example exposed in [**5**] shows that the estimates (6)–(7) are best possible with respect to the order of the parameter λ.

Remark 7. For the sake of simplicity we have supposed that the coefficient $p(x)$ has only one point of discontinuity. But all stated results remain valid when this function has an arbitrary finite number of such points. In that case definitions 1 and 2 should be formulated in the corresponding way.

3. Estimates of eigenfunctions and associated functions. In the proof of Theorem 1 we will essentially use the following estimates for eigenfunctions and associated functions of the operator (1), which were announced in [**3**] and proved in [**4**].

LEMMA 1. (a) *If $q(x) \in L_1^{\mathrm{loc}}(G)$, then for any compact set $K \subset G$ there exist a number $R \in (0, \rho(K, \partial G))$ and constants $C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ such that*

$$\max_{x\in K} |\overset{i}{u}_\lambda(x)| \le C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\ \|\overset{i}{u}_\lambda\|_{L_2(K_R)}. \tag{10}$$

(b) *Suppose that $q(x) \in L_1(G)$, and that $\overset{i}{u}_\lambda(x) \in L_2(G)$ if G is an infinite interval. If $p_1(x)$ and $p_2(x)$ are bounded along with their first derivatives, then there exist constants $C_i(G, p, q, \operatorname{Im}\sqrt{\lambda})$ $(i = 0, 1, 2, \dots)$ such that*

$$\sup_{x\in G} |\overset{i}{u}_\lambda(x)| \le C_i(G, p, q, \operatorname{Im}\sqrt{\lambda})\ \|\overset{i}{u}_\lambda\|_{L_2(G)}. \tag{11}$$

LEMMA 2. (a) *If $q(x) \in L_1^{\mathrm{loc}}(G)$, then for any compact set $K \subset G$ there exist a number $R \in (0, \rho(K, \partial G))$ and constants $A_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})$, $A_i(K_R, p, q)$ $(i = 1, 2, \dots)$ such that*

$$\begin{aligned} \max_{x\in K} |\overset{i-1}{u}_\lambda(x)| &\le A_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\,|\sqrt{\lambda}|\cdot \max_{x\in K_R} |\overset{i}{u}_\lambda(x)| \quad \mathit{for} \quad \lambda \ne 0,\\ \max_{x\in K} |\overset{i-1}{u}_\lambda(x)| &\le A_i(K_R, p, q)\cdot \max_{x\in K_R} |\overset{i}{u}_\lambda(x)| \quad \mathit{for} \quad \lambda = 0. \end{aligned} \tag{12}$$

(b) *Suppose that $q(x) \in L_1(G)$, and that $\overset{i}{u}_\lambda(x) \in L_2(G)$ if G is an infinite interval. If $p_1(x)$ and $p_2(x)$ are bounded along with their first derivatives, then there* *exist constants* $A_i(G,p,q,\operatorname{Im}\sqrt{\lambda})$, $A_i(G,p,q)(\,i=1,2,\dots)$ *such that*

(13)
$$\sup_{x\in G}|\overset{i-1}{u}_{\lambda}(x)|\le A_i(G,p,q,\operatorname{Im}\sqrt{\lambda})\;|\sqrt{\lambda}|\cdot\sup_{x\in G}|\overset{i}{u}_{\lambda}(x)|\quad for\,\lambda\neq 0,$$
$$\sup_{x\in G}|\overset{i-1}{u}_{\lambda}(x)|\le A_i(G,p,q)\cdot\sup_{x\in G}|\overset{i}{u}_{\lambda}(x)|\quad for\ \lambda=0.$$

3.1. If G is a finite interval, then the condition imposed on the functions $p_1'(x)$ and $p_2'(x)$ in the propositions (b) of the previous lemmas can be replaced by the following condition: $p_1'(x)\in L_1(a,x_0), p_2'(x)\in L_1(x_0,b)$.

Also, the global estimate (11) may be sharpened in the following sense: If G is a finite interval, then for any closed interval $K\subset G$ there exist constants $C_i(K,p,q,\operatorname{Im}\sqrt{\lambda})$ such that

$$\sup_{x\in G}|\overset{i}{u}_{\lambda}(x)|\le C_i(K,p,q,\operatorname{Im}\sqrt{\lambda})\cdot\max_{x\in K}|\overset{i}{u}_{\lambda}(x)|.$$

3.2. Having in mind the specific applications of estimates (10)–(13), we note that the constants appearing in these estimates have the following properties of independence of the parameters λ and i:

1) If the condition (8) is satisfied, then it is possible to make the constants $C_0(\cdot)$ independent of the numbers $\lambda\in\sigma(\mathcal{L})$.

2) If the numbers $\lambda\in\sigma(\mathcal{L})$ satisfy (8) and zero is not a limit point of the set $\{|\operatorname{Re}\sqrt{\lambda}|\,|\,\lambda\in\sigma(\mathcal{L})\}$, then the constants $C_i(\cdot)$ and $A_i(\cdot)(\,i\in\mathbb{N})$ do not depend on those numbers.

3) The constants $C_i(\cdot)$, $A_i(\cdot)(i\in\mathbb{N})$ are independent of the parameter i.

As will be shown in the proof of Theorem 1, the statements from Remark 3 are actually consequences of 1)–2).

3.3. The estimates (6)–(7) are proved by using only estimates (10)–(13). They play a basic role in study of the uniform equiconvergence (on compact subsets of G) of the first derivatives of partial sums of spectral expansions (for any absolutely continuous function) corresponding to two nonselfadjoint Sturm–Liouville (or Schrödinger) operators.

1. Estimates for the integrals of an eigenfunction

1. The estimate (7). In this section the proof of Theorem 1 in the case $i=0$ will be given.

1.1. Let us establish first the estimate (7). Let $\overset{\circ}{u}_{\lambda}(\xi)$ be an eigenfunction of the operator (1) corresponding to the (complex) eigenvalue $\lambda\neq 0$. We need a convenient form for the integral

(14)
$$\int_{y_1}^{y_2}\overset{\circ}{u}_{\lambda}(y)\,dy,$$

where $[y_1, y_2] \subset (a, b)$ is an arbitrary closed interval.

1.2. Let $x \in G$ be an arbitrary fixed point. Suppose that $t > 0$ is a number satisfying conditions $x + \rho_2(x,t) \in G$ and $x_0 \in (x, x + \rho_2(x,t))$. We will start from the integral

$$\begin{aligned} &\int_x^{x+\rho_2(x,t)} \big(p(\xi)\, \mathring{u}'_\lambda(\xi)\big)' \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi = \\ (15)\qquad &= \int_x^{x_0} \big(p_1(\xi)\, \mathring{u}'_\lambda(\xi)\big)' \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi + \\ &+ \int_{x_0}^{x+\rho_2(x,t)} \big(p_2(\xi)\, \mathring{u}'_\lambda(\xi)\big)' \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi. \end{aligned}$$

Using the partial integration twice, by virtue of the junction condition and the differential equations (2)–(3), we get from (15) that the following equality holds:

$$\begin{aligned} (16)\qquad &\sqrt{p_2(x + \rho_2(x,t))}\; \mathring{u}_\lambda(x + \rho_2(x,t)) = \sqrt{p_1(x)}\; \mathring{u}_\lambda(x)\, \cos\sqrt{\lambda}\, t + \\ &+ \big(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\,\big)\, \mathring{u}_\lambda(x_0)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x, x_0 - x) - t) + \\ &+ p_1(x)\, \mathring{u}'_\lambda(x)\, \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} + \int_x^{x+\rho_2(x,t)} \frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\, \mathring{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi - \\ &- \frac{1}{\sqrt{\lambda}} \int_x^{x+\rho_2(x,t)} q(\xi)\, \mathring{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi. \end{aligned}$$

1.3. If we suppose that $t > 0$ is such that $x + \rho_2(x,t) < x_0$, then instead of (16) we have the equality

$$\begin{aligned} &\sqrt{p_1(x + \rho_2(x,t))}\; \mathring{u}_\lambda(x + \rho_2(x,t)) = \sqrt{p_1(x)}\; \mathring{u}_\lambda(x)\, \cos\sqrt{\lambda}\, t + \\ &\qquad + p_1(x)\, \mathring{u}'_\lambda(x)\, \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} + \\ (17)\qquad &+ \int_x^{x+\rho_2(x,t)} \frac{p'_1(\xi)}{2\sqrt{p_1(\xi)}}\, \mathring{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi - \\ &- \frac{1}{\sqrt{\lambda}} \int_x^{x+\rho_2(x,t)} q(\xi)\, \mathring{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\, d\xi. \end{aligned}$$

1.4. Let $y_1, y_2 \in (a,b)$ be arbitrary numbers such that $x_0 \in (y_1, y_2)$. Put $x = y_1$ in (16)–(17), and suppose that $t \in [0, \overline{\rho}_2(y_1, y_2 - y_1)]$. Then integrate (17) with respect to the variable $t \in [0, \overline{\rho}_2(y_1, x_0 - y_1)]$ and (16) with respect to $t \in [\overline{\rho}_2(y_1, x_0 - y_1), \overline{\rho}_2(y_1, y_2 - y_1)]$. Therefore, we get

$$\int\limits_0^{\overline{\rho}_2(y_1,y_2-y_1)} \sqrt{p_j(y_1+\rho_2(y_1,t))}\; \mathring{u}_\lambda(y_1+\rho_2(y_1,t))\,dt =$$

$$= \sqrt{p_1(y_1)}\; \mathring{u}_\lambda(y_1)\, \frac{\sin\sqrt{\lambda}\,\overline{\rho}_2(y_1,y_2-y_1)}{\sqrt{\lambda}} -$$

$$-\left(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\;\right) \mathring{u}_\lambda(x_0)\, \frac{\sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,x_0-y_1) - \overline{\rho}_2(y_1,y_2-y_1))}{\sqrt{\lambda}} -$$

$$-p_1(y_1)\,\mathring{u}'_\lambda(y_1)\, \frac{\cos\sqrt{\lambda}\,\overline{\rho}_2(y_1,y_2-y_1) - 1}{\lambda} + \tag{18}$$

$$+ \int\limits_0^{\overline{\rho}_2(y_1,y_2-y_1)} \left(\int\limits_{y_1}^{y_1+\rho_2(y_1,t)} \frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\, \mathring{u}_\lambda(\xi)\, \cos\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_1) - t)\,d\xi \right) dt -$$

$$-\frac{1}{\sqrt{\lambda}} \int\limits_0^{\overline{\rho}_2(y_1,y_2-y_1)} \left(\int\limits_{y_1}^{y_1+\rho_2(y_1,t)} q(\xi)\, \mathring{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_1) - t)\,d\xi \right) dt.$$

Transforming the first integral on the right-hand side of (18) by Fubini's theorem, and introducing a new variable by $y_1 + \rho_2(y_1,t) = y$ in the integral on the left-hand side of (18), we obtain the desired form of the integral (14):

$$\int\limits_{y_1}^{y_2} \mathring{u}_\lambda(y)\,dy = \sqrt{p_1(y_1)}\; \mathring{u}_\lambda(y_1)\, \frac{\sin\sqrt{\lambda}\,\overline{\rho}_2(y_1,y_2-y_1)}{\sqrt{\lambda}} - \tag{19}$$

$$-\left(\sqrt{p_2(x_0)} - \sqrt{p_1(x_0)}\;\right) \mathring{u}_\lambda(x_0)\, \frac{\sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,x_0-y_1) - \overline{\rho}_2(y_1,y_2-y_1))}{\sqrt{\lambda}} -$$

$$-\,p_1(y_1)\,\mathring{u}'_\lambda(y_1)\, \frac{\cos\sqrt{\lambda}\,\overline{\rho}_2(y_1,y_2-y_1) - 1}{\lambda} -$$

$$-\frac{1}{\sqrt{\lambda}} \int\limits_{y_1}^{y_2} \frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\, \mathring{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_2) - \overline{\rho}_2(y_1,y_2-y_1))\,d\xi -$$

$$-\frac{1}{\sqrt{\lambda}} \int\limits_0^{\overline{\rho}_2(y_1,y_2-y_1)} \left(\int\limits_{y_1}^{y_1+\rho_2(y_1,t)} q(\xi)\, \mathring{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_1) - t)\,d\xi \right) dt.$$

1.5. If we suppose *that* $x_0 \notin (x, x+\rho_2(x,t))$ (or, equivalently, $x_0 \notin (y_1, y_2)$), then the following equality corresponds to (19):

$$\begin{aligned}
\int_{y_1}^{y_2} \overset{\circ}{u}_\lambda(y)\,dy &= \sqrt{p_{j_1}(y_1)}\;\overset{\circ}{u}_\lambda(y_1)\,\frac{\sin\sqrt{\lambda}\,\overline{\rho}_2(y_1, y_2-y_1)}{\sqrt{\lambda}} - \\
&- p_{j_1}(y_1)\,\overset{\circ}{u}{}'_\lambda(y_1)\,\frac{\cos\sqrt{\lambda}\,\overline{\rho}_2(y_1, y_2-y_1)-1}{\lambda} - \\
&- \frac{1}{\sqrt{\lambda}}\int_{y_1}^{y_2}\frac{p'_j(\xi)}{2\sqrt{p_j(\xi)}}\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_1)-\overline{\rho}_2(y_1,y_2-y_1))\,d\xi - \\
&- \frac{1}{\sqrt{\lambda}}\int_0^{\overline{\rho}_2(y_1,y_2-y_1)}\left(\int_{y_1}^{y_1+\rho_2(y_1,t)} q(\xi)\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1,\xi-y_1)-t)\,d\xi\right)dt,
\end{aligned} \tag{20}$$

where $j_1 = 2$ if $x_0 < y_1$, and $j_1 = 1$ if $y_2 < x_0$.

1.6. When *the point* x *is equal to* x_0, then $y_1 = x_0$, and for integral (14) the equality (20) holds, with $p_{j_1}(y_1)$, $\overset{\circ}{u}{}'_\lambda(y_1)$ replaced by $p_2(x_0)$, $\overset{\circ}{u}{}'_\lambda(x_0+0)$ respectively.

In *the case when* $y_2 = x_0$ we have to consider the integral

$$\int_{x_0-\rho_1(x_0,t)}^{x_0} \left(p_1(\xi)\,\overset{\circ}{u}{}'_\lambda(\xi)\right)'\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(x_0, x_0-\xi)-t)\,d\xi$$

(where $t > 0$ is such that $x_0 - \rho_1(x_0,t) \in G$). The procedure analogous to the one used in 1.2–1.3 implies that

$$\begin{aligned}
\int_{y_1}^{y_2} \overset{\circ}{u}_\lambda(y)\,dy &= \sqrt{p_1(y_2)}\;\overset{\circ}{u}_\lambda(y_2)\,\frac{\sin\sqrt{\lambda}\,\overline{\rho}_1(y_2, y_2-y_1)}{\sqrt{\lambda}} + \\
&+ p_1(y_2)\,\overset{\circ}{u}{}'_\lambda(x_0-0)\,\frac{\cos\sqrt{\lambda}\,\overline{\rho}_1(y_2, y_2-y_1)-1}{\lambda} + \\
&+ \frac{1}{\sqrt{\lambda}}\int_{y_1}^{y_2}\frac{p'_1(\xi)}{2\sqrt{p_1(\xi)}}\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(y_2,y_2-\xi)-\overline{\rho}_1(y_2,y_2-y_1))\,d\xi - \\
&- \frac{1}{\sqrt{\lambda}}\int_0^{\overline{\rho}_1(y_2,y_2-y_1)}\left(\int_{y_2-\rho_1(y_2,t)}^{y_2} q(\xi)\,\overset{\circ}{u}_\lambda(\xi)\,\sin\sqrt{\lambda}\,(\overline{\rho}_1(y_2,y_2-\xi)-t)\,d\xi\right)dt.
\end{aligned} \tag{21}$$

1.7. Now, we can estimate the integral (14). In order to do that, introduce the notations $\gamma(G,p)$ and $\gamma'(G,p)$, where

$$\gamma(G,p) \stackrel{\text{def}}{=} \max\Big\{ \sup_{x\in(a,x_0]} \sqrt{p_1(x)}\,,\ \sup_{x\in[x_0,b)} \sqrt{p_2(x)}\Big\},$$
$$\gamma'(G,p) \stackrel{\text{def}}{=} \max\Big\{ \sup_{x\in(a,x_0]} |\,p_1'(x)\,|\,,\ \sup_{x\in[x_0,b)} |\,p_2'(x)\,|\Big\}.$$

It follows then from (19) that

$$\Big|\int\limits_{y_1}^{y_2} \overset{\circ}{u}_\lambda(y)\,dy\Big| \le \Big(3\,\gamma(G,p) + \frac{\gamma'(G,p)}{2\,\alpha}\,(b-a) +$$
$$+\frac{b-a}{\alpha}\,\|\,q\,\|_{L_1(G)}\Big)\sqrt{1+\operatorname{sh}^2\big(\frac{b-a}{\alpha}\,\operatorname{Im}\sqrt{\lambda}\,\big)}\cdot\frac{1}{|\sqrt{\lambda}\,|}\cdot\sup_{y\in G}|\,\overset{\circ}{u}_\lambda(y)\,|+ \tag{22}$$
$$+\gamma^2(G,p)\sqrt{2+\operatorname{sh}^2\big(\frac{b-a}{\alpha}\,\operatorname{Im}\sqrt{\lambda}\,\big)}\cdot\frac{1}{|\,\lambda\,|}\cdot\sup_{y\in G}|\,\overset{\circ}{u}{}'_\lambda(y)\,|.$$

Let $\tilde{K}$, $r(G,\operatorname{Im}\sqrt{\lambda}\,)$ be the closed interval and the number defined in 1.1–1.2 §3 [5]. According to the content of 3.1 § 3 [5], the estimate

$$\sup_{y\in G}|\,\overset{\circ}{u}{}'_\lambda(y)\,| \le C_{01}(\tilde{K}_{R_0},p,q,\operatorname{Im}\sqrt{\lambda}\,)\,|\sqrt{\lambda}\,|\cdot\max_{y\in\tilde{K}_{R_0}}|\,\overset{\circ}{u}_\lambda(y)\,|$$

holds if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G,\operatorname{Im}\sqrt{\lambda}\,)$. On the other hand, from 3.1 in Introduction it follows the existence of a constant $C_0(\tilde{K}_{R_0},p,q,\operatorname{Im}\sqrt{\lambda}\,)$ such that

$$\sup_{y\in G}|\,\overset{\circ}{u}_\lambda(y)\,| \le C_0(\tilde{K}_{R_0},p,q,\operatorname{Im}\sqrt{\lambda}\,)\cdot\max_{y\in\tilde{K}_{R_0}}|\,\overset{\circ}{u}_\lambda(y)\,|. \tag{23}$$

Finally, if we put $K \stackrel{\text{def}}{=} \tilde{K}_{R_0}$, then Lemma 1 gives us the estimate

$$\max_{y\in K}|\,\overset{\circ}{u}_\lambda(y)\,| \le C_0(K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,\|\,\overset{\circ}{u}_\lambda\,\|_{L_2(K_R)}, \tag{24}$$

for some number $R\in(0,\rho(K,\partial G))$.

Hence, applying (in the appropriate order) the mentioned three estimates to the right-hand side of (22), we get that the inequality

$$\Big|\int\limits_{y_1}^{y_2} \overset{\circ}{u}_\lambda(y)\,dy\Big| \le \Big[\Big(3\,\gamma(G,p) + \frac{\gamma'(G,p)}{2\,\alpha}\,(b-a) + \tag{25}$$
$$+\frac{b-a}{\alpha}\,\|\,q\,\|_{L_1(G)}\Big)\,C_0(K,p,q,\operatorname{Im}\sqrt{\lambda}\,) + \gamma^2(G,p)\,C_{01}(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)\Big]\times$$
$$\times\sqrt{2+\operatorname{sh}^2\big(\frac{b-a}{\alpha}\,\operatorname{Im}\sqrt{\lambda}\,\big)}\cdot C_0(K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,\frac{1}{|\sqrt{\lambda}\,|}\,\|\,\overset{\circ}{u}_\lambda\,\|_{L_2(K_R)}$$

holds if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G, \operatorname{Im}\sqrt{\lambda}\,)$ and $x_0 \in (y_1, y_2)$.

1.8. Comparing (19) with (20)–(21), we see that estimate (25) also holds in cases considered in 1.5 and 1.6. Therefore, we may conclude that **the estimate** (7) **is proved**, with $D_{01}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,)$ denoting the constant from (25).

Note that $D_{01}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,)$ does not depend on the numbers y_1, y_2.

1.9. Throughout this section we have been assuming that $[y_1, y_2] \subset (a, b)$. If $y_1 = a$ or/and $y_2 = b$, then the corresponding estimates (7) can be established in the following way. Fix some closed interval $[\tilde{y}_1, \tilde{y}_2] \subset (a, b)$ and write estimate (7) for this interval. Then pass to the limit(s) $\lim\limits_{\tilde{y}_1 \to a+0}$ or/and $\lim\limits_{\tilde{y}_2 \to b-0}$ (in this inequality). By $\overset{\circ}{u}_\lambda(\xi) \in L_1(G)$ and the independence of $D_{01}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,)$ of $\tilde{y}_1$, $\tilde{y}_2$, it results that the estimate (7) also holds in the considered cases.

2. The estimate (6). Proof of this estimate is very simple. It will directly show that the two statements formulated in Remark 2 are valid in the considered case.

2.1. Let $\overset{\circ}{u}_\lambda(\xi)$ be an eigenfunction of the operator (1) corresponding to the eigenvalue λ. Then, using estimates (23) and (24), we have the inequalities

$$\begin{aligned}\left|\int\limits_{y_1}^{y_2} \overset{\circ}{u}_\lambda(y)\,dy\right| &\le (b-a)\cdot\sup_{y\in G}|\,\overset{\circ}{u}_\lambda(y)\,| \le \\ &\le (b-a)\,C_0(K, p, q, \operatorname{Im}\sqrt{\lambda}\,)\cdot\max_{y\in K}|\,\overset{\circ}{u}_\lambda(y)\,| \le \\ &\le (b-a)\,C_0(K, p, q, \operatorname{Im}\sqrt{\lambda}\,)\,C_0(K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,)\,\|\,\overset{\circ}{u}_\lambda\,\|_{L_2(K_R)},\end{aligned}$$

where $y_1, y_2 \in [a, b](y_1 < y_2)$ are arbitrary numbers, and $K \subset G$ is the closed interval defined in 1.7.

Hence, **the estimate** (6) **holds true** for any eigenvalue λ, with

$$D_{01}(G, K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,) \overset{\text{def}}{=} (b-a)\,C_0(K, p, q, \operatorname{Im}\sqrt{\lambda}\,)\,C_0(K_R, p, q, \operatorname{Im}\sqrt{\lambda}\,). \tag{26}$$

3. On Remarks 1–3. In order to verify Remark 1, it is sufficient to replace $\sup\limits_{x\in(a,x_0]}(\cdot)$, $\sup\limits_{x\in[x_0,b)}(\cdot)$ in definition of $\gamma'(G,p)$ by integrals $\int\limits_a^{x_0}(\cdot)\,dx$ and $\int\limits_{x_0}^{b}(\cdot)\,dx$ respectively.

3.1 It follows from estimates (22)–(23) that the first statement from Remark 2 holds true.

3.2. As we know, if $\sigma(\mathcal{L})$ is a set of eigenvalues satisfying condition (8), then it is possible to define constants $C_0(K, p, q, \cdot)$, $C_0(K_R, p, q, \cdot)$ and $C_{01}(K, p, q, \cdot)$

uniformly with respect to $\lambda \in \sigma(\mathcal{L})$. Therefore, replacing $\operatorname{Im}\sqrt{\lambda}$ by A in $\operatorname{sh}^2(\cdot)$, we conclude from (25)–(26) that the constants $D_{01}(G, K_R, p, q, \cdot)$ can be chosen independently of numbers $\lambda \in \sigma(\mathcal{L})$.

2. Estimates for the integrals of an associated function

1. The estimate (7). We begin consideration of the case $i \in \mathbb{N}$ by establishing the corresponding estimate (7).

1.1. Let $\overset{i}{u}_\lambda(\xi)$be an associated function of the i–th order corresponding to the eigenfunction $\overset{\circ}{u}_\lambda(\xi)$ and the eigenvalue $\lambda \neq 0$. As in the case of eigenfunctions, the major step in the proof is getting a convenient form for the integral

$$\int\limits_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy, \tag{27}$$

where $[y_1, y_2] \subset (a, b)$ is an arbitrary closed interval.

1.2. Let $x \in G$ be an arbitrary fixed point. Let $t > 0$ be a number such that $x + \rho_2(x, t) \in G$ and $x_0 \in (x, x + \rho_2(x, t))$. Consider the integral

$$\int\limits_{x}^{x+\rho_2(x,t)} \big(p(\xi)\,\overset{i}{u}{}'_\lambda(\xi)\big)' \sin\sqrt{\lambda}\,(\overline{\rho}_2(x, \xi - x) - t)\,d\xi,$$

and apply to it the procedure described in 1.2–1.4 § 1. (Instead of differential equations (2)–(3) it is necessary to use equations (4)–(5).) In that way we obtain the following:

$$\begin{gathered}\int\limits_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy = R_{(19)}(y_1; y_2; \lambda; \overset{i}{u}_\lambda) - \\ -\frac{1}{\sqrt{\lambda}} \int\limits_{0}^{\overline{\rho}_2(y_1, y_2 - y_1)} \left(\int\limits_{y_1}^{y_1+\rho_2(y_1,t)} \overset{i-1}{u}_\lambda(\xi)\, \sin\sqrt{\lambda}\,(\overline{\rho}_2(y_1, \xi - y_1) - t)\,d\xi \right) dt,\end{gathered} \tag{28}$$

where $R_{(19)}(\cdot)$ denotes the right-hand side of equality (19), with $\overset{\circ}{u}_\lambda$ replaced by $\overset{i}{u}_\lambda$.

Using the Fubini's theorem, transforme the integral on the right-hand side of (28). Therefore, it follows from (28) that

$$\begin{gathered}\int\limits_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy = R_{(19)}(y_1; y_2; \lambda; \overset{i}{u}_\lambda) - \\ -\frac{1}{\lambda} \int\limits_{y_1}^{y_2} \overset{i-1}{u}_\lambda(\xi)\, \big[\cos\sqrt{\lambda}\,(\overline{\rho}_2(y_1, \xi - y_1) - \overline{\rho}_2(y_1, y_2 - y_1)) - 1\big]\,d\xi.\end{gathered} \tag{29}$$

1.3. It is not necessary to write explicitly the expressions for integral (27) in the other cases: $x_0 \notin (y_1, y_2), y_1 = x_0$ or $y_2 = x_0$. Namely, analysing the content of 1.5–1.6 § 1, we can conclude that in those cases the mentioned integral has the same form, "up to" the indices and the second member on the right-hand side of (19), as in (29). Thus, any upper-bound estimate of the integral (29) will be valid for the other ones.

1.4. Now, let us estimate integral (27). We obtain from (29) the inequality

$$\left| \int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\, dy \right| \le R_{(22)}(G; p; q; \lambda; \overset{i}{u}_\lambda) + \tag{30}$$
$$+(b-a)\sqrt{2 + \operatorname{sh}^2\left(\frac{b-a}{\alpha}\ \operatorname{Im}\sqrt{\lambda}\right)} \cdot \frac{1}{|\lambda|} \cdot \sup_{y\in G} |\overset{i-1}{u}_\lambda(y)|,$$

where $R_{(22)}(\cdot)$ denotes the right–hand side of inequality (22), with $\overset{\circ}{u}_\lambda$ replaced by $\overset{i}{u}_\lambda$.

Let $\tilde{K}$, $r(G, \operatorname{Im}\sqrt{\lambda})$ be the closed interval and the number defined in 1.1–1.2 § 3 [5]. According to the content of 3.1 § 3 [5], the estimate

$$\sup_{y\in G} |\overset{i}{u}{}'_\lambda(y)| \le C_{i1}(\tilde{K}_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})\, |\sqrt{\lambda}| \cdot \max_{y\in \tilde{K}_{R_0}} |\overset{i}{u}_\lambda(y)|$$

holds if $|\operatorname{Re}\sqrt{\lambda}| > r(G, \operatorname{Im}\sqrt{\lambda})$. On the other–hand, from 3.1 in Introduction it follows the existence of a constant $C_i(\tilde{K}_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda})$ such that

$$\sup_{y\in G} |\overset{i}{u}_\lambda(y)| \le C_i(\tilde{K}_{R_0}, p, q, \operatorname{Im}\sqrt{\lambda}) \cdot \max_{y\in \tilde{K}_{R_0}} |\overset{i}{u}_\lambda(y)|. \tag{31}$$

Also, if we put $K \overset{\text{def}}{=} \tilde{K}_{R_0}$, then Lemma 1 gives us the estimate

$$\max_{y\in K} |\overset{i}{u}_\lambda(y)| \le C_i(K_R, p, q, \operatorname{Im}\sqrt{\lambda})\, \|\overset{i}{u}_\lambda\|_{L_2(K_R)}, \tag{32}$$

for some number $R \in (0, \rho(K, \partial G))$. Finally, by the proposition (b) of Lemma 2 it follows the existence of a constant $A_i(G, p, q, \operatorname{Im}\sqrt{\lambda})$ such that

$$\sup_{y\in G} |\overset{i-1}{u}_\lambda(y)| \le A_i(G, p, q, \operatorname{Im}\sqrt{\lambda})\, |\sqrt{\lambda}| \cdot \sup_{y\in G} |\overset{i}{u}_\lambda(y)|$$

if $\lambda \ne 0$.

Using the listed above estimates in the appropriate order, we obtain from (30) that the inequality

$$\begin{aligned}\Big|\int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy\Big| \le \Big[\Big(3\,\gamma(G,p)+\frac{\gamma'(G,p)}{2\,\alpha}\,(b-a)+\\ +\frac{b-a}{\alpha}\,\|\,q\,\|_{L_1(G)}\Big)\,C_i(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)+\gamma^2(G,p)\,C_{i1}(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)+\\ +2\,(b-a)\,A_i(G,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,C_i(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)\Big]\times\\ \times\sqrt{2+\operatorname{sh}^2(\frac{b-a}{\alpha}\operatorname{Im}\sqrt{\lambda}\,)}\cdot C_i(K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,\frac{1}{|\sqrt{\lambda}\,|}\,\|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}\end{aligned} \tag{33}$$

holds if $|\operatorname{Re}\sqrt{\lambda}\,| > r(G,\operatorname{Im}\sqrt{\lambda}\,)$ and $x_0 \in (y_1, y_2)$.

This inequality also holds in all cases of other positions of the point x_0 with respect to the points y_1, y_2.

Hence, it results that **the estimate** (7) **holds**, with $D_{i1}(G,K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,)$ denoting the constant from (33). Note that this constant does not depend on the numbers y_1, y_2.

1.5. If $y_1 = a$ or/and $y_2 = b$, then the proof of the corresponding estimates (7) is the same as in the case of an eigenfunction (see 1.9 § 1).

2. The estimate (6). As in the case of an eigenfunction, the following proof of estimate (6) directly shows that Remark 2 holds true.

2.1. If $y_1, y_2 \in [a,b]\,(y_1 < y_2)$ are arbitrary numbers and K is the closed interval defined in 1.6 § 1, then using estimates (31) and (32), we have the inequalities

$$\begin{aligned}\Big|\int_{y_1}^{y_2} \overset{i}{u}_\lambda(y)\,dy\Big| &\le (b-a)\cdot\sup_{y\in G}|\,\overset{i}{u}_\lambda(y)\,| \le\\ &\le (b-a)\,C_i(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)\cdot\max_{y\in K}|\,\overset{i}{u}_\lambda(y)\,| \le\\ &\le (b-a)\,C_i(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,C_i(K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,\|\,\overset{i}{u}_\lambda\,\|_{L_2(K_R)}.\end{aligned}$$

By these inequalities we conclude that **the estimate** (6) **is valid**, with

$$D_{i1}(G,K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,) \overset{\text{def}}{=} (b-a)\,C_i(K,p,q,\operatorname{Im}\sqrt{\lambda}\,)\,C_i(K_R,p,q,\operatorname{Im}\sqrt{\lambda}\,). \tag{34}$$

Proof of Theorem 1 is completed.

3. On Remarks 1–4. The statement from Remark 1 holds true; the necessary changes in the proof are described in 3 § 1.

3.1. It is not difficult to verify, by virtue of 1.4, that Remark 2 is valid.

3.2. Under the two conditions on the set $\sigma(\mathcal{L})$ described in Remark 3, all the constants $C_{i1}(\cdot)$, $C_i(\cdot)$, $A_i(\cdot)$ appearing in (33)–(34) do no depend on the numbers $\lambda \in \sigma(\mathcal{L})$. Thus, using the replacement $\operatorname{Im}\sqrt{\lambda} \to A$ in (33), we conclude that the constants $D_{i1}(G, K_R, p, q, \cdot)$ can be defined uniformly with respect to $\lambda \in \sigma(\mathcal{L})$.

3.3. It has been already shown that the constants $C_{i1}(\cdot)$, $C_i(\cdot)$, $A_i(\cdot)$ appearing in (33)–(34) do not depend on the parameter i. That is why the constants $D_{i1}(\cdot)$ have the same property.

References

1. В.А. Ильин, И.Йо, *Оценка разности частичных сумм разложений, отвечающих двум произвольным неотрицательным самосопряженным расширениям двух операторов типа Штурма–Лиувиля, для абсолютно непрерывенойфункции*, Дифф. уравнения **15**(7) (1979), 1175–1193.
2. N.L. Lažetić, *On the convegrence of spectral decompositions corresponding to ordinary differential operators of second order*, Dissertation, University of Beograd, 1980. (Serbian)
3. Н. Лажетич, *Оценки собственных и присоединенных функцийоператора Штурма–Лиувиля с разрывными коэффициентами*, ДАН СССР **258**(3) (1981), 541–544.
4. Н. Лажетич, *Оценки собственных и присоединенных функцийоператора Штурма–Лиувиля с разрывными коэффициентами (I),* Bull. Acad. Serbe Sci. Arts **84** Cl. Sci. Math. Natur. **13** (1984), 75–101.
5. N.L. Lažetić, *Estimates for derivatives and integrals of eigenfunctions and associated functions of nonselfadjoint Sturm–Liouville operator with discontinuous coefficients (I),* Publ. Inst. Math. (Beograd)(N.S.) **59**(**73**) (1996), 49–76

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