

\mathcal{O} -REGULARLY VARYING FUNCTIONS AND SOME ASYMPTOTIC RELATIONS

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Abstract. We prove that in the class of measurable positive functions defined on the interval $I_a = [a, +\infty)$ ($a > 0$), the class of functions which preserve the strong asymptotic equivalence on the set of functions $\{x: I_a \mapsto \mathbb{R}^+, x(t) \rightarrow +\infty, t \rightarrow +\infty\}$, is a class of \mathcal{O} -regularly varying functions with continuous index function. We also prove a representation theorem for functions from this class, and a morphism-theorem for some asymptotic relations.

1. Introduction and auxiliary results

Throughout this paper, we shall use the following denotations:

$$I = (0, +\infty), \quad I_a = [a, +\infty) \quad (a > 0), \quad F: I_a \mapsto I, \\ K_F(t) = \overline{\lim}_{x \rightarrow +\infty} \frac{F(tx)}{F(x)} \quad (t \in I).$$

Consider arbitrary functions x, y, F which are positive on the interval I_a , F is measurable on I_a , and $x(t), y(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Consider the relation

$$(1) \quad x(t) \sim y(t) \quad (t \rightarrow +\infty) \implies F(x(t)) \sim F(y(t)) \quad (t \rightarrow +\infty).$$

The class of all functions F satisfying relation (1) is denoted in this paper by CRV . This abbreviation is motivated by Theorem 2, where we proved that all functions $F \in CRV$ have *continuous* index functions.

Definition 1. A function $F: I_a \rightarrow I$ belongs to the class ORV if it is measurable and $K_F(t) < +\infty$ for every $t \in I$.

LEMMA 1. *For a positive and measurable function $F: I_a \mapsto I$, the next relations are mutually equivalent:*

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- (a) F satisfies relation (1);
- (b) If (a_n) and (b_n) are arbitrary sequences tending to $+\infty$, then $a_n \sim b_n$ as $n \rightarrow \infty$ implies $F(a_n) \sim F(b_n)$ as $n \rightarrow \infty$;
- (c) It holds true

$$\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \frac{F(\lambda x)}{F(x)} = 1.$$

Proof. (a) \implies (b). Taking $x(t) = a_n$ and $y(t) = b_n$ for $n \leq t < n + 1$, we have

$$\lim_{n \rightarrow \infty} \frac{F(a_n)}{F(b_n)} = \lim_{t \rightarrow +\infty} \frac{F(a_{[t]})}{F(b_{[t]})} = 1.$$

(b) \implies (c). Take any sequence $x_n > 0$ tending to $+\infty$, and any sequence $\lambda_n > 0$ tending to 1, as $n \rightarrow \infty$. Putting $a_n = \lambda_n x_n$, $b_n = x_n$ and using (b), we find

$$\lim_{n \rightarrow +\infty} \frac{F(\lambda_n x_n)}{F(x_n)} = 1,$$

whence we get (c).

(c) \implies (a). By relation (c), we conclude that for every $\epsilon > 0$, there exist some $\Delta(\epsilon)$ and $\delta(\epsilon) > 0$ such that $|F(\lambda x)/F(x) - 1| < \epsilon$ for $x \geq \Delta(\epsilon)$ and $|1 - \lambda| \leq \delta(\epsilon)$. By assumptions from (a), we have $|x(t)/y(t) - 1| \leq \delta(\epsilon)$ if $t \geq t_1$, and $y(t) \geq \Delta(\epsilon)$ if $t \geq t_0$. Taking $t_2 = \max\{t_0, t_1\}$, we finally find $|F(x(t))/F(y(t)) - 1| < \epsilon$ for $t \geq t_2$. \square

Examples. I. If F is an arbitrary function from the class RV with index $\rho \in \mathbb{R}$, then $F(x) = x^\rho L(x)$ for some function $L \in SRV$ and all $x \in I_a$. By Lemma 1(c) and the Uniform convergence theorem for slowly varying functions [3], we obtain $F \in CRV$.

II. The function $F(x) = 2 + \sin(\log x)$ ($x \in I_a$) belongs to the class ORV , but does not belong to the class RV . Nevertheless we have $\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} (F(\lambda x)/F(x)) = 1$. The similar is true for the function $F(x) = \exp\{\sin(2 \log x)\}$ ($x \in I_a$).

III. If ϵ is an arbitrary bounded, measurable function on the interval I_B , then the function $F(x) = \exp\left\{\int_B^x \frac{\epsilon(t)}{t} dt\right\}$ ($x \in I_B$) belongs to the class ORV , and we have the inequality $\lambda^a \leq F(\lambda x)/F(x) \leq \lambda^b$ ($x \in I_B, \lambda \geq 1$), whenever $a \leq \epsilon(t) \leq b$ ($t \in I_B$). For any value $\lambda < 1$, we have the reverse inequalities. Hence, we have $\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} (F(\lambda x)/F(x)) = 1$, i.e. $F \in CRV$.

Remark. By a straightforward calculation, it can be easily checked that

- (a) if $F, G \in CRV$, then $F \cdot G \in CRV$;
- (b) if $F \in CRV$, then $1/F \in CRV$;
- (c) if $F, G \in CRV$, $F \circ G$ is a measurable function, and $G(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, then $F \circ G \in CRV$.

Hence we conclude the class CRV is a multiplicative group, and the class $CRV^\infty = \{F \in CRV \mid \lim_{x \rightarrow +\infty} F(x) = +\infty\}$ is a groupoid with respect to the composition of functions.

IV. Let F be any function from the Matuszewska class [3]. Then for some $c \in \mathbb{R}$, and some measurable, bounded functions μ and ϵ on I , such that $\lim_{x \rightarrow \infty} \mu(x) = 0$, we have the representation

$$(2) \quad F(x) = \exp \left\{ c + \mu(x) + \int_1^x \frac{\epsilon(t)}{t} dt \right\}.$$

From this representation, we conclude that $F \in CRV$.

V. The function $F(x) = 2 + \sin x$ ($x \in I$) belongs to the class ORV , but the limit $\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} (F(\lambda x)/F(x))$ does not exist. Hence $F \notin CRV$.

LEMMA 2. *Every function F from the class CRV defined on I_a , belongs to the class ORV .*

Proof. If $F \in CRV$ and $\epsilon > 0$, then there are an $x_0 \in I_a$ and $\delta > 0$ such that $1 - \epsilon \leq F(\lambda x)/F(x) \leq 1 + \epsilon$ whenever $x \geq x_0$, $|1 - \lambda| \leq \delta$. Hence $F \in ORV$. \square

Since the class CRV is a proper subclass of the class ORV , and any function $F \in ORV$ has the representation

$$(3) \quad F(x) = \exp \left\{ \mu(x) + \int_B^x \frac{\epsilon(t)}{t} dt \right\} \quad (x \in I_B),$$

for some measurable, bounded functions μ and ϵ on I_B , by the remark above and Example III, we have

$$(4) \quad F \in CRV \iff \lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} (\mu(\lambda x) - \mu(x)) = 0.$$

LEMMA 3. *For a continuous function $h: I_{\log A} \rightarrow \mathbb{R}$, the next conditions are equivalent:*

- (a) h is uniformly continuous on $I_{\log A}$;
- (b) $\lim_{\substack{\lambda \rightarrow 0 \\ x \rightarrow +\infty}} (h(\lambda + x) - h(x)) = 0$.

Proof. (a) \implies (b). If h is uniformly continuous on $I_{\log A}$ and $\epsilon > 0$, then there is a $\delta > 0$ such that $|h(x') - h(x'')| < \epsilon$ for arbitrary $x', x'' \in I_{\log A}$ such that $|x' - x''| \leq \delta$. Thus, we have $|h(\lambda + x) - h(x)| < \epsilon$ whenever $x, \lambda + x \in I_{\log A}$, and $|\lambda| \leq \delta$. Hence, for sufficiently large $x \in I_{\log A}$ and $|\lambda| \leq \delta$ we have $|h(\lambda + x) - h(x)| < \epsilon$, i.e. we have (b).

(b) \implies (a). From (b) it follows that for any $\epsilon > 0$ there are x_0, λ_0 such that $|h(\lambda + x) - h(x)| < \epsilon$ whenever $x \geq x_0$ and $|\lambda| \leq \lambda_0$. Hence, h is a uniformly continuous function on I_{x_0} . If $x_0 > \log A$, then h is uniformly continuous on the interval $[\log A, x_0]$, so h is uniformly continuous on $[\log A, +\infty)$. \square

COROLLARY 1. *For any function F , positive and continuous on the interval I_A , the next conditions are mutually equivalent:*

- (a) $F \in CRV$;
- (b) $F(x) = \exp\{h(\log x)\}$ ($x \geq A$), where the function $h(x) = \log F(e^x)$ ($x \in I_{\log A}$) is uniformly continuous on $I_{\log A}$.

LEMMA 4. *If $h: I_{\log A} \mapsto \mathbb{R}$ is a measurable and bounded function on $I_{\log A}$, then the next conditions are mutually equivalent:*

- (a) $h(x) = \mu(x) + r(x)$ ($x \in I_{\log A}$), where μ is a measurable, bounded function such that $\mu(x) \rightarrow 0$ as $x \rightarrow +\infty$, and r is bounded and uniformly continuous on $I_{\log A}$;

$$(b) \quad \lim_{\substack{\lambda \rightarrow 0 \\ x \rightarrow +\infty}} (h(\lambda + x) - h(x)) = 0.$$

Proof. (a) \implies (b) is trivial.

(b) \implies (a). For any $\delta > 0$, denote $\omega_\delta(x) = \sup\{|h(x') - h(x)| : x, x' \in I_{\log A}\}$. Then for every $\epsilon > 0$, there is an $M = M(\epsilon)$ and a $\delta_0 = \delta_0(\epsilon) > 0$ such that $\omega_\delta(x) < \epsilon$ for $x \geq M, \delta \leq \delta_0$. Next for $t \geq B \geq \log A$, there is a function $\varphi_t(x) \in C(\mathbb{R})$ such that $\text{supp } \varphi_t(x) \subseteq (t - 1/t, t + 1/t)$, $\varphi_t \geq 0$ on \mathbb{R} , $\int_{\mathbb{R}} \varphi_t(x) dx = 1$ and $F(t, x) = \varphi_t(x)$ is a continuous function with respect to the variables (t, x) . Then $f(t) = \int_{I_B} \varphi_t(x) h(x) dx = \int_{I_B} F(t, x) h(x) dx$ is a continuous function on $t \in I_B$. Now $\nu(t) = h(t) - f(t)$ is a measurable function on I_B , and we have

$$\begin{aligned} |\nu(t)| &= |h(t) - f(t)| = \left| \int_{I_B} \varphi_t(x) (h(x) - h(t)) dx \right| \leq \\ &= \int_{I_B \cap (t-1/t, t+1/t)} \varphi_t(x) |h(x) - h(t)| dx \leq \omega_{2/t}(t). \end{aligned}$$

Hence we obtain $\nu(t) \rightarrow 0$ as $t \rightarrow +\infty$, and consequently $h(t) = f(t) + \nu(t)$ ($t \in I_B$). Since $\nu(t) \rightarrow 0$ as $t \rightarrow +\infty$, we find ν and consequently f satisfy condition (b). Denoting $B = \log A$, $f(x) = r(x)$ and $\nu(x) = \mu(x)$ for $x \in I_{\log A}$, we see that both r and μ are bounded on $I_{\log A}$. By Lemma 3, we immediately get condition (a). \square

Remark. By a straightforward calculation, one can check the function

$$\varphi_t(x) = F(t, x) = \begin{cases} 0, & x \in (-\infty, t - 1/2t] \cup [t + 1/2t, +\infty) \\ 2t(1 - 2t|x - t|), & |x - t| < 1/2t \end{cases}$$

defined for $t \geq B$, satisfies all the conditions from Lemma 4.

2. Representation and characterization theorems for the class CRV

THEOREM 1 (Representation theorem). *A function F belongs to the class CRV if and only if we have*

$$(5) \quad F(x) = \exp \left\{ \tilde{\mu}(x) + r(\log x) + \int_B^x \frac{\epsilon(t)}{t} dt \right\} \quad (x \in I_B),$$

where ϵ is a measurable, bounded function on some interval I_B ($B \in I$), $\tilde{\mu}$ is a measurable, bounded function on the same interval I_B such that $\tilde{\mu}(x) \rightarrow 0$ as $x \rightarrow +\infty$, and r is a uniformly continuous, bounded function on the interval $I_{\log B}$.

Proof. By relation (4), a function $F \in CRV$ if and only if

$$\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \left(\mu_1(\log \lambda + \log x) - \mu_1(\log x) \right) = 0,$$

where $\mu_1(\log x) = \mu(x)$ ($x \in I_B$), thus $\mu_1(x) = \mu(e^x)$ ($x \in I_{\log B}$). Using the substitution $\log \lambda = k$, $\log x = t$, the limit above can be written in the form

$$\lim_{\substack{k \rightarrow 0 \\ t \rightarrow +\infty}} \left(\mu_1(k + t) - \mu_1(t) \right) = 0.$$

By Lemma 4, the previous condition is satisfied if and only if the measurable and bounded function $\mu_1(x) = \mu(e^x)$ can be written as $\mu_1(x) = \bar{\mu}(x) + r(x)$ on $I_{\log B}$, where $\bar{\mu}$ and r are bounded, $\bar{\mu}$ is measurable, $\bar{\mu}(x) \rightarrow 0$ as $x \rightarrow +\infty$, and r is uniformly continuous on $I_{\log B}$. Here we can obviously use $\mu(x) = \mu_1(\log x) = \bar{\mu}(\log x) + r(\log x)$ for $x \in I_B$ and $\bar{\mu}(\log x) = \tilde{\mu}(x)$ for $x \in I_B$. \square

THEOREM 2 (Characterization theorem). *A function F belongs to the class CRV if and only if the index function $K_F(\lambda)$ is continuous in $\lambda \in I$.*

Proof. Assume $F \in CRV$. Then for every $\epsilon > 0$, there are a $\delta > 0$ and $\Delta > 0$ so that

$$1 - \epsilon < \frac{F(\lambda x)}{F(x)} < 1 + \epsilon,$$

if $x \geq \Delta$ and $|1 - \lambda| \leq \delta$. Hence we get $1 - \epsilon < K_F(\lambda) < 1 + \epsilon$ if $|1 - \lambda| \leq \delta$, thus $\lim_{\lambda \rightarrow 1} K_F(\lambda) = 1$. By a result from [2], it follows the function $K_F(\lambda)$ is continuous on I .

Next assume $F \in ORV$ and the function $K_F(\lambda)$ is continuous on I . By a result from [2], we then have

$$\lim_{u \rightarrow +\infty} \sup_{\lambda \in T} \left(\sup_{x \geq u} \frac{F(\lambda x)}{F(x)} - K_F(\lambda) \right) = 0,$$

for any compact $T \subseteq I$. Hence, for any $\epsilon > 0$, there are a $u_0 > 0$ and a $p > 1$ such that $F(\lambda x)/F(x) \leq K_F(\lambda) + \epsilon$ whenever $x \geq u_0$ and $1/p \leq \lambda \leq p$. Consequently, $\overline{\lim}_{\lambda \rightarrow 1, x \rightarrow +\infty} (F(\lambda x)/F(x)) \leq 1 + \epsilon$. If we put $y = \lambda x$ and $\mu \lambda = 1$, we obtain $F(y)/F(\mu y) \leq K_F(1/\mu) + \epsilon$ whenever $1/p \leq \mu \leq p$ and $y \geq pu_0$. Consequently, we obtain $\overline{\lim}_{\mu \rightarrow 1, y \rightarrow +\infty} (F(y)/F(\mu y)) \leq 1 + \epsilon$. Hence we get

$$\frac{1}{1 + \epsilon} \leq \lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \frac{F(\lambda x)}{F(x)} \leq \overline{\lim}_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \frac{F(\lambda x)}{F(x)} \leq 1 + \epsilon,$$

and consequently $\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} (F(\lambda x)/F(x)) = 1$. This means that $F \in CRV$. \square

COROLLARY 2. *If F is a measurable, positive function on the interval I_α , then the following is true:*

- (a) $\overline{\lim}_{\lambda \rightarrow 1} \overline{\lim}_{x \rightarrow +\infty} \frac{F(\lambda x)}{F(x)} = 1 \implies \lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \frac{F(\lambda x)}{F(x)} = 1;$
- (b) $\lim_{\substack{\lambda \rightarrow 1 \\ x \rightarrow +\infty}} \frac{F(\lambda x)}{F(x)} = 1 \implies \lim_{\lambda \rightarrow p} \overline{\lim}_{x \rightarrow +\infty} \frac{F(\lambda x)}{F(x)} = K_F(p), p > 0$
 $(K_F(1) = 1).$

For implication (a), one can consult [2].

COROLLARY 3. *If F is any continuous positive function on the interval I_A , then the following conditions are equivalent:*

- (a) $\overline{\lim}_{\lambda \rightarrow 1} \overline{\lim}_{x \rightarrow +\infty} \frac{F(\lambda x)}{F(x)} = 1;$
- (b) *The function $h(x) = \log F(e^x)$ is uniformly continuous on the interval $I_{\log A}$.*

COROLLARY 4. *Every function from the Matuszewska class ERV has the continuous index function $K_F(\lambda)$ on the interval I .*

By the next example, we prove that $ERV \subsetneq CRV$.

Example VI. Notice that the function $F(x) = \exp(|\sin(\log x)|)^{1/2}$ has the following properties:

- (a) $F: I \mapsto I; F(1) = 1;$
- (b) For arbitrary $s, t \in I$,

$$\begin{aligned} F(st) &= \exp(|\sin(\log s + \log t)|)^{1/2} \leq \exp(|\sin \log s| + |\sin \log t|)^{1/2} \leq \\ &\leq \exp((\sin \log s)^{1/2} + (\sin \log t)^{1/2}) = F(s) \cdot F(t). \end{aligned}$$

- (c) If $p = e^{2\pi} > 1$, then $F(pt) = \exp(|\sin(\log t + 2\pi)|)^{1/2} = F(t).$

Hence, by a result from [2], we have $\overline{\lim}_{x \rightarrow +\infty} (F(\lambda x)/F(x)) = F(\lambda)$ for every $\lambda \in I$, which gives $F \in CRV$. On the other side, assuming $F \in ERV$, we find a real number c such that $\overline{\lim}_{x \rightarrow +\infty} (F(\lambda x)/F(x)) = K_F(\lambda) \leq \lambda^c$ ($\lambda \geq 1$), that is $(\log K_F(\lambda))/(\log \lambda) \leq c$, for every $\lambda > 1$. Then

$$\frac{\log K_F(\lambda)}{\log \lambda} = \frac{\log F(\lambda)}{\log \lambda} = \frac{(|\sin(\log \lambda)|)^{1/2}}{\log \lambda} \leq c$$

for all λ close to 1. But since

$$\lim_{\lambda \rightarrow 1^+} \frac{(|\sin \log \lambda|)^{1/2}}{\log \lambda} = \lim_{\lambda \rightarrow 1^+} \frac{1}{(\log \lambda)^{1/2}} = +\infty,$$

we get the contradiction. This proves that $F \notin ERV$. \square

Since

$$SRV \subsetneq RV \subsetneq ERV \subsetneq CRV \subsetneq ORV.$$

using the corresponding representation theorems for classes ERV , CRV and ORV , we get the following conclusion: If ϵ is a bounded, measurable function on the interval I_B , and

$$F(x) = \exp \left\{ \nu(x) + \int_B^x \frac{\epsilon(t)}{t} dt \right\} \quad (x \in I_B),$$

then $F(x) \in ORV$ if and only if ν is a measurable and bounded function on I_B , while $F(x) \in CRV$ if and only if $\nu(x) = r(\log x) + \mu(x)$ ($x \in I_B$) where $r \circ \log$, μ are bounded on I_B , r is uniformly continuous on $I_{\log B}$, μ is measurable on I_B , and $\mu(x) \rightarrow 0$ as $x \rightarrow +\infty$. We also recall that $F(x) \in ERV$ if and only if $\nu(x) = c + \mu(x)$ ($x \in I_B$), where $c \in \mathbb{R}$ and μ is a measurable and bounded on I_B such that $\mu(x) \rightarrow 0$ as $x \rightarrow +\infty$.

3. Morphism theorem

Let $H = \{x | x: I_a \mapsto I\}$, $H_1 = \{x \in H | x(t) \rightarrow +\infty, t \rightarrow +\infty\}$. If \simeq denotes the asymptotic similarity of functions, and ρ_1, ρ_2 are arbitrary relations from the set $\{\sim, \simeq, \asymp\}$, then let $\text{Hom}((H_1, \rho_1); (H, \rho_2))$ be the set of all measurable functions $F: I_a \mapsto I$ such that

$$x(t) \rho_1 y(t) \quad (t \rightarrow +\infty) \implies F(x(t)) \rho_2 F(y(t)) \quad (t \rightarrow +\infty)$$

for any two functions $x, y \in H_1$.

THEOREM 3. *Let $F: I_a \mapsto I$ be a measurable function. Then:*

- (a) $F \in ORV \implies F \in \text{Hom}((H_1, \asymp); (H, \asymp));$
- (b) $F \in \text{Hom}((H_1, \sim); (H, \asymp)) \implies F \in ORV;$
- (c) $F \in CRV \implies F \in \text{Hom}((H_1, \sim); (H, \sim));$
- (d) $F \in \text{Hom}((H_1, \sim); (H, \simeq)) \implies F \in CRV;$
- (e) $F \in RV \iff F \in \text{Hom}((H_1, \simeq); (H, \simeq));$
- (f) $F \in SRV \implies F \in \text{Hom}((H_1, \asymp); (H, \sim));$
- (g) $F \in \text{Hom}((H_1, \asymp); (H, \simeq)) \implies F \in SRV;$
- (h) $F \in \text{Hom}((H_1, \simeq); (H, \sim)) \implies F \in SRV.$

Proof. (a) If a function $F \in ORV$, then using some results from [1] we have $\overline{\lim}_{x \rightarrow +\infty} \sup_{\lambda \in K} (F(\lambda x)/F(x)) < +\infty$ for any compact interval $K \subseteq I$. Hence, for any functions $x, y \in H_1$ such that $x(t) \asymp y(t)$ as $t \rightarrow +\infty$, there is an $M > 0$ such that

$$\frac{F(x(t))}{F(y(t))} = \frac{F\left(\frac{x(t)}{y(t)}y(t)\right)}{F(y(t))} \leq M < +\infty \quad (t \geq t_0).$$

Similarly, since $\underline{\lim}_{x \rightarrow +\infty} (F(\lambda x)/F(x)) > 0$ holds uniformly in $\lambda \in K$, we find

$$\frac{F(x(t))}{F(y(t))} = \frac{F\left(\frac{x(t)}{y(t)}y(t)\right)}{F(y(t))} \geq m > 0,$$

for some $m > 0$ and all $t \geq t_0$. Consequently, $F \in \text{Hom}((H_1, \asymp); (H, \asymp))$.

(b) If $F \in \text{Hom}((H_1, \sim); (H, \asymp))$, then similarly as in Lemmas 1 and 2 we obtain $F \in ORV$.

(c) If $F \in CRV$, then by Theorem 2 we have $F \in \text{Hom}((H_1, \sim), (H, \sim))$.

(d) Assume $F \in \text{Hom}((H_1, \sim); (H, \simeq))$. Then for arbitrary functions $x, y \in H_1$ such that $x(t) \sim y(t)$ as $t \rightarrow +\infty$ we have $F(x(t)) \simeq F(y(t))$ as $t \rightarrow +\infty$. Taking two arbitrary sequences $(a'_n), (b'_n)$ tending to $+\infty$ as $n \rightarrow \infty$, such that $a'_n \sim b'_n$ as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} (F(a'_n)/F(b'_n)) = c'$. Taking any other sequences $(a''_n), (b''_n)$ with the similar properties, we get $\lim_{n \rightarrow \infty} (F(a''_n)/F(b''_n)) = c''$. If a_n is the general term of the sequence $a'_1, a''_1, a'_2, a''_2, \dots$, and b_n is the general term of the sequence $b'_1, b''_1, b'_2, b''_2, \dots$, then we obviously have $\lim_{n \rightarrow \infty} (F(a_n)/F(b_n)) = c$. Hence we get $c = c' = c'' = 1$, thus $F \in CRV$.

(e) If $F \in RV$, then there is a function $L \in SRV$ and a number $\rho \in \mathbb{R}$ such that $F(t) = L(t)t^\rho$ for all $t \in I_a$. If x, y are arbitrary functions from the class H_1 such that $x(t)/y(t) \rightarrow c > 0$ as $t \rightarrow +\infty$, then

$$\lim_{t \rightarrow +\infty} \frac{F(x(t))}{F(y(t))} = \lim_{t \rightarrow +\infty} \left(\frac{x(t)}{y(t)}\right)^\rho \cdot \lim_{t \rightarrow +\infty} \frac{L\left(\frac{x(t)}{y(t)}y(t)\right)}{L(y(t))} = c^\rho,$$

which means that $F \in \text{Hom}((H_1, \simeq); (H, \simeq))$. Here we used the Uniform convergence theorem for slowly varying functions.

Conversely, suppose $F \in \text{Hom}((H_1, \simeq); (H, \simeq))$, $\lambda > 0$, $x \in H_1$ and $y = \lambda x$. Then

$$\lim_{t \rightarrow +\infty} \frac{F(y(t))}{F(x(t))} = \lim_{t \rightarrow +\infty} \frac{F(\lambda x(t))}{F(x(t))} = d(\lambda) < +\infty.$$

Using the characterization theorem for functions from the class RV , we have $d(\lambda) = \lambda^\rho$ for some $\rho \in \mathbb{R}$, which gives $F \in RV$.

(f) If $F \in SRV$, then by the Uniform convergence theorem for slowly varying functions, we obtain $F \in \text{Hom}((H_1, \asymp); (H, \sim))$.

(g) Let $F \in \text{Hom}((H_1, \asymp); (H, \simeq))$. Then for arbitrary sequences (a'_n) , (b'_n) such that $a'_n, b'_n \rightarrow +\infty$ and $a'_n \asymp b'_n$, we have $\lim_{n \rightarrow +\infty} (F(a'_n)/F(b'_n)) = c'$. For any other sequences $(a''_n), (b''_n)$ with the similar properties, we obtain $\lim_{n \rightarrow +\infty} (F(a''_n)/F(b''_n)) = c''$. Denoting the general term of the sequence $a'_1, a''_1 a'_2, a''_2, \dots$ by a_n , and the general term of the sequence $b'_1, b''_1 b'_2, b''_2, \dots$ by b_n , we have $a_n \asymp b_n$ as $n \rightarrow \infty$, whence $\lim_{n \rightarrow +\infty} (F(a_n)/F(b_n)) = c$. Hence we get $c = c' = c'' = 1$. If next (a_n) is an arbitrary sequence such that $a_n \rightarrow +\infty$ as $n \rightarrow \infty$, $\lambda > 0$ and $b_n = \lambda a_n$, then we get $\lim_{n \rightarrow +\infty} (F(\lambda a_n)/F(a_n)) = 1$. This means that $F \in \text{SRV}$.

(h) Finally, assume $F \in \text{Hom}((H_1, \simeq); (H, \sim))$, $x \in H_1$ and $y = \lambda x$ for some $\lambda > 0$. Then obviously

$$\lim_{t \rightarrow +\infty} \frac{F(y(t))}{F(x(t))} = \lim_{t \rightarrow +\infty} \frac{F(\lambda x(t))}{F(x(t))} = 1,$$

which means that $F \in \text{SRV}$. \square

COROLLARY 5. *Let $F: I_a \mapsto I$ be a measurable function. Then:*

- (a) $F \in \text{ORV} \implies F \in \text{Hom}((H_1, \simeq); (H, \asymp)) \cap \text{Hom}((H_1, \sim); (H, \asymp));$
- (b) $F \in \text{Hom}((H_1, \asymp); (H, \asymp)) \cup \text{Hom}((H_1, \simeq); (H, \asymp)) \implies F \in \text{ORV};$
- (c) $F \in \text{CRV} \implies F \in \text{Hom}((H_1, \sim); (H, \simeq));$
- (d) $F \in \text{Hom}((H_1, \sim); (H, \sim)) \implies F \in \text{CRV};$
- (e) $F \in \text{SRV} \implies F \in \text{Hom}((H_1, \asymp); (H, \simeq)) \cap \text{Hom}((H_1, \simeq); (H, \sim));$
- (f) $F \in \text{Hom}((H_1, \asymp); (H, \sim)) \implies F \in \text{SRV}.$

Remark. The proposition " $F \in \text{ORV}$ if and only if $F \in \text{Hom}((H_1, \simeq); (H, \asymp))$ " is closely related to the definition of the class ORV , which have been introduced by V. Avakumović 1935.

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