

LITTLEWOOD-PALEY TYPE INEQUALITIES FOR \mathcal{M} -HARMONIC FUNCTIONS

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Abstract. We prove several Littlewood-Paley type inequalities for \mathcal{M} -harmonic and analytic functions on the unit ball B of C^n . Further, we give some characterizations of \mathcal{M} -harmonic and analytic Hardy spaces on B .

1. Introduction

Let B denote the open unit ball in C^n , $n > 1$, with boundary S . We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability measure on S .

The main purpose of this paper is to prove the following theorem.

THEOREM 1.1. *If f is a function in $L^p(S)$, $1 < p < \infty$, and u is a function on B defined via the invariant Poisson integral of f , then*

$$(1.1) \quad \left(\int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^n d\tau(z) + |u(0)|^p \right) \succeq \int_S |f(\xi)|^p d\sigma(\xi), \text{ for } 1 < p \leq 2,$$

and

$$(1.2) \quad \left(\int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^n d\tau(z) + |u(0)|^p \right) \preceq \int_S |f(\xi)|^p d\sigma(\xi), \text{ for } 2 \leq p < \infty,$$

where $\tilde{\nabla}$ and τ denote the invariant gradient and invariant measure on B .

We also show that $|\tilde{\nabla}u(z)|$ in (1.1) and (1.2) may be replaced by $(1 - |z|^2) \times |\nabla u(z)|$, where ∇ denotes the real gradient of u and by $(1 - |z|^2)(|Ru(z)| + |\bar{R}u(z)|)$, where, as usual, $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ is the radial derivative.

THEOREM 1.2. *If $f \in L^p(S)$, $1 < p < \infty$, and u is the invariant Poisson integral of f , then for $1 < p \leq 2$ we have*

$$(1.3) \quad \left(\int_B |\nabla u(z)|^p (1 - |z|^2)^{p-1} d\nu(z) + |u(0)|^p \right) \succeq \int_S |f(\xi)|^p d\sigma(\xi)$$

$$(1.4) \quad \left(\int_B (|Ru(z)| + |\bar{R}u(z)|)^p (1 - |z|^2)^{p-1} d\nu(z) + |u(0)|^p \right) \succeq \int_S |f(\xi)|^p d\sigma(\xi),$$

and for $2 \leq p < \infty$ we have

$$(1.5) \quad \left(\int_B (|Ru(z)| + |\bar{R}u(z)|)^p (1 - |z|^2)^{p-1} d\nu(z) + |u(0)|^p \right) \preceq \int_S |f(\xi)|^p d\sigma(\xi),$$

$$(1.6) \quad \left(\int_B |\nabla u(z)|^p (1 - |z|^2)^{p-1} d\nu(z) + |u(0)|^p \right) \preceq \int_S |f(\xi)|^p d\sigma(\xi).$$

For $n = 1$ (1.3) and (1.6) are well known inequalities of Littlewood and Paley [11]. Various generalizations of their result are referred to as a Littlewood-Paley type inequalities.

The method of proof of Theorem 1.1 we will present is based on local estimates for \mathcal{M} -harmonic functions (which will be defined in Section 2) and the following theorems that allow us to express the L^p norm of f in terms of some area integrals, and which are of interest on their own right.

THEOREM 1.3. *Let u be \mathcal{M} -harmonic function on B . If $1 < p < \infty$, then*

$$(1.7) \quad \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi) = \frac{c_p r^{1-2n} (1-r^2)^{n-1}}{2n} \int_{rB} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 d\tau(z),$$

where $c_p = p(p-1)$.

For $1 < p < \infty$, let $\mathcal{H}^p = \mathcal{H}^p(B)$ denote the set of \mathcal{M} -harmonic functions u on B , $u \in \mathcal{M}$, for which $|u|^p$ has an \mathcal{M} -harmonic majorant on B . It is well known that $u \in \mathcal{H}^p(B)$ if and only if $\|u\|_{\mathcal{H}^p} = \sup_{0 < r < 1} M_p(r, u) < \infty$, where, as usual, $M_p^p(r, u) = \int_S |u(r\xi)|^p d\sigma(\xi)$.

THEOREM 1.4. *A function u \mathcal{M} -harmonic on B belongs to \mathcal{H}^p , $1 < p < \infty$, if and only if*

$$\int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 (1 - |z|^2)^{-1} d\nu(z) < \infty$$

Furthermore, if $u \in \mathcal{H}^p$, $1 < p < \infty$, then

$$(1.8) \quad \|u\|_{\mathcal{H}^p}^p = \int_B |u(z)|^p d\nu(z) + \frac{p(p-1)}{4n^2} \int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 (1 - |z|^2)^{-1} d\nu(z),$$

Let

$$(1.9) \quad G(\rho, r) = \frac{1}{2n} \int_{\rho}^r t^{1-2n} (1-t^2)^{n-1} dt, \quad 0 \leq \rho \leq r \leq 1,$$

As a corollary of Theorem 1.3 we have another characterization of the Hardy space \mathcal{H}^p .

THEOREM 1.5. *Let $1 < p < \infty$. A function $u \in \mathcal{M}$ belongs to \mathcal{H}^p if and only if*

$$\int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 G(|z|, 1) d\tau(z) < \infty.$$

Moreover, if $u \in \mathcal{H}^p$, $1 < p < \infty$, then

$$(1.10) \quad \|u\|_{\mathcal{H}^p}^p = |u(0)|^p + p(p-1) \int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 G(|z|, 1) d\tau(z).$$

The method of proof of Theorems concerning \mathcal{M} -harmonic functions we will present can also be applied to Hardy spaces H^p of holomorphic functions. Recall that a holomorphic function f on B , $f \in H(B)$, belongs to the Hardy space H^p , $0 < p < \infty$, if and only if $\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty$. An analogue of the identity (1.7) of the Hardy-Stein-Spencer type for analytic functions is as follows.

THEOREM 1.6. *Let $f \in H(B)$. If $0 < p < \infty$, then*

$$(1.11) \quad \frac{d}{dr} \int_S |f(r\xi)|^p d\sigma(\xi) = \frac{p^2}{4} \frac{r^{1-2n} (1-r^2)^{n-1}}{2n} \int_{rB} |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 d\tau(z),$$

An application of the identity (1.11) gives the following characterization of the Hardy spaces H^p .

THEOREM 1.7. *A function f holomorphic on B belongs to H^p , $0 < p < \infty$, if and only if*

$$\int_B |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{-1} d\nu(z) < \infty.$$

Furthermore, if $f \in H^p$, $0 < p < \infty$, then

$$(1.12) \quad \|f\|_{H^p}^p = \int_B |f(z)|^p d\nu(z) + \frac{p^2}{8n} \int_B |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{-1} d\nu(z).$$

THEOREM 1.8. *Let $0 < p < \infty$. A function $f \in H(B)$ belongs to H^p if and only if*

$$\int_B |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 G(|z|, 1) d\tau(z) < \infty$$

Furthermore, if $f \in H^p$, $0 < p < \infty$, then

$$(1.13) \quad \|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{4} \int_B |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 G(|z|, 1) d\tau(z).$$

The characterizations of Hardy spaces \mathcal{H}^p , $1 < p < \infty$, and H^p , $0 < p < \infty$, given in Theorems 1.4 and 1.7 are known (see [14]). The proofs we will present are based on the new identities (1.8) and (1.12) and they are simpler than the proofs given in [14].

For another proof of (1.13) see [4]. See also [14].

The following identity due to Beatrous and Burbea [2] was first proved by Hardy, Stein and Spenser (see [6, p. 42]) for $n = 1$.

THEOREM 1.9. *Let $f \in H(B)$, $0 < p < \infty$ and $0 < r < 1$. Then*

$$(1.14) \quad r \frac{d}{dr} M_p^p(r, f) = \frac{p^2}{2n} \int_{rB} |z|^{-2n} |f(z)|^{p-2} |Rf(z)|^2 d\nu(z).$$

An application of the identity (1.14) gives the criteria for f holomorphic in B to belong to the Hardy space H^p .

THEOREM 1.10. [2] *Let $0 < p < \infty$ and $f \in H(B)$. Then the following statements are equivalent:*

- (i) $f \in H^p$,
- (ii) $\int_B |f(z)|^{p-2} |Rf(z)|^2 (1 - |z|^2) d\nu(z) < \infty$.

Since $|\tilde{\nabla} f(z)| \geq (1 - |z|^2) |\nabla f(z)| \geq (1 - |z|^2) |Rf(z)|$, (see Section 2) the following theorem is an immediate consequence of Theorems 1.7 and 1.10.

THEOREM 1.11. *A function f holomorphic on B belongs to H^p , $0 < p < \infty$, if and only if*

$$\int_B |\nabla f(z)|^2 |f(z)|^{p-2} (1 - |z|^2) d\nu(z) < \infty.$$

Remark 1. It is authors belief that the results of Theorems 1.10 and 1.11 should hold for the Hardy spaces \mathcal{H}^p , $1 < p < \infty$ of \mathcal{M} -harmonic functions

As a final result we have the inequalities of Littlewood-Paley type for analytic functions.

THEOREM 1.12. *Let $f \in H(B)$. If $0 < p \leq 2$, then*

$$(1.15) \quad \|f\|_p^p \preceq \left(\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{-1} d\nu(z) + |f(0)|^p \right),$$

$$(1.16) \quad \|f\|_p^p \preceq \left(\int_B |\nabla f(z)|^p (1 - |z|^2)^{p-1} d\nu(z) + |f(0)|^p \right),$$

$$(1.17) \quad \|f\|_p^p \preceq \left(\int_B |Rf(z)|^p (1 - |z|^2)^{p-1} d\nu(z) + |f(0)|^p \right),$$

and if $2 \leq p < \infty$, then

$$(1.18) \quad \left(|f(0)|^p + \int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{-1} d\nu(z) \right) \leq \|f\|_p^p,$$

$$(1.19) \quad \left(|f(0)|^p + \int_B |\nabla f(z)|^p (1 - |z|^2)^{p-1} d\nu(z) \right) \leq \|f\|_p^p,$$

$$(1.20) \quad \left(|f(0)|^p + \int_B |Rf(z)|^p (1 - |z|^2)^{p-1} d\nu(z) \right) \leq \|f\|_p^p.$$

The usual method of proof of the inequality (1.20) ((1.19), resp.) is to apply the Riesz Convexity Theorem to the operator $f \rightarrow |Rf(z)|(1 - |z|^2)$ ($f \rightarrow |\nabla f(z)|(1 - |z|^2)$) acting on functions f on the measure space (S, σ) and taking them to functions on $(B, (1 - |z|^2)^{-1} d\nu(z))$. It is relatively easy to show that this operator is of type (2.2) as well as (∞, ∞) and the Riesz theorem produces (1.20) ((1.19), resp.). By duality we have (1.17) and (1.16) for $1 < p \leq 2$. Additional considerations show that (1.17) holds also for $0 < p \leq 1$. See [3]. Obviously (1.15) and (1.16) are consequences of (1.17). For the inequality (1.18) see [5].

Our argument is different. We show that

$$\int_B |\tilde{\nabla} f(z)|^p (1 - |z|^2)^{-1} d\nu(z) \leq \int_B |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2 (1 - |z|^2)^{-1} d\nu(z),$$

for $2 \leq p < \infty$, and that the reverse inequality holds for $0 < p \leq 2$, and then we apply the identity (1.12) established in Theorem 1.7.

2. Notations and preliminary results

Let $\tilde{\Delta}$ be the invariant Laplacian on B . That is, $\tilde{\Delta}u(z) = \Delta(u \circ \varphi_z)(0)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z [13]. (It can be shown that $\tilde{\Delta}$ is equal $(n + 1)$ times the Laplacian with respect to the Bergman metric). As in [13],

$$\tilde{\Delta} = 4(1 - |z|^2) \sum_{i,j} (\delta_{i,j} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

The real valued functions annihilated by $\tilde{\Delta}$ are called invariantly harmonic or \mathcal{M} -harmonic.

The operator $\tilde{\Delta}$ is \mathcal{M} -invariant in the sense that $\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta}f) \circ \varphi$, for $\varphi \in \text{Aut}(B)$ [13, Theorem 4.1.2]. This implies that the class \mathcal{M} of \mathcal{M} -harmonic functions on B is \mathcal{M} -invariant.

For a function $u \in C^1(B)$ let $\tilde{D}u(a) = D(u \circ \varphi_a)(0)$ and $\tilde{\nabla}u(a) = \nabla(u \circ \varphi_a)(0)$ be the invariant complex and invariant real gradient respectively, where

$Du = \left(\frac{\partial u}{\partial z_1}, \dots, \frac{\partial u}{\partial z_n} \right)$ denotes the complex gradient and $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{2n}} \right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, \dots, n$, is the real gradient of u .

For $u \in C^1(B)$ and $\phi \in \text{Aut}(B)$ we have $|\tilde{D}(u \circ \phi)| = |(\tilde{D}u) \circ \phi|$ and $|\tilde{\nabla}(u \circ \phi)| = |(\tilde{\nabla}u) \circ \phi|$. In other words, $|\tilde{D}|$ and $|\tilde{\nabla}|$ are \mathcal{M} -invariant. See [12].

The length of the invariant gradient of a function $u \in C^1(B)$ is given in coordinates by

$$(2.1) \quad |\tilde{\nabla}u(z)|^2 = 2(1 - |z|^2)(|Du(z)|^2 - |Ru(z)|^2 + |D\bar{u}(z)|^2 - |R\bar{u}(z)|^2).$$

It is easy to check that

$$|z|^2 |Du(z)|^2 = |Ru(z)|^2 + \sum_{i < j} |T_{i,j}u(z)|^2, \quad u \in C^2(B),$$

where $T_{i,j}u = \bar{z}_i \frac{\partial u}{\partial z_j} - \bar{z}_j \frac{\partial u}{\partial z_i}$ are tangential derivatives of u . Using this and (2.1) we find that

$$(2.2) \quad \begin{aligned} |z|^2 |\tilde{\nabla}u(z)|^2 &= 2(1 - |z|^2)^2 [(1 - |z|^2)(|Ru(z)|^2 + |R\bar{u}(z)|^2) \\ &\quad + \sum_{i < j} |T_{i,j}u(z)|^2 + \sum_{i < j} |T_{i,j}\bar{u}(z)|^2], \end{aligned}$$

For $z \in B$ and r between 0 and 1, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

We write $A \preceq B$, or equivalently $B \succeq A$, when there is a constant C such that $A \leq CB$, and $A \cong B$ when $A \preceq B$ and $B \preceq A$.

LEMMA 2.1. [12], [1] *Let $0 < p < \infty$ and $0 < r < 1$. There exist constants $C_1 > 0$ and $C_2 > 0$ so that if $u \in \mathcal{M}$, then*

- (a) $|\tilde{\nabla}u(z)|^p \leq \frac{C_1}{(1 - |z|^2)^{n+1}} \int_{E_r(z)} |\tilde{\nabla}u(w)|^p d\nu(w), \quad z \in B, \text{ and}$
- (b) $|\nabla u(z)|^p \leq \frac{C_2}{(1 - |z|^2)^{n+1}} \int_{E_r(z)} |\nabla u(w)|^p d\nu(w), \quad z \in B.$

LEMMA 2.2. [9], [8] *Let $0 < r < 1$. There is a constant C such that if $u \in \mathcal{M}$ then*

- (a) $|T_{i,j}Ru(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |Ru(w)| d\tau(w), \quad z \in B,$
- (b) $|T_{i,j}\bar{R}u(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |\bar{R}u(w)| d\tau(w), \quad z \in B,$
- (c) $|T_{i,j}u(z)| \leq C(1 - |z|^2)^{-1/2} \int_{E_r(z)} |u(w)| d\tau(w), \quad z \in B.$

Here, $d\tau(z) = (1 - |z|^2)^{-n-1} d\nu(z)$ is the Mobius invariant measure on B .

The following lemma gives some basic properties of G which will be needed later.

LEMMA 2.3. *Let $0 < \delta < \frac{1}{2}$ be fixed. Then $G(|z|, 1)$ satisfies the following:*

(a) $G(|z|, 1) \geq c_n(1 - |z|^2)^n$, for all $z \in B$.

(b) $G(|z|, 1) \leq c_\delta(1 - |z|^2)^n$ for all $z \in B$, $|z| \geq \delta$, where c_δ is a positive constant depending only on δ .

Furthermore, for all z , $|z| \leq \delta$, $G(|z|, 1) \cong |z|^{2-2n}$, $n > 1$.

The proof is a routine estimation of the integral in (1.9), and thus is omitted.

LEMMA 2.4. *Let $0 < \delta < 1$. There exists a constant $K = K(\delta) > 0$ such that $|\varphi_a(z)| \geq K|z - a|$, for all a , $|a| \leq \delta$, and $z \in B$.*

Proof. By [13, p. 26]

$$|\varphi_a(z)|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - z\bar{a}|^2} = \frac{|z - a|^2 + |z\bar{a}|^2 - |z|^2|a|^2}{|1 - z\bar{a}|^2}$$

Thus

$$\begin{aligned} |\varphi_a(z)|^2 &\geq \frac{|z - a|^2 + |z\bar{a}|^2 - |z|^2|a|^2}{(1 + \delta)^2} \\ &= \frac{|z - a|^2 + |(z - a)\bar{a}|^2 - |a|^2|z - a|^2}{(1 + \delta)^2} \\ &\geq \frac{(1 - \delta^2)|z - a|^2}{(1 + \delta)^2} \end{aligned}$$

3. Characterizations of the Hardy space \mathcal{H}^p

Proof of Theorem 1.3. Let $u_\epsilon = (u^2 + \epsilon^2)^{p/2}$, $\epsilon > 0$. Then $u_\epsilon \in C^2(B)$ and

$$\tilde{\Delta}u_\epsilon = p(p - 2)(u^2 + \epsilon^2)^{p/2-2}u^2|\tilde{\nabla}u|^2 + p(u^2 + \epsilon^2)^{p/2-1}(|\tilde{\nabla}u|^2 + u\tilde{\Delta}u).$$

By the Green formula we have

$$(3.1) \quad \frac{d}{dr} \int_S u_\epsilon(r\xi) d\sigma(\xi) = \frac{r^{1-2n}(1 - r^2)^{n-1}}{2n} \int_{rB} \tilde{\Delta}u_\epsilon(z) d\tau(z), \quad 0 < r < 1,$$

(see [12])

Since u is \mathcal{M} -harmonic, we have

$$(3.2) \quad \tilde{\Delta}u_\epsilon = p(p - 2)(u^2 + \epsilon^2)^{p/2-2}u^2|\tilde{\nabla}u|^2 + p(u^2 + \epsilon^2)^{p/2-1}|\tilde{\nabla}u|^2,$$

and

$$(3.3) \quad \lim_{\epsilon \rightarrow 0} \frac{d}{dr} \int_S u_\epsilon(r\xi) d\sigma(\xi) = \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi),$$

The latter holds because the function

$$(\epsilon, z) \rightarrow (u^2(z) + \epsilon^2)^{p/2} = |u(z) + i\epsilon|^p$$

is of class C^1 , $(\epsilon, z) \in (-\infty, \infty) \times B$.

If $1 < p \leq 2$, then it follows from (3.2) that

$$\begin{aligned} \tilde{\Delta}u_\epsilon &\geq p(p-2)(u^2 + \epsilon^2)^{p/2-2}(u^2 + \epsilon^2)|\tilde{\nabla}u|^2 \\ &\quad + p(u^2 + \epsilon^2)^{p/2-1}|\tilde{\nabla}u|^2 \\ (3.4) \quad &= p(p-1)(u^2 + \epsilon^2)^{p/2-1}|\tilde{\nabla}u|^2 \end{aligned}$$

Hence, by (3.1),

$$c_p \frac{r^{1-2n}(1-r^2)^{n-1}}{2n} \int_{rB} (u(z)^2 + \epsilon^2)^{p/2-1} |\tilde{\nabla}u(z)|^2 d\tau(z) \leq \frac{d}{dr} \int_S u_\epsilon(r\xi) d\sigma(\xi)$$

where $c_p = p(p-1)$.

Using Fatou's lemma we find that

$$(3.5) \quad \frac{p(p-1)}{2n} r^{1-2n}(1-r^2)^{n-1} \int_{rB} |u(z)|^{p-2} |\tilde{\nabla}u(z)|^2 d\tau(z) \leq \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi).$$

If $2 \leq p < \infty$, we find that

$$(3.6) \quad \frac{p}{2n} (1-r^2)^{n-1} r^{1-2n} \int_{rB} |u(z)|^{p-2} |\tilde{\nabla}u(z)|^2 d\tau(z) \leq \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi).$$

Hence by (3.5) and (3.6) we conclude that the function $|u(z)|^{p-2} |\tilde{\nabla}u(z)|^2$ is locally integrable.

By (3.4) and (3.2) $\tilde{\Delta}u_\epsilon \geq 0$, for every $z \in B$. (Note that $p > 1$).

It is easy to see that

$$\tilde{\Delta}u_\epsilon(z) \leq c_p p (u^2(z) + \epsilon^2)^{p/2-1} |\tilde{\nabla}u(z)|^2, \quad z \in B,$$

where $c_p = 1$ if $1 < p \leq 2$ and $c_p = p-1$ if $p \geq 2$.

Hence by (3.1), (3.2), (3.3) and the Lebesgue theorem, we obtain

$$\begin{aligned} \frac{d}{dr} \int_S |u(r\xi)|^p d\sigma(\xi) &= \lim_{\epsilon \rightarrow 0} \int_{rB} (\tilde{\Delta}u_\epsilon(z)) d\tau(z) \\ &= \frac{p(p-1)}{2n} (1-r^2)^{n-1} r^{1-2n} \int_{rB} |u(z)|^{p-2} |\tilde{\nabla}u(z)|^2 d\tau(z) \end{aligned}$$

To prove Theorem 1.4 we need the following proposition

PROPOSITION 3.1. *Let u be \mathcal{M} -harmonic and $p > 1$. Then $u \in \mathcal{H}^p$ if and only if*

$$\overline{\lim}_{r \rightarrow 1} [M_p^p(r, u) - r^{-2n} \int_{rB} |u(z)|^p d\nu(z)] < \infty$$

Proof. “Only if” is trivial. To prove the “if” let

$$\varphi(r) = \int_{rB} |u(z)|^p d\nu(z) = 2n \int_0^r \rho^{2n-1} M_p^p(\rho, u) d\rho.$$

Hence,

$$M_p^p(r, u) - r^{-2n} \int_{rB} |u(z)|^p d\nu(z) = \frac{r^{1-2n}}{2n} \varphi'(r) - r^{-2n} \varphi(r) = \frac{r}{2n} (\varphi(r) r^{-2n})'.$$

From this the result follows easily.

Proof of Theorem 1.4. By (1.7) we have

$$r^{2n} \frac{d}{dr} M_p^p(r, u) = \frac{p(p-1)}{2n} (1-r^2)^{n-1} r \int_{rB} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 d\tau(z)$$

Integrating this from 0 to ϵ we get, by integration by parts and Fubini's theorem,

$$\begin{aligned} & \epsilon^{2n} M_p^p(\epsilon, u) - \int_{\epsilon B} |u(z)|^p d\nu(z) \\ &= \frac{c_p}{2n} \int_0^\epsilon [r(1-r^2)^{n-1} \int_{rB} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 d\tau(z)] dr \\ &= \frac{c_p}{2n} \int_{\epsilon B} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 d\tau(z) \int_{|z|}^\epsilon (1-r^2)^{n-1} r dr \\ (3.7) \quad &= \frac{c_p}{2n} \int_{\epsilon B} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 \frac{(1-|z|^2)^n - (1-\epsilon^2)^n}{2n} d\tau(z), \end{aligned}$$

where $c_p = p(p-1)$.

If $u \in \mathcal{H}^p$, by taking limit in (3.7), $\epsilon \rightarrow 1$, we obtain (1.8).

Conversely, if

$$\int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 (1-|z|)^{-1} d\nu(z) < \infty$$

then

$$\overline{\lim}_{\epsilon \rightarrow 1} [M_p^p(\epsilon, u) - \epsilon^{-2n} \int_{\epsilon B} |u(z)|^p d\nu(z)] < \infty$$

and $u \in \mathcal{H}^p$, by Proposition 3.1.

Proof of Theorem 1.5. We have, by (1.7) and Fubini's theorem,

$$(3.8) \quad M_p^p(r, u) = |u(0)|^p + p(p-1) \int_{rB} |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 G(|z|, r) d\tau(z).$$

If $M_p^p(r, u) \leq C < \infty$, then we apply Fatou's lemma to get that

$$|u(0)|^p + p(p-1) \int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 G(|z|, 1) d\tau(z) \leq C$$

In the opposite direction we use the Lebesgue theorem and the inequality

$$G(|z|, r) \leq G(|z|, 1).$$

Remark 2. An upper-semicontinuous function $u : B \rightarrow [-\infty, \infty)$ is \mathcal{M} -subharmonic or invariant subharmonic if for each $a \in B$

$$u(a) \leq \int_S u(\varphi_a(r\xi)) d\sigma(\xi), \quad 0 < r < 1,$$

provided that none of the integrals is $-\infty$.

Recall that the Riesz measure of \mathcal{M} -subharmonic function u on B is the non-negative regular Borel measure μ_u satisfying $\int_B \phi d\mu_u = \int_B u \tilde{\Delta} \phi d\tau$, for all $\phi \in C_c^2(B)$, the class of twice continuously differentiable functions on B with compact support.

If u is \mathcal{M} -harmonic on B and $1 < p < \infty$, then the Riesz measure of $|u(z)|^p$ is given by $u_p^* d\tau$, where $u_p^*(z) = p(p-1)|u(z)|^{p-2} |\tilde{\nabla} u(z)|^2$, (see [14]).

If $u \in \mathcal{H}^p(B)$, $1 < p < \infty$, then the least \mathcal{M} -harmonic majorant of $|u|^p$ is given by

$$P[u^*](z) = \int_S P(z, \xi) u^*(\xi) d\sigma(\xi),$$

where $P(z, \xi) = |1 - z\bar{\xi}|^{-2n} (1 - |z|^2)^n$ is the invariant Poisson kernel on B , and $u^*(\xi) = \lim_{r \rightarrow 1} u(r\xi)$ a.e. on S , [13]. Thus by the Riesz Decomposition Theorem [14], [15] with $z = 0$, we again have the equality (1.10):

$$\|u\|_{\mathcal{H}^p}^p = |u(0)|^p + p(p-1) \int_B |u(z)|^{p-2} |\tilde{\nabla} u(z)|^2 G(|z|, 1) d\tau(z)$$

For a similar argument see [4], [14]. Our argument in proving (1.10) is much simpler. We only use Green's theorem, as well as in the proof of Theorem 1.4. We note that the characterization of the Hardy space \mathcal{H}^p given in Theorem 1.4 is a part of Proposition 5, [14]. The proof given in [14] is also based on the computation of the Riesz measure of the function $|u|^p$ and the Riesz decomposition theorem.

4. A Littlewood-Paley type inequalities

Proof of Theorem 1.1. Without loss of generality we may suppose that f is real. It is well known that a \mathcal{M} -harmonic function v belongs to \mathcal{H}^p , $1 < p < \infty$, if and only if $v = P[g]$, for some $g \in L^p(S)$, $1 < p < \infty$. Furthermore, $\|v\|_{\mathcal{H}^p} = \|g\|_{L^p(S)}$. Thus to prove (1.1), by (1.8), it is sufficient to show that

$$I_1 = \int_B |u(z)|^p d\nu(z) \leq \left(|u(0)|^p + \int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^{-1} d\nu(z) \right)$$

and

$$I_2 = \int_B |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} (1 - |z|^2)^{-1} d\nu(z) \leq \int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^{-1} d\nu(z).$$

Let I denote the last integral. Using Lemma 2.1, (b), we find that

$$\begin{aligned} |u(z)|^p &= \left| \int_0^1 \frac{d}{dt} u(tz) dt + u(0) \right|^p \\ &\leq C \left(\int_0^1 |\nabla u(tz)|^p dt + |u(0)|^p \right) \\ &\leq C \left(\int_0^1 \left(\int_{E_r(tz)} |\nabla u(w)|^p d\tau(w) \right) dt + |u(0)|^p \right) \end{aligned}$$

Since $1 - |w|^2 \cong 1 - t^2|z|^2 \cong |1 - tz\bar{w}|$ for $w \in E_r(tz)$, we obtain

$$\begin{aligned} |u(z)|^p &\leq C \left(\int_0^1 \left(\int_{E_r(tz)} \frac{|\nabla u(w)|^p (1 - |w|^2)^{\alpha-n-1}}{|1 - tz\bar{w}|^\alpha} d\nu(w) \right) dt + |u(0)|^p \right) \\ &\leq C \left(\int_0^1 \left(\int_B \frac{|\nabla u(w)|^p (1 - |w|^2)^{\alpha-n-1}}{|1 - tz\bar{w}|^\alpha} d\nu(w) \right) dt + |u(0)|^p \right) \\ &\leq C \left(\int_B \frac{|\nabla u(w)|^p (1 - |w|^2)^{\alpha-n-1}}{|1 - z\bar{w}|^{\alpha-1}} d\nu(w) + |u(0)|^p \right) \end{aligned}$$

We may suppose that $\alpha > n + 2$.

By (2.1) $|\nabla u(w)| \leq \frac{|\tilde{\nabla}u(w)|}{1 - |w|^2}$. Thus

$$\begin{aligned} I_1 &\leq C \left(|u(0)|^p + \int_B d\nu(z) \int_B \frac{|\tilde{\nabla}u(w)|^p (1 - |w|^2)^{\alpha-n-1-p}}{|1 - z\bar{w}|^{\alpha-1}} d\nu(w) \right) \\ &= C \left(|u(0)|^p + \int_B |\tilde{\nabla}u(w)|^p (1 - |w|^2)^{\alpha-n-1-p} d\nu(w) \int_B \frac{d\nu(z)}{|1 - z\bar{w}|^{\alpha-1}} \right) \end{aligned}$$

By standard estimates (see [13, p. 17])

$$\int_B \frac{d\nu(z)}{|1 - z\bar{w}|^{\alpha-1}} \leq \frac{C}{(1 - |w|^2)^{\alpha-n-2}}.$$

Hence,

$$I_1 \leq C \left(|u(0)|^p + \int_B |\tilde{\nabla} u(w)|^p (1 - |w|^2)^{1-p} d\nu(w) \right)$$

Since $1 < p \leq 2$, we have $(1 - |w|^2)^{1-p} \leq (1 - |w|^2)^{-1}$. Therefore, $I_1 \preceq (|u(0)|^p + I)$.

Let $r \in (0, 1/2)$ be fixed. Set $\epsilon = r/4$ and $\delta = r/2$. By Fubini's theorem we have

$$(4.1) \quad I_2 \leq C \int_B (1 - |a|^2)^n d\tau(a) \int_{E_\epsilon(a)} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z).$$

It is easy to see that if $|z| \leq r/4$ then $G(|z|, r/2) \geq C > 0$ and therefore

$$\int_{\epsilon B} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \leq C \int_{\delta B} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|z|, r/2) d\tau(z).$$

Applying this to $u \circ \varphi_a$ and using the \mathcal{M} -invariance of $|\tilde{\nabla}|$, we see that

$$(4.2) \quad \int_{E_\epsilon(a)} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} d\tau(z) \leq C \int_{E_\delta(a)} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|\varphi_a(z)|, \delta) d\tau(z).$$

Let $c_p = p(p-1)$. Using (3.8) and Jensen's inequality we get

$$(4.3) \quad \begin{aligned} c_p \int_{\delta B} |\tilde{\nabla} u(z)|^2 |u(z)|^{p-2} G(|z|, r/2) d\tau(z) &= \int_S |u(\delta\xi)|^p d\sigma(\xi) - |u(0)|^p \\ &= \int_S (|u(\delta\xi)|^2)^{p/2} - (|u(0)|^2)^{p/2} \\ &\leq \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) \right)^{p/2} - (|u(0)|^2)^{p/2}. \end{aligned}$$

If $0 < \alpha \leq 1$ and $0 \leq b \leq a$ then $a^\alpha - b^\alpha \leq (a - b)^\alpha$. Thus

$$(4.4) \quad \begin{aligned} \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) \right)^{p/2} - (|u(0)|^2)^{p/2} &\leq \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) - |u(0)|^2 \right)^{p/2} \\ &= \left(2 \int_{\delta B} |\tilde{\nabla} u(z)|^2 G(|z|, r/2) d\tau(z) \right)^{p/2} \\ &\leq C \sup_{z \in \delta B} |\tilde{\nabla} u(z)|^p \left(\int_{\delta B} G(|z|, r/2) d\tau(z) \right)^{p/2} \\ &\leq C \sup_{z \in \delta B} |\tilde{\nabla} u(z)|^p. \end{aligned}$$

Using \mathcal{M} -harmonic behaviour of $|\tilde{\nabla}u(z)|^p$, (Lemma 2.1), we find that

$$(4.5) \quad \sup_{z \in \delta B} |\tilde{\nabla}u(z)|^p \leq C \sup_{z \in \delta B} \int_{E_\delta(z)} |\tilde{\nabla}u(w)|^p d\tau(w) \leq C \int_{rB} |\tilde{\nabla}u(w)|^p d\tau(w).$$

Combining (4.3), (4.4) and (4.5) we obtain

$$\int_{\delta B} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} G(|z|, r/2) d\tau(z) \leq C \int_{rB} |\tilde{\nabla}u(z)|^p d\tau(z)$$

Applying this again to $u \circ \varphi_a$ we get

$$(4.6) \quad \int_{E_\delta(a)} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} G(|\varphi_a(z)|, r/2) d\tau(z) \leq C \int_{E_r(a)} |\tilde{\nabla}u(z)|^p d\tau(z)$$

Finally, from (4.1), (4.2) and (4.6) we see that

$$\begin{aligned} I_2 &\leq C \int_B (1 - |a|^2)^n d\tau(a) \int_{E_r(a)} |\tilde{\nabla}u(z)|^p d\tau(z) \\ &= C \int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^{-1} d\nu(z) = CI, \end{aligned}$$

by Fubini's theorem.

This finishes the proof of the inequality (1.1).

Since the function $|u|^p$ is \mathcal{M} -subharmonic we have

$$|u(0)|^p \leq \int_B |u(z)|^p d\nu(z).$$

Hence by (1.8) to prove (1.2) it suffices to show that $I_2 \geq CI$.

By Fubini's theorem

$$(4.7) \quad I_2 \cong \int_B (1 - |a|^2)^n d\tau(a) \int_{E_r(a)} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} d\tau(z)$$

Using (3.8) and Jensen's inequality we find that

$$\begin{aligned} c_p \int_{\delta B} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} G(|z|, \delta) d\tau(z) &= \int_S |u(\delta\xi)|^p d\sigma(\xi) - |u(0)|^p \\ &= \int_S (|u(\delta\xi)|^2)^{p/2} d\sigma(\xi) - |u(0)|^p \\ &\geq \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) \right)^{p/2} - (|u(0)|^2)^{p/2}. \end{aligned}$$

For $1 \leq \alpha < \infty$ and $0 \leq b \leq a$, we have $(a - b)^\alpha \leq a^\alpha - b^\alpha$.

Hence, by using (3.8) and Lemma 2.1,

$$\begin{aligned} \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) \right)^{p/2} - (|u(0)|^2)^{p/2} &\geq \left(\int_S |u(\delta\xi)|^2 d\sigma(\xi) - |u(0)|^2 \right)^{p/2} \\ &= \left(2 \int_{\delta B} |\tilde{\nabla}u(z)|^2 G(|z|, \delta) d\tau(z) \right)^{p/2} \\ &\geq C \left(\int_{\delta B} |\tilde{\nabla}u(z)|^2 d\tau(z) \right)^{p/2} \geq C |\tilde{\nabla}u(0)|^p \end{aligned}$$

Therefore,

$$|\tilde{\nabla}u(0)|^p \leq C \int_{\delta B} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} G(|z|, \delta) d\tau(z)$$

Since $G(|z|, \delta) \leq C|z|^{2-2n}$, for $|z| \leq \delta$, (note that $n > 1$), we get

$$|\tilde{\nabla}u(0)|^p \leq C \int_{\delta B} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} |z|^{2-2n} d\tau(z).$$

Applying this to $u \circ \varphi_a$, where $|a| < \delta$, we find that

$$|\tilde{\nabla}u(a)|^p \leq C \int_{E_\delta(a)} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} |\varphi_a(z)|^{2-2n} d\tau(z)$$

By Lemma 2.4 $|\varphi_a(z)| \geq C|z - a|$. Thus

$$|\tilde{\nabla}u(a)|^p \leq C \int_{E_\delta(a)} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} |z - a|^{2-2n} d\tau(z).$$

Integrating the last inequality over $|a| < \delta$ with respect to ν we obtain

$$\int_{\delta B} |\tilde{\nabla}u(a)|^p d\nu(a) \leq C \int_{rB} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} d\tau(z) \int_{\delta B} |z - a|^{2-2n} d\nu(a).$$

Since, for $z \in rB$,

$$\int_{\delta B} |z - a|^{2-2n} d\nu(a) \leq \int_{\{a: |z-a| < 2r\}} |z - a|^{2-2n} d\nu \leq C$$

where C is independent of z , we have

$$|\tilde{\nabla}u(0)|^p \leq C \int_{\delta B} |\tilde{\nabla}u(a)|^p d\tau(a) \leq C \int_{rB} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} d\tau(z)$$

Applying this again to $u \circ \varphi_a$ we get

$$(4.8) \quad |\tilde{\nabla}u(a)|^p \leq C \int_{E_r(a)} |\tilde{\nabla}u(z)|^2 |u(z)|^{p-2} d\tau(z)$$

From (4.7) and (4.8) it follows that $I_2 \geq CI$. This finishes the proof of Theorem 1.1

Now, using Theorem 1.1 we get another version of Littlewood-Paley theorem.

THEOREM 4.1. *Let $f \in L^p(S)$ and $u = P[f]$. Then*

$$(4.9) \quad \|f\|_p^p \leq \left(|u(0)|^p + \int_B |\tilde{\nabla}u(z)|^p G(|z|, 1) d\tau(z) \right), \quad 1 < p \leq 2$$

and

$$(4.10) \quad \|f\|_p^p \geq \left(|u(0)|^p + \int_B |\tilde{\nabla}u(z)|^p G(|z|, 1) d\tau(z) \right), \quad 2 \leq p < \infty.$$

Proof. By Lemma 2.3 $G(|z|, 1) \geq c_n(1 - |z|)^n$ for all $z \in B$. Hence, (4.9) is a consequence of (1.1).

Let $0 < \delta < 1/2$ be fixed. Then using again Lemma 2.3 we find that

$$\begin{aligned} & \int_B |\tilde{\nabla}u(z)|^p G(|z|, 1) d\tau(z) \\ & \leq C \left(\int_{\frac{1}{2}\delta B} |\tilde{\nabla}u(z)|^p |z|^{2-2n} d\tau(z) + \int_{(\frac{1}{2}\delta B)^c} |\tilde{\nabla}u(z)|^p (1 - |z|^2)^n d\tau(z) \right). \end{aligned}$$

Thus, (4.10) follows from (1.2) since

$$\int_{\frac{1}{2}\delta B} |\tilde{\nabla}u(z)|^p |z|^{2-2n} d\tau(z) \leq C \int_{\delta B} |\tilde{\nabla}u(z)|^p (1 - |z|^2)^{-1} d\nu(z).$$

Theorem 1.2 is a corollary of Theorem 1.1 and the following theorem

THEOREM 4.2. *Let $1 < p < \infty$ and $u \in \mathcal{M}$. Then the following statements are equivalent:*

- (i) $\int_B |\tilde{\nabla}u(z)|^p (1 - |z|^2)^{-1} d\nu(z) < \infty$.
- (ii) $\int_B |\nabla u(z)|^p (1 - |z|^2)^{p-1} d\nu(z) < \infty$.
- (iii) $\int_B |Ru(z)|^p (1 - |z|^2)^{p-1} d\nu(z) < \infty$.

Proof. Since $|\tilde{\nabla}u(z)| \geq (1 - |z|^2)|\nabla u(z)| \geq (1 - |z|^2)|Ru(z)|$, we have (i) \Rightarrow (ii) \Rightarrow (iii). So it remains to show that (iii) \Rightarrow (i).

By (2.2) to prove that (iii) \Rightarrow (i) it is sufficient to prove that

$$\int_B (1 - |z|^2)^{p/2-1} |T_{i,j}u(z)|^p d\nu(z) \leq C \int_B (1 - |z|^2)^{p-1} |Ru(z)|^p d\nu(z), \quad 1 \leq i < j \leq n.$$

An integration by parts show that

$$u(z) = \int_0^1 [Ru(tz) + \bar{R}u(tz) + u(tz)] dt$$

From this we conclude that it is sufficient to prove that

$$I = \int_B (1 - |z|^2)^{p/2-1} \left(\int_0^1 |T_{i,j}v(tz)| dt \right)^p d\nu(z) \leq C \int_B (1 - |z|^2)^{p-1} |Ru(z)|^p d\nu(z)$$

where v is either Ru or $\bar{R}u$ or u . We prove this for $v = Ru$. The remaining cases may be treated analogously.

Let $J = \int_0^1 |T_{i,j}Ru(tz)| dt$. Using Lemma 2.2 and Fubini's theorem we find that for any $s > 0$

$$\begin{aligned} J &\leq C \int_0^1 \left(\int_{E_r(tz)} \frac{|Ru(w)|(1 - |w|^2)^s d\nu(w)}{|1 - tz\bar{w}|^{n+s+3/2}} \right) dt \\ &\leq C \int_0^1 \left(\int_B \frac{|Ru(w)|(1 - |w|^2)^s d\nu(w)}{|1 - tz\bar{w}|^{n+s+3/2}} \right) dt \\ &= \int_B |Ru(w)|(1 - |w|^2)^s \left(\int_0^1 \frac{dt}{|1 - tz\bar{w}|^{n+s+3/2}} \right) d\nu(w) \\ &\leq C \int_B \frac{|Ru(w)|(1 - |w|^2)^s d\nu(w)}{|1 - z\bar{w}|^{n+s+1/2}}. \end{aligned}$$

Now we apply Lemma 4.1 [3] to conclude that

$$\begin{aligned} I &\leq C \left(\int_B (1 - |z|^2)^{p/2-1} \left(\int_B \frac{|Ru(w)|(1 - |w|^2)^s d\nu(w)}{|1 - z\bar{w}|^{n+s+1/2}} \right)^p d\nu(z) \right) \\ &\leq C \int_B (1 - |w|^2)^{p-1} |Ru(w)|^p d\nu(w) \end{aligned}$$

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