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WEAK PROBABILITY LOGIC WITH INFINITARY PREDICATES

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Abstract. We construct the logic $L(V, \nu, \mathfrak{m}, R)$, as a logic with infinitary predicates, generalized ordinary and probability quantifiers and propositional connectives. An important feature of this logic is that infinitely many variables can occur in a single formula, but only finitely many quantifiers and connectives. We prove the weak completeness theorem for this logic.

Let V and Pr be disjoint sets of variables and predicates symbols, respectively, ν a function from the set Pr to the set of all ordinals and let $\mathfrak{m}$ be a cardinal. In [**4**] Keisler introduce a formal system $L(V, \nu, \mathfrak{m})$ which has predicates with infinitely many argument places and quantifiers over infinite sets of variables, but which has only finitary propositional connectives and no identity symbol, and whose proofs are finite. We suppose that V, ν, $\mathfrak{m}$ satisfy the conditions I, II and III from [**4**].

Let R be a Łukasiewicz chain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ together with the operations

$$x \oplus y = \min\{x + y, 1\} \qquad \text{and} \qquad \neg x = 1 - x.$$

In [**3**] Keisler introduced several probability logics and developed model theory for them together with Hoover (see [**2**]). The notion of probability logic is designed to permit a logical and model–theoretic approach to probability theory. We construct similar weak probability logic with infinitary predicates $L(V, \nu, \mathfrak{m}, R)$ (briefly L) by adding probability quantifiers $Px \geq r$, where $r \in R$ and $x \in V^\alpha$ is a sequence of different variables of the length $\alpha, \overline{\overline{\alpha}} < \mathfrak{m}$. The set R is taken to be finite in order to preserve the finiteness of proofs. The set F of all formulas of L is the set of all expressions that are built from atomic formulas $p(x)$ ($p \in \mathrm{Pr}$ and $x \in V^{\nu(p)}$), using negation $\neg$, finite disjunction $\vee$, quantifier $(\forall x)$ and probability quantifier $(Px \geq r)$ $(x \in V^\alpha, \overline{\overline{\alpha}} < \mathfrak{m})$. The formula $(Px \geq r)\Phi(x)$ means that the set $\{x : \Phi(x)\}$ has probability greater than or equal to r.

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The notions of set $V_f(\Phi)$ of free variables and set $V_b(\Phi)$ of bound variables of $\Phi \in F$ are defined as usual, with the quantifiers $(\forall x)$ and $(Px \geq r)$ binding all the variables in the sequence x.

For each $\tau \in V^V$, the substitution $S(\tau)\Phi$ of each variable v in a formula Φ by $\tau(v)$ is defined as usual (see [**4**]), with $S(\tau)(\forall x)\Phi = (\forall \tau \circ x)S(\tau)\Phi$,

$S(\tau)(Px \geq r)\Phi = (P\tau \circ x \geq r)S(\tau)\Phi$,

in the quantifier case. Similarly, the substitution $S_f(\tau)\Phi$ of free variables is defined by:

$S_f(\tau)(\forall x)\Phi = (\forall x)S_f(\sigma)\Phi$,

$S_f(\tau)(Px \geq r)\Phi = (Px \geq r)S_f(\sigma)\Psi$,

where $\sigma \in V^V$ and $\sigma(v) = \begin{cases} \tau(v), & v \in V \setminus \operatorname{range} x \\ v, & v \in \operatorname{range} x. \end{cases}$

We can write $\sigma = (\tau \restriction (V \setminus \operatorname{range} x)) \restriction V$ where $\restriction$ is defined as in [**4**].

If $\tau \in V^W$, $W \subset V$, let $S(\tau)\Phi = S(\tau \restriction V)\Phi$ and $S_f(\tau)\Phi = S_f(\tau \restriction V)\Phi$.

Abbreviations $\wedge$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\bot$, $\top$, $Px < r$, $Px > r$ and $Px \leq r$ are introduced as usual.

The rules of inference for L are those from [**4**].

The axioms for L are:

A_1 The axioms of propositional logic,

A_2 $(\forall x)(\Phi \rightarrow \Psi) \rightarrow (\Phi \rightarrow (\forall x)\Psi)$, $\operatorname{range} x \subset V \setminus V_f(\Phi)$,

A_3 $(\forall x)\Phi \rightarrow S_f(\tau)\Phi$, $\tau : \operatorname{range} x \rightarrow V \setminus V_b(\Phi)$,

A_4 $(\forall x)\Phi \leftrightarrow (\forall \tau \circ x)\Phi$, where $\tau : \operatorname{range} x \underset{1-1}{\overset{na}{\rightarrow}} \operatorname{range} x$

A_5 $(Px \geq 0)\Phi$,

A_6 $(Px \geq s)\Phi \rightarrow (Px \geq r)\Phi$, for $s > r$,

A_7 $(Px \geq s)\Phi \wedge (Px \geq r)\Psi \wedge (Px \geq 1)(\neg\Phi \vee \neg\Psi) \rightarrow (Px \geq s \oplus r)(\Phi \vee \Psi)$,

A_8 $(Px \leq s)\Phi \wedge (Px < r)\Psi \rightarrow (Px < s \oplus r)(\Phi \vee \Psi)$,

A_9 $(Px < s)\Phi \rightarrow (Px \leq s)\Phi$,

A_{10} $(Px \geq s)\Phi \rightarrow (Px > r)\Phi$, for $s > r$,

A_{11} $(Px > s)\Phi \rightarrow (Px \geq s^+)\Phi$, where $s^+ = s \oplus \frac{1}{n}$,

A_{12} $(\forall x)\Phi \rightarrow (Px \geq 1)\Phi$,

where Φ, $\Psi \in F$, $s, r \in R$ and $x \in V^\alpha$ is the sequence of different variables, for $\overline{\overline{\alpha}} < \mathfrak{m}$.

A weak probability structure of type ν is

$$\mathfrak{A} = (A, R_p, \mu_\alpha)_{p \in \mathrm{Pr}, \overline{\overline{\alpha}} \geq \overline{\overline{V}}},$$

where A is a nonempty set, $R_p \subset A^{\nu(p)}$, μ_α is a finitely additive probability measure on A^α with range R, such that the set $\{b \circ x \mid \models_{\mathfrak{A}} \Phi[b],\ b \restriction (V \setminus \operatorname{range} x) = a \restriction$

$(V \setminus \mathrm{range}\, x)$, $b \in A^V\}$ is μ_α measurable, for each sequence of variables $x \in V^\alpha$, any $a \in A^V$ and any formula $\Phi \in F$, where the satisfiability relation is defined as usual (see [**4**]), with:

$\models_{\mathfrak{A}} (\forall x)\Phi[a]$ if for each $b \in A^V$ such that $b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)$, we have $\models_{\mathfrak{A}} \Phi[b]$,

$\models_{\mathfrak{A}} (Px \geq r)\Phi[a]$ if $\mu_\alpha\{b \circ x \mid \models_{\mathfrak{A}} \Phi[b]$ and $b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\} \geq r$.

Theorem 1. *Let $\Gamma \subset F$, $\Phi, \Psi \in \Gamma$, $\mathfrak{A}$ is a structure of type ν, $a \in A^V$. Then:*

(i) If $\tau : V(\Phi) \xrightarrow{1-1} V$, then $\models_{\mathfrak{A}} S(\tau)\Phi[a]$ iff $\models_{\mathfrak{A}} \Phi[a \circ (\tau \restriction V)]$.

(ii) If $\tau : V_f(\Phi) \longrightarrow V \setminus V_b(\Phi)$, then $\models_{\mathfrak{A}} S_f(\tau)\Phi[a]$ iff $\models_{\mathfrak{A}} \Phi[a \circ (\tau \restriction V)]$.

(iii) If Φ is a theorem in L, then Φ is valid.

(iv) If the set of formulas Γ is satisfiable, then Γ is consistent in L.

Let V^* be a set of symbols such that $V^* \supset V$ and $V^* \cap \mathrm{Pr} = \emptyset$. Let $L^* = L(V^*, \nu, \mathfrak{m}, R)$ and let F^* be a set of formula of L^*.

A formula Φ is a V-formula in L^* if $\Phi \in F^*$ and $V_b(\Phi) \subset V$. A formula Φ is V-sentence in L^* if $\Phi \in F^*$, $V_b(\Phi) \subset V$ and $V_f(\Phi) \subset V^* \setminus V$.

Assume that $V^* \neq V$ and Γ is a maximal consistent set of V-sentences in L^*. Let $\mathfrak{A}(\Gamma, V)$ be a structure defined as follows:

$$A = V^* \setminus V,$$

$$R_p = \{x \in A^{\nu(p)} \mid p(x) \in \Gamma\}, \text{ for each } p \in \mathrm{Pr},$$

$$\mu_\alpha\{b \circ x \mid S_f(b)\Phi \in \Gamma, b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\} = \max\{r \mid S_f(a)(Px \geq r)\Phi \in \Gamma\},$$

for each V-formula Φ, each $a \in A^V$ and for each sequence of variables $x \in V^\alpha$.

Lemma 1. *Suppose that*

(i) Γ is a maximal consistent set of V-sentences in L^*,

(ii) for any V-sentence $(\forall x)\Psi$ there exists a $\tau : \mathrm{range}\, x \longrightarrow V^* \setminus V$ such that $S_f(\tau)\Psi \to (\forall x)\Psi \in \Gamma$.

Then

(a) $\mathfrak{A}(\Gamma, V)$ is a weak probability structure of type ν,

(b) for each V-formula Φ in L^* and each function $b \in (V^* \setminus V)^V$ we have

$$\models_{\mathfrak{A}} \Phi[b \restriction V^*] \text{ iff } S_f(b)\Phi \in \Gamma.$$

Proof. (a) First, we shall prove that μ_α is a well defined finitely additive measure on A^α, for any ordinal α, $\overline{\overline{\alpha}} \leq \overline{\overline{V}}$.

(1) μ_α is a well defined function.

We shall show that the measure of any subset of A^α does not depend on the defining formula of that subset. Let

$$\{b\circ x \mid S_f(b)\Phi \in \Gamma,\ b\restriction(V\setminus \operatorname{range} x) = a\restriction(V\setminus \operatorname{range} x)\} =$$
$$\{b\circ x \mid S_f(b)\Psi \in \Gamma,\ b\restriction(V\setminus \operatorname{range} x) = a\restriction(V\setminus \operatorname{range} x)\}.$$

Then, for every $b\in A^V$ such that $b\restriction(V\setminus \operatorname{range} x) = a\restriction(V\setminus \operatorname{range} x)$, we have

$$S_f(b)(\Phi\to\Psi)\in\Gamma.$$

By (ii), for V-sentence $(\forall x)S_f(a\restriction(V\setminus \operatorname{range} x))(\Phi\to\Psi)$ there exists a $\tau\in A^{\operatorname{range} x}$ such that

$$S_f(\tau)S_f(a\restriction(V\setminus \operatorname{range} x))(\Phi\to\Psi)\to(\forall x)S_f(a\restriction(V\setminus \operatorname{range} x))(\Phi\to\Psi)\in\Gamma.$$

This means that for $b=\tau\cup a\restriction(V\setminus \operatorname{range} x)$ we have

$$S_f(b)(\Phi\to\Psi)\to(\forall x)S_f(a\restriction(V\setminus \operatorname{range} x))(\Phi\to\Psi)\in\Gamma.$$

By using R_1 (see [**4**]) and A_{12}, it follows

$$(Px\geq 1)S_f(a\restriction(V\setminus \operatorname{range} x))(\Phi\to\Psi)\in\Gamma.$$

By assuming

$$\mu_\alpha\{b\circ x \mid S_f(b)\Phi \in \Gamma,\ b\restriction(V\setminus \operatorname{range} x) = a\restriction(V\setminus \operatorname{range} x)\} \neq$$
$$\mu_\alpha\{b\circ x \mid S_f(b)\Psi \in \Gamma,\ b\restriction(V\setminus \operatorname{range} x) = a\restriction(V\setminus \operatorname{range} x)\},$$

we obtain the existence of an $s\in R$ such that

$$(Px\geq s)S_f(a\restriction(V\setminus \operatorname{range} x))\Phi\in\Gamma \text{ and } (Px\geq s)S_f(a\restriction(V\setminus \operatorname{range} x))\Psi\notin\Gamma.$$

Thus

$$(Px\geq s)S_f(a\restriction(V\setminus \operatorname{range} x))\Phi\wedge\neg(Px\geq s)S_f(a\restriction(V\setminus \operatorname{range} x))\Psi\wedge$$
$$(Px\geq 1)S_f(a\restriction(V\setminus \operatorname{range} x))(\neg\Phi\vee\Psi)\in\Gamma.$$

But, by A_8

$$(Px\leq\neg s)S_f(a\restriction(V\setminus \operatorname{range} x))\neg\Phi\wedge(Px< s)S_f(a\restriction(V\setminus \operatorname{range} x))\Psi\wedge$$
$$\neg(Px< 1)S_f(a\restriction(V\setminus \operatorname{range} x))(\neg\Phi\vee\Psi)\notin\Gamma.$$

A contradiction.

(2) $\mu_\alpha(A^\alpha)=1$ and $\mu_\alpha(\emptyset)=0$.

Since $A^\alpha = \{b\circ x \mid S_f(\Phi \vee \neg\Phi) \in \Gamma,\ b\restriction(V\setminus \mathrm{range}\, x) = a\restriction(V\setminus \mathrm{range}\, x)\}$, for any $a \in A^V$ and $\Phi \in F$, we have

$$\mu_\alpha(A^\alpha) = \max\{r \mid S_f(a)(Px \geq r)(\Phi \vee \neg\Phi) \in \Gamma\}.$$

It follows from $S_f(a)(Px \geq 1)(\Phi \vee \neg\Phi) \in \Gamma$ that $\mu_\alpha(A^\alpha) = 1$.

Now, we shall prove that $\mu_\alpha(\emptyset) = 0$. Obviously,

$$\emptyset = \{b\circ x \mid S_f(b)(\Phi \wedge \neg\Phi) \in \Gamma,\ b\restriction(V\setminus \mathrm{range}\, x) = a\restriction(V\setminus \mathrm{range}\ x)\}.$$

Assuming that there exists $r > 0$ such that $S_f(a)(Px \geq r)(\Phi \wedge \neg\Phi) \in \Gamma$ we obtain

$$(Px > \neg r)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \vee \neg S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \notin \Gamma.$$

It follows from A_9 and A_{10} that

$$(Px \geq 1)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \vee \neg S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \notin \Gamma.$$

Hence, we have $S_f(a)(Px \geq 1)(\Phi \vee \neg\Phi) \notin \Gamma$, which contradicts the first part of (1).

(3) It follows from A_5 and (7.3) that μ_α is a nonnegative function.

(4) $\mu_\alpha(A^\alpha \setminus B) = \neg\mu_\alpha(B)$, for every set B of the form $B = \{b\circ x \mid S_f(b)\Phi \in \Gamma,$ $b\restriction(V\setminus \mathrm{range}\, x) = a\restriction(V\setminus \mathrm{range}\, x)\}$.

If B is of the above form, then

$$A^\alpha \setminus B = \{b\circ x \mid S_f(b)\neg\Phi \in \Gamma,\ b\restriction(V\setminus \mathrm{range}\, x) = a\restriction(V\setminus \mathrm{range}\, x)\}.$$

Let $\mu_\alpha(B) = s$, $0 < s < 1$. Then $\max\{r \mid S_f(a)(Px \geq r)\Phi \in \Gamma\} = s$ and hence

$$(Px \geq s)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \in \Gamma \text{ and } (Px \geq s^+)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \notin \Gamma.$$

It follows from A_{11} that

$$\neg(Px > s)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \in \Gamma.$$

So,

$$S_f(a)(Px \geq \neg s)\neg\Phi \in \Gamma.$$

It means that $\mu_\alpha(A^\alpha \setminus B) \geq \neg s$. Assuming that there exists a $t > \neg s$ such that $S_f(a)(Px \geq t)\neg\Phi \in \Gamma$ and by using A_{10} we obtain

$$(Px > \neg s)\neg S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \in \Gamma,$$

which contradicts $\mu_\alpha(B) = s$.

Now, let $\mu_\alpha(B) = 0$ and $\mu_\alpha(A^\alpha \setminus B) \neq 1$. Then $S_f(a)(Px \geq 1)\neg\Phi \notin \Gamma$, and hence

$$(Px > 0)S_f(a\restriction(V\setminus \mathrm{range}\, x))\Phi \in \Gamma.$$

It follows from A_{11} that

$$(Px \geq 0^+)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \in \Gamma,$$

contradicting our assumption $\mu_\alpha(B) = 0$. Similarly, for $\mu_\alpha(B) = 1$.

(5) μ_α is an increasing function.

Let $C = \{b \circ x \mid S_f(b)\Phi \in \Gamma,\ b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\}$, $D = \{b \circ x \mid S_f(b)\Psi \in \Gamma,\ b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\}$ and $C \subset D$. Then, as in (1),

$$(Px \geq 1)(S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \to S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \in \Gamma.$$

Putting $\mu_\alpha(C) = s, \mu_\alpha(D) = t$ and $t < s$, we have

$$(Px \leq \neg s)\neg S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \in \Gamma \text{ and } (Px < s)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi \in \Gamma.$$

It follows from A_8 that

$$(Px < 1)(\neg S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \vee S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \in \Gamma.$$

A contradiction.

(6) μ_α is a finitely additive function.

Let $C = \{b \circ x \mid S_f(b)\Phi \in \Gamma,\ b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\}, D = \{b \circ x \mid S_f(b)\Psi \in \Gamma,\ b \restriction (V \setminus \mathrm{range}\, x) = a \restriction (V \setminus \mathrm{range}\, x)\}, C \cap D = \emptyset$, $\mu_\alpha(C) = s$and $\mu_\alpha(D) = t$. Then

$$(Px \geq s)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \in \Gamma \text{ and } (Px \geq t)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi \in \Gamma.$$

From $C \cap D = \emptyset$ it follows

$$(Px \geq 1)\neg(S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \wedge S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \in \Gamma.$$

By A_7, we have

$$(Px \geq s \oplus t)(S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \vee S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \in \Gamma.$$

Since $C \subset A^\alpha \setminus D$, we have, by (5), $\mu_\alpha(C) \leq \neg t$, and hence $s + t \leq 1$. So,

$$(Px \geq s + t)(S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \vee S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \in \Gamma.$$

Now, we prove that

$$(Px \geq (s + t)^+)(S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \vee S_f(a\restriction(V \setminus \mathrm{range}\, x))\Psi) \notin \Gamma.$$

From $(Px \geq s^+)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \notin \Gamma$, by A_{11}, we obtain

$$(Px \leq s)S_f(a\restriction(V \setminus \mathrm{range}\, x))\Phi \in \Gamma.$$

From this and

$$(Px < t^+) S_f(a\restriction(V \setminus \operatorname{range} x))\Psi \in \Gamma,$$

by A_8, we have

$$(7) \qquad (Px < s \oplus t^+)(S_f(a\restriction(V \setminus \operatorname{range} x))\Phi \vee S_f(a\restriction(V \setminus \operatorname{range} x))\Psi) \in \Gamma.$$

Suppose $s < \neg t$. Then $t^+ \leq \neg s$, and thus $s \oplus t^+ = s + t^+$. It follows that

$$(Px < s + t^+)(S_f(a\restriction(V \setminus \operatorname{range} x))\Phi \vee S_f(a\restriction(V \setminus \operatorname{range} x))\Psi) \in \Gamma, \text{ i.e.,}$$

$$S_f(a)(Px \geq s + t^+)(\Phi \vee \Psi) \notin \Gamma.$$

We complete the proof in this case because $(s+t)^+ = s + t^+$.

In the case $s = 1 - t$, we have $s \oplus t^+ = 1$. It follows from (7) that

$$(Px < 1)(S_f(a\restriction(V \setminus \operatorname{range} x))\Phi \vee S_f(a\restriction(V \setminus \operatorname{range} x))\Psi) \in \Gamma.$$

On the other hand, since $t + s = 1$, from

$$(Px \geq t + s)(S_f(a\restriction(V \setminus \operatorname{range} x))\Phi \vee S_f(a\restriction(V \setminus \operatorname{range} x))\Psi) \in \Gamma$$

we have

$$(Px \geq 1)(S_f(a\restriction(V \setminus \operatorname{range} x))\Phi \vee S_f(a\restriction(V \setminus \operatorname{range} x))\Psi) \in \Gamma.$$

A contradiction.

(b) By induction on the complexity of the formula Φ we shall prove

$$\models_{\mathfrak{A}} \Phi[b\restriction V^*] \quad \text{iff} \quad S_f(b)\Phi \in \Gamma.$$

From this fact, the definition of μ_α and the finiteness of R it will follow that the set $\{a \circ x \mid a\restriction(V^* \setminus \operatorname{range} x) = b\restriction(V^* \setminus \operatorname{range} x),\ \models_{\mathfrak{A}} \Phi[a]\}$ is μ_α-measurable, for each V-formula Φ, each $b \in A^{V^*}$ and every sequence of variables $x \in V^\alpha$. It will mean that $\mathfrak{A}(\Gamma, V)$ is a weak probability structure.

Let $\Phi = p(x), b \in A^V, x \in V^{\nu(p)}$. Then

$$\begin{aligned}\models_{\mathfrak{A}} p(x)[b\restriction V^*] &\text{ iff } b\restriction V^* \circ x \in R_p, \text{ by definition of the satisfiability relation,}\\ &\text{ iff } b \circ x \in R_p\\ &\text{ iff } p(b \circ x) \in \Gamma, \text{ by definition of } R_p,\\ &\text{ iff } S_f(b)p(x) \in \Gamma, \text{ by definition of } S_f.\end{aligned}$$

The steps $\Phi = \Psi \vee \Theta$ and $\Phi = \neg\Psi$ are easy.

Let $\Phi = (\forall x)\Psi$ and $\models_{\mathfrak{A}} (\forall x)\Psi[b\restriction V^*]$. Then for each $a \in A^{V^*}$, such that $a\restriction (V^* \setminus \operatorname{range} x) = b\restriction(V^* \setminus \operatorname{range} x)$ we have $\models_{\mathfrak{A}} \Psi[a]$. We must show $S_f(b)(\forall x)\Psi \in \Gamma$.

The formula $(\forall x)S_f(b\restriction V\setminus \mathrm{range}\, x)\Psi$ is a V-sentence, and hence, by (ii), there exists a $\tau \in (V^* \setminus V)^{\mathrm{range}\, x}$ such that

$$S_f(\tau)S_f(b\restriction V \setminus \mathrm{range}\, x)\Psi \to (\forall x)S_f(b\restriction V \setminus \mathrm{range}\, x)\Psi \in \Gamma.$$

Let $d = \tau \cup b\restriction (V^* \setminus \mathrm{range}\, x)$. Then $d \in (V^* \setminus V)^{V^*}$ and $d\restriction (V^* \setminus \mathrm{range}\, x) = b\restriction(V^* \setminus \mathrm{range}\, x)$. So, $\models_{\mathfrak{A}} \Psi[d]$. By induction hypothesis,

$$S_f(d\restriction V)\Psi \in \Gamma.$$

Since

$$S_f(d\restriction V)\Psi = S_f(\tau)S_f\,(b\restriction(V \setminus \mathrm{range}\, x))\,\Psi,$$

it follows from R_1 that

$$(\forall x)S_f(b\restriction(V \setminus \mathrm{range}\, x)\Psi \in \Gamma.$$

Conversely, suppose that

$$S_f(b)(\forall x)\Psi \in \Gamma, \text{ i.e.,}$$

$$(\forall x)S_f\,(b\restriction(V \setminus \mathrm{range}\, x))\,\Psi \in \Gamma.$$

Let $c \in (V^* \setminus V)^{V^*}$ be such that $c\restriction(V^* \setminus \mathrm{range}\, x) = b\restriction(V^* \setminus \mathrm{range}\, x)$. Then

$$S_f(c)\Psi = S_f(c\restriction \mathrm{range}\, x)S_f\,(b\restriction(V^* \setminus \mathrm{range}\, x))\,\Psi.$$

By A_3, we have

$$(\forall x)S_f\,(b\restriction(V \setminus \mathrm{range}\, x)) \to S_f(c\restriction \mathrm{range}\, x)S_f\,(b\restriction(V^* \setminus \mathrm{range}\, x))\,\Psi \in \Gamma.$$

It follows from R_1 that

$$S_f(c)\Psi \in \Gamma.$$

By induction hypothesis, we have

$$\models_{\mathfrak{A}} \Psi[c].$$

Finally, by the definition of the satisfiability relation, we have

$$\models_{\mathfrak{A}} (\forall x)\Psi[b\restriction V^*].$$

Now, let $\Phi = (Px \geq r)\Psi$. Then

$$\begin{aligned}
&\models_{\mathfrak{A}} (Px \geq r)\Psi[b\restriction V^*]\\
&\text{iff } \mu_\alpha\{c\circ x \mid c\restriction(V \setminus \mathrm{range}\, x) = b\restriction(V \setminus \mathrm{range}\, x), \models_{\mathfrak{A}} \Psi[c\restriction V^*], c \in A^V\} \geq r\\
&\text{iff } \mu_\alpha\{c\circ x \mid c\restriction(V \setminus \mathrm{range}\, x) = b\restriction(V \setminus \mathrm{range}\, x), S_f(c)\Psi \in \Gamma, c \in A^V\} \geq r\\
&\text{iff } \max\{s \mid S_f(b)(Px \geq s)\Psi \in \Gamma\} \geq r\\
&\text{iff } S_f(b)(Px \geq r)\Psi \in \Gamma.
\end{aligned}$$

The proof is complete.

Theorem 2. *If Γ is a consistent set of formulas in L, then Γ is satisfiable in some weak probability structure $\mathfrak{A}$ of type ν.*

Moreover, if $\mathfrak{n}$ is a cardinal and we have

$$\overline{\overline{F}} \leq \mathfrak{n},$$

and

$$\text{for each } p \in \mathrm{Pr}, \quad \mathfrak{n} = \mathfrak{n}^{\overline{\overline{\nu(p)}}},$$

then $\mathfrak{A}$ may be taken to be of power $\mathfrak{n}$.

The proof may be found in [**4**].

Corollary 1. *Let $\Gamma \subset F$. Then, Γ is satisfiable iff every finite subset of Γ is satisfiable.*

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