PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 66(80) (1999), 3–15

THE GENERALIZED BAUES PROBLEM FOR CYCLIC POLYTOPES II

CHRISTOS A. ATHANASIADIS, JÖRG RAMBAU, AND FRANCISCO SANTOS

ABSTRACT. Given an affine surjection of polytopes $\pi : P \to Q$, the Generalized Baues Problem asks whether the poset of all proper polyhedral subdivisions of Q which are induced by the map π has the homotopy type of a sphere. We extend earlier work of the last two authors on subdivisions of cyclic polytopes to give an affirmative answer to the problem for the natural surjections between cyclic polytopes $\pi : C(n, d') \to C(n, d)$ for all $1 \leq d < d' < n$.

1. Introduction

The Generalized Baues Problem, posed by Billera, Kapranov and Sturmfels [4], is a question in combinatorial geometry and topology, motivated by the theory of fiber polytopes [5], [18, Lecture 9]. Given an affine surjection of polytopes $\pi : P \to Q$, the problem asks to determine whether the *Baues poset* $\omega(P \xrightarrow{\pi} Q)$ of all proper polyhedral subdivisions of Q which are induced in a certain way by the map π, endowed with a standard topology [6], has the homotopy type of a sphere of dimension $\dim(P) - \dim(Q) - 1$. We refer to [11] for a concise introduction and [15] for a recent survey.

Although the Generalized Baues Problem is known to have a negative answer in general [14], various special cases have remained of interest in the literature; see [15, Section 4]. One such relates to subdivisions of cyclic polytopes. Another is the case where P is a simplex, in which $\omega(P \xrightarrow{\pi} Q)$ is the poset of *all* proper polyhedral subdivisions of Q and is simply denoted $\omega(Q)$. In [9] an affirmative answer to the problem was given in the case of the poset of all subdivisions of cyclic polytopes of dimension at most 3. This was recently improved in [13] to all dimensions, as follows.

1991 *Mathematics Subject Classification.* Primary 52B11; Secondary 52B12, 05E25.

Work of Francisco Santos is partially supported by CajaCantabria and by grant PB97–0358 of Spanish *Dirección General de Enseñanza Superior e Investigación Científica.*

Theorem 1.1. [13, Theorem 1.1] *For all $1 \leq d < n$, the Baues poset $\omega(C(n,d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n,d)$ is homotopy equivalent to an $(n-d-2)$-sphere.*

For $1 \leq d < d' < n$, one can consider the natural projections $\pi : C(n,d') \to C(n,d)$ between cyclic polytopes [1]. The Baues poset $\omega(C(n,d))$ in Theorem 1.1 is the Baues poset of the projection π for $d' = n-1$. In this paper we use the "sliding" technique of [13] to give an affirmative answer to the Generalized Baues Problem for π for all d, d' and n.

Theorem 1.2. *For $1 \leq d < d' < n$, the Baues poset $\omega(C(n,d') \xrightarrow{\pi} C(n,d))$ of all proper polyhedral subdivisions of the cyclic polytope $C(n,d)$ which are induced by π is homotopy equivalent to a $(d'-d-1)$-sphere.*

Theorem 1.2 was conjectured by Reiner [15] on the basis of the following special cases:

- $d = 2$, $d' = n-2$ [1, Corollary 6.3],
- $d' = n-1$ (Theorem 1.1),
- $d = 2$, $n < 2d'+2$, $d' \geq 9$ [16, Corollary 15].

Other previously known special cases are those of $d = 1$ and $d' - d \leq 2$, which follow from more general results of [4] and [14], respectively: for any polytope projection $\pi : P \to Q$, the poset $\omega(P \xrightarrow{\pi} Q)$ of all proper π-induced subdivisions of Q is homotopy equivalent to a sphere whenever $\dim(Q) = 1$ or $\dim(P) - \dim(Q) \leq 2$.

Our argument is a modification of the one used in [13, Section 4] to prove Theorem 1.1 and therefore relies heavily on the constructions of [13]. In the next section we review some basic definitions and facts. In Section 3 we give a sketch of the proof of Theorem 1.2, thereby recalling those constructions from [13] that will be essential here. Section 4 contains the remaining details, which amount to proving that two certain posets of subdivisions are contractible.

2. Preliminaries

2.1. **Polyhedral subdivisions.** By a point configuration $\mathcal{A}$ in $\mathbb{R}^d$ we mean a finite labeled subset of $\mathbb{R}^d$. We allow $\mathcal{A}$ to have repeated points which are distinguished by their labels. The convex hull $\mathrm{conv}(\mathcal{A})$ of $\mathcal{A}$ is a polytope.

A face of a subconfiguration $\sigma \subseteq \mathcal{A}$ is a subconfiguration $F^\omega \subseteq \sigma$ consisting of *all* points on which some linear functional $\omega \in (\mathbb{R}^d)^*$ takes its minimum over σ.

We say that two subconfigurations σ_1 and σ_2 of $\mathcal{A}$ *intersect properly* if the following two conditions are satisfied:

- $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2;
- $\mathrm{conv}(\sigma_1) \cap \mathrm{conv}(\sigma_2) = \mathrm{conv}(\sigma_1 \cap \sigma_2)$.

A subconfiguration of $\mathcal{A}$ is said to be *full-dimensional*, or *spanning*, if it affinely spans $\mathbb{R}^d$. In that case we call it a *cell*. Following [3] and [10, Section 7.2] we say that a collection S of cells of $\mathcal{A}$ is a *(polyhedral) subdivision* of $\mathcal{A}$ if the elements of

S intersect pairwise properly and cover $\mathrm{conv}(\mathcal{A})$ in the sense that

$$\bigcup_{\sigma \in S} \mathrm{conv}(\sigma) = \mathrm{conv}(\mathcal{A}).$$

Cells that share a common facet are *adjacent*. The set of subdivisions of $\mathcal{A}$ is partially ordered by the *refinement* relation

$$S_1 \leq S_2 \qquad :\Longleftrightarrow \qquad \forall \sigma_1 \in S_1,\ \exists \sigma_2 \in S_2 :\ \sigma_1 \subset \sigma_2.$$

The poset of subdivisions of $\mathcal{A}$ has a unique maximal element which is the trivial subdivision $\{\mathcal{A}\}$. The minimal elements are the subdivisions all of whose cells are affinely independent, which are called *triangulations* of $\mathcal{A}$. We call subdivisions of a polytope Q the subdivisions of its vertex set.

2.2. **Induced subdivisions.** Now let $P \subset \mathbb{R}^p$ be a polytope, and let $\pi : \mathbb{R}^p \to \mathbb{R}^d$ be a linear projection map. We can consider the point configuration $\mathcal{A} = \pi(\mathrm{vert}(P))$ arising from the projection of the vertex set of P. An element in $\mathcal{A}$ is labeled by the vertex of P of which it is considered to be the image. In other words, π induces a bijection between the vertex set of P and $\mathcal{A}$, even if different vertices of P have the same projection.

A subdivision S of $\mathcal{A}$ is said to be *π-induced* if every cell of S is the projection of the vertex set of a face of P. If P is a simplex then all subdivisions of $\mathcal{A}$ are π-induced. This concept of π-induced subdivisions was introduced in [5].

A π-induced subdivision S contains the same information as the collection of faces of P whose vertex sets are in S. In this sense one can say that a π-induced subdivision of $\mathcal{A}$ is a polyhedral subdivision whose cells are projections of faces of P (this statement is not accurate; see [11, 14, 18] for an accurate definition of π-induced subdivisions in terms of faces of P).

The poset of π-induced subdivisions excluding the trivial one is denoted by $\omega(P \xrightarrow{\pi} \pi(P))$. Its minimal elements are the subdivisions for which every cell comes from a $\dim(\mathcal{A})$-dimensional face of P. They are called *tight* π-induced subdivisions.

In [4] it was conjectured that the Baues poset $\omega(P \xrightarrow{\pi} \pi(P))$ is homotopy equivalent to a sphere of dimension $p-d-1$. Evidence for this were the cases $p-d=1$ (trivial) and $d=1$ (proved in [4]) together with the fact that $\omega(P \xrightarrow{\pi} \pi(P))$ always contains a subposet *homeomorphic* to a sphere of dimension $p-d-1$ (the poset of *coherent* π-induced subdivisions [5]). The conjecture was known as the *generalized Baues conjecture* since the case $d=1$ had been conjectured by J. Baues in a different form, until it was disproved in [14]. Still, several cases remain of interest. Theorem 1.1 is the case where π is the natural projection from a simplex to a cyclic polytope and our Theorem 1.2 is the case where π is the natural projection between two cyclic polytopes. Other cases where the statement is known to be true are when $p-d=2$ [14] and when P is a simplex and $d=2$ [8].

See [5, 15, 18] for more information on π-induced subdivisions and the Baues problem.

2.3. **Poset topology.** When refering to the topology of a finite poset we mean the topology of its *order complex*, i.e., the simplicial complex of chains in the poset

[6]. For a poset P and $x \in P$ we denote by $P_{\leq x}$ the set $\{ y \in P : y \leq x \}$. We will use the following tool from [2] to relate the homotopy type of two posets. A proof is given in [17, Section 3].

Lemma 2.1. (Babson) *Let $f : \omega \to \omega'$ be an order preserving map of posets. If*

(i) *$f^{-1}(y)$ is contractible for every $y \in \omega'$ and*
(ii) *$\omega_{\leq x} \cap f^{-1}(y)$ is contractible for every $x \in \omega$ and $y \in \omega'$ with $f(x) > y$*

then f induces a homotopy equivalence.

2.4. **Cyclic polytopes.** The cyclic polytope $C(n,d)$ is the convex hull of any n points on the moment curve $\{(t,t^2,\dots,t^d) : t \in \mathbb{R}\}$ in $\mathbb{R}^d$. We consider it as the point configuration consisting of these n points, which are the vertices for $d \geq 2$. Hence, all the notions for induced subdivisions make sense for cyclic polytopes. Also, we extend the usual definition by the trivial case of $d = 0$: the cyclic polytope $C(n,0)$ is just the set of n copies of the only point in $\mathbb{R}^0$. The cyclic polytope $C(n,1)$ consists of n distinct points in the real line $\mathbb{R}$.

As usual, we label the vertices of $C(n,d)$ with the numbers $1,\dots,n$, in the order they appear along the moment curve and refer to faces of $C(n,d)$ by the index sets of their vertices, i.e. as subsets of $[n] := \{1,2,\dots,n\}$.

The face lattice of $C(n,d)$ is known to be independent of the choice of points on the curve and is characterized by Gale's evenness criterion, which is as follows (see also [18, p. 14] or [1, Theorem 5.2]). For a subset $F \subset [n]$ with complement $[n] \setminus F = \{a_1,a_2,\dots,a_k\}$, we divide F in its *initial interval* $\{1,\dots,a_1-1\}$, its *final interval* $\{a_k+1,\dots,n\}$ and its *interior intervals* $\{a_i+1,\dots,a_{i+1}-1\}$, $i = 1,\dots,k-1$. The initial and final intervals may be empty. An interval is called odd if it has an odd number of elements and even otherwise. Then, F is a face of $C(n,d)$ if and only if the cardinality of F plus the number of odd interior intervals does not exceed d. Two obvious consequences of this description are that cyclic polytopes are simplicial and that faces of $C(n,d)$ are also faces of $C(n,d')$ for $d' > d$.

Moreover, if d is the smallest integer for which F is a face of $C(n,d)$, then F is an *upper* face of $C(n,d)$ (meaning that its normal cone contains only vectors with last coordinate positive) if the final interval in F is odd and F is a *lower* face (meaning that its normal cone contains only vectors with last coordinate negative) if the final interval in F is even (or empty).

2.5. **The canonical projections between cyclic polytopes.** For a fixed pair of dimensions $d' > d$ we will be interested in the surjection $\pi : C(n,d') \to C(n,d)$, induced by the map $\pi : \mathbb{R}^{d'} \to \mathbb{R}^d$ which forgets the last $d'-d$ coordinates. The fiber polytopes for this family of surjections were studied in [1]. The associated Baues posets were studied in the special case $d = 2$ in [16]. For the ease of notation, we will write $\omega_{d'}(C(n,d))$ for the Baues poset $\omega(C(n,d') \xrightarrow{\pi} C(n,d))$. This poset is also independent of the choice of points used to define $C(n,d')$. Note that the Baues poset $\omega_{d'}(C(n,0)) = \omega(C(n,d') \xrightarrow{\pi} C(n,0))$ is isomorphic to the poset of proper faces of $C(n,d')$ for all $d' > 0$, hence homeomorphic to a $(d'-1)$-sphere.

3. Structure of the proof

The idea for proving Theorem 1.2 is as follows. Let us fix the dimensions $2 \leq d < d'$ and then use induction on the number n of vertices. The result is already known in the cases $d = 0, 1$. The base case $n = d' + 1$ for the induction is provided by Theorem 1.1. For the inductive step, we will use the same approach as in [13]: via the *deletion operation* of vertex n from a subdivision of $C(n, d)$, we will define a map between the posets $\omega_{d'}(C(n, d))$ and $\omega_{d'}(C(n-1, d))$ and will prove it to be a homotopy equivalence. This deletion operation is a generalization of the deletion operation on triangulations of $C(n, d)$ from [12].

For two collections S and T of finite pointsets in $\mathbb{R}^d$ we define

$$\begin{aligned} \text{spanning}(S) &:= \{\, \sigma \in S \,:\, \sigma \text{ is spanning} \,\} \\ \text{ast}_S(i) &:= \{\, \sigma \in S \,:\, i \notin \sigma \,\} \\ \text{lk}_S(i) &:= \{\, \sigma - \{i\} \,:\, \sigma \in S,\, i \in \sigma \,\} \\ S * T &:= \{\, \sigma \cup \tau \,:\, \sigma \in S, \tau \in T \,\}. \end{aligned}$$

As was discussed in [13, Section 4], if S is a subdivision of $C(n, d)$ then $\text{lk}_S(n)$ is a subdivision of $C(n-1, d-1)$. Moreover, Gale's evenness criterion easily implies that if S is in $\omega_{d'}(C(n, d))$ then $\text{lk}_S(n)$ is in $\omega_{d'-1}(C(n-1, d-1))$.

Definition 3.1. ([13]) *Given a subdivision S of $C(n, d)$, the deletion $S \backslash n$ is*

$$S \backslash n := \text{spanning}\left(\{\sigma \backslash n : \sigma \in S\}\right),$$

where

$$\sigma \backslash n := \begin{cases} (\sigma - \{n\}) \cup \{n-1\}, & \text{if } n \in \sigma, \\ \sigma, & \text{otherwise.} \end{cases}$$

Equivalently,

$$S \backslash n := \text{ast}_S(n) \,\cup\, \text{spanning}\left(\text{lk}_S(n) * \{n-1\}\right).$$

Using the idea of "sliding" vertex n to $n-1$, it is proved in [13, Theorem 3.2] that $S \backslash n$ is a subdivision of $C(n-1, d)$. The deletion of n defines a map between the posets $\omega_{d'}(C(n, d))$ and $\omega_{d'}(C(n-1, d))$:

Proposition 3.2. *Let $n \geq d' + 2$. The deletion map $\Pi_{d'} : \omega_{d'}(C(n, d)) \to \omega_{d'}(C(n-1, d))$*

$$\Pi_{d'}(S) = S \backslash n$$

between the Baues posets of proper π-induced subdivisions is well-defined and order preserving.

Proof. In order to see that $\Pi_{d'}$ is well-defined we just need to check that if σ is a proper face of $C(n, d')$ then $\sigma \backslash n$, introduced in Definition 3.1, is a proper face of $C(n-1, d')$. It follows easily from Gale's evenness criterion that $\sigma \backslash n$ is a face of $C(n-1, d')$. Moreover, since σ is proper and $C(n, d')$ is simplicial, σ has at most $d' \leq n-2$ vertices. Thus $\sigma \backslash n$ has at most $n-2$ vertices and is a proper face of $C(n-1, d')$.

That $\Pi_{d'}$ is order preserving follows trivially from the fact that if $\sigma \subset \sigma'$ then $\sigma \backslash n \subset \sigma' \backslash n$. $\square$

In order to apply Lemma 2.1 to the map $\Pi_{d'}$ we need to understand its fibers. The following concept of subdivisions of $C(n,d)$ *induced by a certain subdivision* $\overline{S}$ of $C(n,d+1)$ will be crucial for this.

Let $\overline{S}$ be a subdivision of the cyclic polytope $C(n,d+1)$ and S a subdivision of $C(n,d)$. Following [13], we say that S is induced by $\overline{S}$ if every cell $\sigma \in S$ is a face (not necessarily proper) of a cell $\sigma' \in \overline{S}$. We can think of S as a cellular section of the natural projection $C(n,d+1) \to C(n,d)$ which uses only cells in $\overline{S}$ or their faces. Observe that for every cell σ'' of $\overline{S}$ we can tell whether σ'' is above, on or below (the section corresponding to) a subdivision S induced by $\overline{S}$. We will denote by $\mathrm{above}(S,\overline{S})$ and $\mathrm{below}(S,\overline{S})$ the set of cells of $\overline{S}$ which lie above and below S, respectively.

We denote by $\omega(\overline{S})$ the poset of all subdivisions of $C(n,d)$ which are induced by $\overline{S}$, partially ordered by refinement, so that $\omega(\overline{S})$ is a subposet $\omega(C(n,d))$.

From the definition of the deletion $S\backslash n$ it follows trivially that $\mathrm{lk}_S(\{n,n-1\}) := \mathrm{lk}_{\mathrm{lk}_S(n)}(n-1) \subset \mathrm{lk}_{S\backslash n}(n-1)$ for any $S \in \omega(C(n,d))$. Let $T \in \omega_{d'}(C(n-1,d))$ and let $S \in \omega_{d'}(C(n,d))$ be such that $S\backslash n = T$, i.e., $S \in \Pi_{d'}^{-1}(T)$. Then the subdivision $\mathrm{lk}_S(\{n,n-1\})$ of $C(n-2,d-2)$ is induced by the subdivision $\mathrm{lk}_T(n-1) \in \omega(C(n-2,d-1))$. In other words, we have a map $\Pi_{d'}^{-1}(T) \to \omega(\mathrm{lk}_T(n-1))$ defined by $S \mapsto \mathrm{lk}_S(\{n,n-1\})$. The following much stronger statement follows from [13, Lemma 4.7].

Lemma 3.3. *Let $2 \le d < d' \le n-2$ and consider the deletion map $\Pi_{d'}$: $\omega_{d'}(C(n,d)) \to \omega_{d'}(C(n-1,d))$. Let $T \in \omega_{d'}(C(n-1,d))$ and $\overline{S} = \mathrm{lk}_T(n-1) \in \omega(C(n-2,d-1))$. Then:*

1. *The map $\omega(C(n,d)) \to \omega(C(n-2,d-2))$ given by $S \mapsto \mathrm{lk}_S(\{n-1,n\})$ restricts to a poset isomorphism between $\Pi_{d'}^{-1}(T)$ and a subposet $\omega_{d'}(\overline{S})$ of $\omega(\overline{S})$.*
2. *The inverse map $\tau : \omega_{d'}(\overline{S}) \to \Pi_{d'}^{-1}(T)$ is given by*

$$\begin{aligned}\tau(S) := \{\, \sigma \in T \,:\, n-1 \notin \sigma \,\} & \\ \bigcup \{\, \sigma \cup \{n\} \,:\, \sigma \in \overline{S},\ \sigma \textit{ is below } S \,\} & \\ \bigcup \{\, \sigma \cup \{n-1\} \,:\, \sigma \in \overline{S},\ \sigma \textit{ is above } S \,\} & \\ \bigcup \{\, \sigma \cup \{n,n-1\} \,:\, \sigma \in S \,\}. &\end{aligned}$$

   Moreover,

$$\omega_{d'}(\overline{S}) = \{S \in \omega(\overline{S}) : \tau(S) \in \omega_{d'}(C(n,d))\}.$$

3. *Let $T' \in \omega_{d'}(C(n,d))$ be such that $T'\backslash n$ is coarser than T and let $S_0 = \mathrm{lk}_{T'}(\{n,n-1\}) \in \omega(C(n-2,d-2))$. Then, the previous isomorphism restricts to an isomorphism between $\omega_{d'}(C(n,d))_{\le T'} \cap \Pi_{d'}^{-1}(T)$ and*

$$\omega_{d'}(\overline{S})_{\le S_0} := \{S \in \omega_{d'}(\overline{S}) : S \textit{ refines } S_0\} = \omega_{d'}(\overline{S}) \cap \omega(C(n-2,d-2))_{\le S_0}.$$

By Lemma 2.1 applied to the map $\Pi_{d'}$ introduced in Proposition 3.2, Lemma 3.3 implies that in order to prove Theorem 1.2 we just need to show that, under the assumptions of the lemma, both $\omega_{d'}(\overline{S})$ and $\omega_{d'}(\overline{S})_{\leq S_0}$ are contractible. We will do this in the next section, following the ideas of [13].

4. The details

Throughout this section we assume that the hypotheses of Lemma 3.3 hold and we fix an element $T \in \omega_{d'}(C(n-1,d))$ and an element $T' \in \omega_{d'}(C(n,d))$ such that T refines $T' \backslash n$. We also let $\overline{S} = \mathrm{lk}_T(n-1)$ and $S_0 = \mathrm{lk}_{T'}(\{n, n-1\})$. Our task is to prove that both $\omega_{d'}(\overline{S})$ and $\omega_{d'}(\overline{S})_{\leq S_0}$ are contractible. The proof for $\omega_{d'}(\overline{S})_{\leq S_0}$ is easier and we do it in the following proposition. The proof for $\omega_{d'}(\overline{S})$ occupies the rest of this section.

Proposition 4.1. *Under the assumptions of part 3 of Lemma 3.3, let* $\omega(\overline{S})_{\leq S_0} := \omega(\overline{S}) \cap \omega(C(n-2,d-2))_{\leq S_0}$. *Then:*

1. $\omega_{d'}(\overline{S})_{\leq S_0} = \omega(\overline{S})_{\leq S_0}$ *and hence*
2. $\omega_{d'}(\overline{S})_{\leq S_0}$ *is contractible.*

Proof. The second statement follows from [13, Corollary 4.6], where $\omega(\overline{S})_{\leq S_0}$ is proved to be contractible.

For the first statement, let $T' \in \omega_{d'}(C(n,d))$ be such that $T' \backslash n$ is coarser than T and let $S_0 = \mathrm{lk}_{T'}(\{n-1,n\}) \in \omega(C(n-2,d-2))$. Observe that S_0 might not be in $\omega(\overline{S})$ but it is in $\omega(\overline{S'})$, where $\overline{S'} := \mathrm{lk}_{T' \backslash n}(n-1)$ is coarser than $\overline{S}$. By parts 1 and 2 of Lemma 3.3 we have that S_0 is in $\omega_{d'}(\overline{S'})$.

Let $S \in \omega(\overline{S})$ be a refinement of S_0. We will prove that $\tau(S)$ is in $\omega(C(n,d'))$, i.e. $S \in \omega_{d'}(\overline{S})$. Thus $S \in \omega_{d'}(\overline{S})_{\leq S_0}$. For the proof we only use the fact that $S_0 \in \omega_{d'}(\overline{S'})$, that S refines S_0 and that $\overline{S}$ refines $\overline{S'}$.

Let $\sigma \in \mathrm{above}(S, \overline{S})$ and choose $\sigma' \in \overline{S'}$ such that $\sigma \subset \sigma'$. Since S refines S_0, either $\sigma' \in \mathrm{above}(S_0, \overline{S'})$ or $\sigma' \in S_0$. In both cases $\sigma' \cup \{n-1\}$, and hence $\sigma \cup \{n-1\}$, is a face of $C(n,d')$. In the same way, if $\sigma \in \mathrm{below}(S, \overline{S})$ then $\sigma \cup \{n\}$ is a face of $C(n,d')$. Finally, if $\sigma \in S$, then there is $\sigma' \in S_0$ with $\sigma \subset \sigma'$ and since $\sigma' \cup \{n, n-1\}$ is a face of $C(n,d')$, $\sigma \cup \{n, n-1\}$ is a face too. ☐

We are now concerned with the poset $\omega(\overline{S}) \subset \omega(C(n-2,d-2))$ and its subposet

$$\omega_{d'}(\overline{S}) = \{S \in \omega(\overline{S}) : \tau(S) \in \omega_{d'}(C(n,d))\}.$$

Our goal is to prove that $\omega_{d'}(\overline{S})$ is contractible. Actually, we will never use the fact that $\overline{S}$ is a link of a subdivision $T \in C(n-1,d)$ but only that, since $T \in \omega_{d'}(C(n-1,d))$, its link $\overline{S}$ is in $\omega_{d'-1}(C(n-2,d-1))$. In other words, we will prove the following result.

Theorem 4.2. *Let $\overline{S}$ be a proper subdivision of $C(n-2,d-1)$, induced by $C(n-2, d'-1)$, and let $\omega_{d'}(\overline{S}) \subset \omega(\overline{S})$ be the poset of sections S of $\overline{S}$ which have the properties:*

1. *For any* $\sigma \in \mathrm{above}(S, \overline{S})$, $\sigma \cup \{n-1\}$ *is a face of* $C(n,d')$.
2. *For any* $\sigma \in \mathrm{below}(S, \overline{S})$, $\sigma \cup \{n\}$ *is a face of* $C(n,d')$.

3. *For any* $\sigma \in S$, $\sigma \cup \{n, n-1\}$ *is a face of* $C(n, d')$.

Then $\omega_{d'}(\overline{S})$ *is contractible.*

Let us recall the following technical property of subdivisions of cyclic polytopes, proved and called *stackability* in [13]. Let S be a subdivision of a cyclic polytope $C(n, d)$, for arbitrary $n > d$. For any two cells $\sigma_1, \sigma_2 \in S$ which share a facet, their common facet is an upper facet of one of σ_1, σ_2 and a lower facet of the other. If it is a lower facet of σ_2 and an upper facet of σ_1 we say that "σ_2 is above σ_1" and "σ_1 below σ_2".

Lemma 4.3. [13, Lemma 2.16] *The relation "σ_1 is below σ_2" just defined has no cycles. Hence, its transitive closure is a partial order on the collection of all cells of* S.

In the sequel we denote by $<_{\mathrm{st}}$ this partial order on the cells of the subdivision $\overline{S} \in \omega_{d'-1}(C(n-2, d-1))$.

Lemma 4.4. *Let* $\overline{S} \in \omega_{d'-1}(C(n-2, d-1))$. *Let* $\sigma \subset [n-2]$ *be a face of* $C(n-2, d'-1)$ *(not necessarily a cell of* $\overline{S}$*) and let* σ_+ *and* σ_- *be cells in* $\overline{S}$ *such that* $\sigma_- <_{\mathrm{st}} \sigma_+$. *Then:*

1. *At least one of* $\sigma \cup \{n-1\}$ *and* $\sigma \cup \{n\}$ *is a proper face of* $C(n, d')$.
2. *If* $\sigma \cup \{n-1\}$ *and* $\sigma \cup \{n\}$ *are both proper faces of* $C(n, d')$, *then so is* $\sigma \cup \{n, n-1\}$.
3. *If* $\sigma_+ \cup \{n\}$ *and* $\sigma_- \cup \{n-1\}$ *are both proper faces of* $C(n, d')$ *then so is either* $\sigma_- \cup \{n\}$ *or* $\sigma_+ \cup \{n-1\}$.

Proof. 1. If σ is a face of $C(n-2, d'-2)$ then $\sigma \cup \{n, n-1\}$ is a face of $C(n, d')$. Hence, both $\sigma \cup \{n-1\}$ and $\sigma \cup \{n\}$ are faces of $C(n, d')$ as well.

If σ is not a face of $C(n-2, d'-2)$ then σ is either an upper or a lower face of $C(n-2, d'-1)$. In the first case $\sigma \cup \{n-1\}$ is a face of $C(n, d')$ and in the second case $\sigma \cup \{n\}$ is a face of $C(n, d')$, as follows easily from Gale's evenness criterion.

2. We will show that $\sigma \cup \{n, n-1\}$ has at least one interior component of odd length less than either $\sigma \cup \{n\}$ or $\sigma \cup \{n-1\}$. Taking m to be the maximum element in $[n-2] \backslash \sigma$, we observe that this is the case for $\sigma \cup \{n-1\}$ if $n-m$ is even and for $\sigma \cup \{n\}$ if $n-m$ is odd.

3. Let m_+ (respectively m_-) be the maximum in $[n-2] \backslash \sigma_+$ (respectively $[n-2] \backslash \sigma_-$). We will prove that either $\{m_- + 1, \ldots, n-2\}$ has an even number of elements (and then $\sigma_- \cup \{n\}$ is a face of $C(n, d')$), or $\{m_+ + 1, \ldots, n-2\}$ has an odd number of elements (and then $\sigma_+ \cup \{n-1\}$ is a face of $C(n, d')$).

For this let $\sigma_- = \sigma_0 <_{\mathrm{st}} \sigma_1 <_{\mathrm{st}} \cdots <_{\mathrm{st}} \sigma_k = \sigma_+$ be a chain of cells of $\overline{S}$ such that every two consecutive ones share a facet. Let m be the maximum integer in $[n-2] \backslash \bigcap_{i=0}^k \sigma_i$. We will consider separately the following three cases: (i) $m \notin \bigcup_{i=0}^k \sigma_i$, (ii) $m \in \bigcup_{i=0}^k \sigma_i$ and $n-m$ is even and (iii) $m \in \bigcup_{i=0}^k \sigma_i$ and $n-m$ is odd.

(i) If $m \notin \bigcup_{i=0}^k \sigma_i$ then $m = m_+ = m_-$. Obviously, $\{m+1, \ldots, n-2\}$ has either an even or an odd number of elements.

(ii) If $n-m$ is even then the common interval $\{m+1,\dots,n-2\}$ to all the σ_i has an even number of elements. This implies that if $m \in \sigma_i$ for an $i \le k-1$ then $m \in \sigma_{i+1}$ too. Indeed, the common facet τ between σ_i and σ_{i+1} is an upper facet of σ_i and, hence, its last interval has odd length. So it is impossible to have $m \in \sigma_i \backslash \tau$ and $\{m+1,\dots,n-2\} \subset \tau$. In particular, m cannot be in $\sigma_- = \sigma_0$ because then it would be in $\bigcap_{i=0}^k \sigma_i$. Hence, $m = m_-$ and $\{m_- + 1,\dots,n-2\}$ has an even number of elements.

(iii) This case is analogous: If $n-m$ is odd then the common interval $\{m+1, \dots,n-2\}$ to all the σ_i has an odd number of elements. This implies that if $m \in \sigma_i$ for an $i \ge 1$ then $m \in \sigma_{i-1}$ too. Indeed, the common facet τ between σ_i and σ_{i-1} is a lower facet of σ_i and, hence, its last interval has even length. So it is impossible to have $m \in \sigma_i \backslash \tau$ and $\{m+1,\dots,n-2\} \subset \tau$. In particular, m cannot be in $\sigma_+ = \sigma_k$ because then it would be in $\bigcap_{i=0}^k \sigma_i$. Hence, $m = m_+$ and $\{m_+ + 1,\dots,n-2\}$ has an odd number of elements.

□

Our next goal is to prove that $\omega_{d'}(\overline{S})$ is not empty, and hence that the map $\Pi_{d'}$ is surjective. Clearly, $\omega(\overline{S})$ is not empty so we are interested in which elements $S \in \omega(\overline{S})$ lie also in $\omega_{d'}(\overline{S}) \subset \omega(\overline{S})$.

Lemma 4.5. *Let $S \in \omega(\overline{S})$. Then $S \in \omega_{d'}(\overline{S})$ if and only if for any cell σ of $\overline{S}$ we have:*

$$\textit{if } \sigma \in S \cup \text{above}(S,\overline{S}) \textit{ then } \sigma \cup \{n-1\} \textit{ is a face of } C(n,d'),$$
$$\textit{if } \sigma \in S \cup \text{below}(S,\overline{S}) \textit{ then } \sigma \cup \{n\} \textit{ is a face of } C(n,d').$$

Proof. Necessity of the conditions is obvious by the definition of $\omega_{d'}(\overline{S})$ in Lemma 3.3. Sufficiency is not obvious since a cell σ of S might not be a (spanning) cell of $\overline{S}$. We need to prove under the conditions in the statement that for such a cell σ, $\sigma \cup \{n, n-1\}$ is a face of $C(n,d')$.

Let σ be in $S \setminus \overline{S}$. Then σ is a simplex of dimension $d-2$ in $C(n-2,d-1)$ and there is a cell $\sigma_+ \in \overline{S}$ (respectively σ_-) of which σ is a lower (respectively upper) facet unless σ is an upper (respectively lower) facet of $C(n-2,d-1)$. We will prove that $\sigma \cup \{n\}$ and $\sigma \cup \{n-1\}$ are faces of $C(n,d')$. Then by part 2 of Lemma 4.4 we conclude that $\sigma \cup \{n,n-1\}$ is a face of $C(n,d')$.

If σ is an upper (respectively lower) facet of $C(n-2,d-1)$ then $\sigma \cup \{n-1\}$ (respectively $\sigma \cup \{n\}$) is a lower (respectively upper) facet of $C(n,d)$, hence a face of $C(n,d')$. If σ is an upper (respectively lower) facet of $\sigma_- \in \overline{S}$, (respectively of σ_+) then σ_- is below S (respectively σ_+ is above S) and by hypothesis $\sigma_- \cup \{n\}$ (respectively $\sigma_+ \cup \{n-1\}$) is a face of $C(n,d')$. Thus $\sigma \cup \{n\}$ (respectively $\sigma \cup \{n-1\}$) is also a face. □

For the sequel, for $\overline{S} \in \omega_{d'-1}(C(n-2,d-1))$, let us define the following collections, which depend on d':

$$\text{above}(\overline{S}) = \{\sigma \in \overline{S} : \forall \sigma' \in \overline{S} \text{ with } \sigma \le_{\text{st}} \sigma',\ \sigma' \cup \{n-1\} \text{ is a face of } C(n,d')\},$$
$$\text{below}(\overline{S}) = \{\sigma \in \overline{S} : \forall \sigma' \in \overline{S} \text{ with } \sigma' \le_{\text{st}} \sigma,\ \sigma' \cup \{n\} \text{ is a face of } C(n,d')\}.$$

By definition, $\mathrm{below}(\overline{S})$ and $\mathrm{above}(\overline{S})$ are a lower and an upper ideal respectively in $<_{\mathrm{st}}$. This implies that the upper envelope S_{up} of $\mathrm{below}(\overline{S})$ and the lower envelope S_{down} of $\mathrm{above}(\overline{S})$ are valid sections in $\omega(\overline{S})$. We show that they are also in $\omega_{d'}(\overline{S})$. Observe that $\mathrm{above}(\overline{S}) = \mathrm{above}(S_{\mathrm{down}}, \overline{S})$ and $\mathrm{below}(\overline{S}) = \mathrm{below}(S_{\mathrm{up}}, \overline{S})$.

Lemma 4.6. *We have:*

1. $\mathrm{below}(\overline{S}) \cup \mathrm{above}(\overline{S}) = \overline{S}$.
2. *Let* $S \in \omega(\overline{S})$*. Then* $S \in \omega_{d'}(\overline{S})$ *if and only if*
$$\mathrm{above}(S, \overline{S}) \cup \mathrm{below}(\overline{S}) = \overline{S},$$
$$\mathrm{below}(S, \overline{S}) \cup \mathrm{above}(\overline{S}) = \overline{S}.$$
3. *In particular,* S_{up} *and* S_{down} *are in* $\omega_{d'}(\overline{S})$*.*

Proof. 1. Let $\sigma \in \overline{S}$ and suppose $\sigma \notin \mathrm{below}(\overline{S})$. By definition of $\mathrm{below}(\overline{S})$ this means that there is a $\sigma' \leq_{\mathrm{st}} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. Since σ' is a face of $C(n-2, d'-1)$, parts 1 and 3 of Lemma 4.4 imply, respectively, that $\sigma' \cup \{n-1\}$ and any $\sigma'' \cup \{n-1\}$ with $\sigma'' \geq_{\mathrm{st}} \sigma'$ are faces of $C(n, d')$. In particular, $\sigma \in \mathrm{above}(\overline{S})$.

2. We first prove the necessity of the conditions. If $\sigma \in \overline{S} \setminus (\mathrm{above}(S, \overline{S}) \cup \mathrm{below}(\overline{S}))$ then $\sigma \in S \cup \mathrm{below}(S, \overline{S})$ and there is a $\sigma' <_{\mathrm{st}} \sigma$ such that $\sigma' \cup \{n\}$ is not a face of $C(n, d')$. This σ' will also be in $S \cup \mathrm{below}(S, \overline{S})$ and hence $S \notin \omega_{d'}(\overline{S})$ by Lemma 4.5. The case of a $\sigma \in \overline{S} \setminus (\mathrm{below}(S, \overline{S}) \cup \mathrm{above}(\overline{S}))$ is analogous.

For the sufficiency, let $S \in \omega(\overline{S})$ be such that $\mathrm{above}(S, \overline{S}) \cup \mathrm{below}(\overline{S}) = \overline{S}$ and $\mathrm{below}(S, \overline{S}) \cup \mathrm{above}(\overline{S}) = \overline{S}$. We will prove that $S \in \omega_{d'}(\overline{S})$ using Lemma 4.5. Let $\sigma \in \overline{S}$ and suppose that $\sigma \in S \cup \mathrm{above}(S, \overline{S})$. This is equivalent to $\sigma \notin \mathrm{below}(S, \overline{S})$ and hence $\sigma \in \mathrm{above}(\overline{S})$. Hence, $\sigma \cup \{n-1\}$ is a face of $C(n, d')$. In the same way, if $\sigma \in S \cup \mathrm{below}(S, \overline{S})$ we prove that $\sigma \cup \{n\}$ is a face of $C(n, d')$.

3. From the definition of S_{up}, it follows that S_{up} does not contain any cell of $\overline{S}$ (i.e. $\mathrm{above}(S_{\mathrm{up}}, \overline{S}) \cup \mathrm{below}(S_{\mathrm{up}}, \overline{S}) = \overline{S}$) and also that $\mathrm{below}(S_{\mathrm{up}}, \overline{S}) = \mathrm{below}(\overline{S})$. Putting these two facts together and using part 1, we conclude that S_{up} satisfies the conditions of part 2. The same holds for S_{down}. ☐

Remark 4.7. The last result can be interpreted using the following poset structure on the collection of subdivisions induced by $\overline{S}$.

Definition 4.8. Let $\mathrm{St}(\overline{S})$ be the set of subdivisions of $C(n, d)$ induced by $\overline{S}$, partially ordered by $S_1 \leq S_2$ if and only if S_1 lies below S_2 as a cellular section of the natural projection $C(n, d+1) \to C(n, d)$ or, equivalently, if $\mathrm{above}(S_2, \overline{S}) \subset \mathrm{above}(S_1, \overline{S})$ and $\mathrm{below}(S_1, \overline{S}) \subset \mathrm{below}(S_2, \overline{S})$.

Let $\mathrm{St}_{d'}(\overline{S})$ be the induced subposet of $\mathrm{St}(\overline{S})$ on the subset $\omega_{d'}(\overline{S})$. We call $\mathrm{St}(\overline{S})$ and $\mathrm{St}_{d'}(\overline{S})$ the *Stasheff orders* on $\omega(\overline{S})$ and $\omega_{d'}(\overline{S})$.

The above definition reminds of the second higher Stasheff-Tamari order on the set of all triangulations of a cyclic polytope and its characterization as closed sets in dimensions 2 and 3 [7]. In this context the structure is well-behaved in all dimensions.

Using the Stasheff order, Lemma 4.6 can be rewritten as follows.

Lemma 4.9. *An element S of $\omega(\overline{S})$ is in $\omega_{d'}(\overline{S})$ if and only if $S_{\text{down}} \leq_{\text{St}} S \leq_{\text{St}} S_{\text{up}}$. Thus, $\text{St}_{d'}(\overline{S})$ is a nonempty interval in $\text{St}(\overline{S})$.*

It is also easy to see that $\omega_{d'}(\overline{S})$ is a lattice, where for every $S_1, S_2 \in \omega_{d'}(\overline{S})$ the join $S_1 \vee S_2$ and the meet $S_1 \wedge S_2$ are the elements satisfying

$$\begin{aligned}
\text{above}(S_1 \vee S_2, \overline{S}) &:= \text{above}(S_1, \overline{S}) \cap \text{above}(S_2, \overline{S}),\\
\text{below}(S_1 \vee S_2, \overline{S}) &:= \text{below}(S_1, \overline{S}) \cup \text{below}(S_2, \overline{S});\\
\text{above}(S_1 \wedge S_2, \overline{S}) &:= \text{above}(S_1, \overline{S}) \cup \text{above}(S_2, \overline{S}),\\
\text{below}(S_1 \wedge S_2, \overline{S}) &:= \text{below}(S_1, \overline{S}) \cap \text{below}(S_2, \overline{S}).
\end{aligned}$$

In a sense, $S_1 \vee S_2$ and $S_1 \wedge S_2$ are the common upper and lower envelopes of S_1 and S_2, except that if a cell σ is in $S_1 \cap S_2$ then σ (instead of its upper or lower envelope) is also in $S_1 \vee S_2$ and $S_1 \wedge S_2$.

In what follows we argue that the proof of [13, Theorem 4.5], showing that $\omega(\overline{S})$ is contractible, can be modified to prove that $\omega_{d'}(\overline{S})$ is contractible. The original proof is based on a total ordering of the cells of $\overline{S}$ compatible with the partial order $<_{\text{st}}$. Here we also want our total order to behave nicely with respect to above$(\overline{S})$ and below$(\overline{S})$.

Lemma 4.10. *There is a total order, i.e. a numbering $\overline{S} = \{\sigma_1, \dots, \sigma_k\}$, of the cells of $\overline{S}$ such that for every $i, j \in \{1, \dots, k\}$ we have:*

1. *If $\sigma_i <_{\text{st}} \sigma_j$ then $i < j$.*
2. *If $\sigma_i \in \text{below}(\overline{S})$ and $\sigma_j \notin \text{below}(\overline{S})$ then $i < j$.*
3. *If $\sigma_i \in \text{above}(\overline{S})$ and $\sigma_j \notin \text{above}(\overline{S})$ then $i > j$.*

Proof. We first order the cells not in above$(\overline{S})$ with the numbers from 1 to k_1, in a way compatible with the partial order $<_{\text{st}}$. Then we order the cells in above$(\overline{S}) \cap$ below$(\overline{S})$ with numbers $k_1 + 1, \dots, k_2$ and then those not in below$(\overline{S})$ with $k_2 + 1, \dots, k$, both times again in a way compatible with $<_{\text{st}}$. This can be done since $<_{\text{st}}$ is a partial order.

The order so obtained satisfies conditions 2 and 3 by construction and it also satisfies condition 1 since below$(\overline{S})$ is a lower ideal in $<_{\text{st}}$ (so that if $\sigma_i <_{\text{st}} \sigma_j$, it is impossible that $\sigma_j \in \text{below}(\overline{S})$ and $\sigma_i \notin \text{below}(\overline{S})$) and above$(\overline{S})$ is an upper ideal in $<_{\text{st}}$ (so that if $\sigma_i <_{\text{st}} \sigma_j$, it is impossible that $\sigma_i \in \text{above}(\overline{S})$ and $\sigma_j \notin \text{above}(\overline{S})$). □

The proof of the following proposition follows closely the one of [13, Theorem 4.5] but we include it for the sake of completeness. It establishes Theorem 4.2 and finishes the proof of Theorem 1.2.

Proposition 4.11. *The subposet $\omega_{d'}(\overline{S})$ of $\omega(C(n-2, d-2))$ is contractible.*

Proof. Let the cells of $\overline{S}$ be totally ordered as in Lemma 4.10, so that $\{\sigma_1, \dots, \sigma_{k_1}\} =$ below$(\overline{S}) \setminus$ above$(\overline{S})$, $\{\sigma_{k_1}+1, \dots, \sigma_{k_2}\} =$ below$(\overline{S}) \cap$ above$(\overline{S})$ and $\{\sigma_{k_2}+1, \dots, \sigma_k\}$ $=$ above$(\overline{S}) \setminus$ below$(\overline{S})$.

For any $S \in \omega(\overline{S})$ we call *height* of S the maximum index i of a cell σ_i on or below S. For each $i = 0, \dots, k$ we denote by $\omega_{d'}(\overline{S}; i)$ the subposet of $\omega_{d'}(\overline{S})$ consisting of the subdivisions of height at most i.

By definition, S_{down} has height k_1 and S_{up} has height k_2. Moreover, by Lemma 4.6, $\omega_{d'}(\overline{S}) = \omega_{d'}(\overline{S}; k_2)$ and $\omega(\overline{S}; k_1)$ has only the element S_{down}. We will prove that $\omega_{d'}(\overline{S}; i)$ and $\omega_{d'}(\overline{S}; i-1)$ are homotopically equivalent for every $i = k_1 + 1, \dots, k_2$.

Consider first the following situation. Let $S \in \omega_{d'}(\overline{S})$ with $\sigma_i \in S$. We can get two new elements S_{σ_i+} and S_{σ_i-} of $\omega_{d'}(\overline{S})$ by substituting σ_i in S for its upper and lower envelope, respectively.

We now construct the homotopy equivalence $f_i : \omega_{d'}(\overline{S}; i) \to \omega_{d'}(\overline{S}; i-1)$. We define f_i to be the identity on those $S \in \omega_{d'}(\overline{S}; i)$ with height at most $i-1$. If S has height i then either S contains σ_i, in which case we take $f_i(S) = S_{\sigma_i-}$, or S contains the upper envelope of σ_i. In this case $S = T_{\sigma_i^+}$ for some $T \in \omega_{d'}(\overline{S})$. We then define $f_i(T_{\sigma_i^+}) = T_{\sigma_i^-}$. In this way, the inverse image of an element $S \in \omega_{d'}(\overline{S}; i-1)$ is given as follows:

(i) It is S itself, if S does not contain the lower envelope of σ_i.
(ii) If S contains the lower envelope of σ_i, then $S = T_{\sigma_i-}$ for some $T \in \omega_{d'}(\overline{S}; i)$ and $f^{-1}(S) = f^{-1}(T_{\sigma_i-}) = \{T, T_{\sigma_i-}, T_{\sigma_i+}\}$.

Define the following order-preserving map:

$$g_i : \begin{cases} \omega(\overline{S}; i-1) & \to \quad \omega(\overline{S}; i), \\ \qquad\qquad S & \mapsto \quad \begin{cases} S & \text{in case (i)}, \\ T & \text{in case (ii)}. \end{cases} \end{cases}$$

Then $f_i \circ g_i = \mathrm{id}_{\omega_{d'}(\overline{S}; i-1)}$ and $g_i \circ f_i \geq \mathrm{id}_{\omega_{d'}(\overline{S}; i)}$, which means that f_i and g_i are homotopy inverses to each other by Quillen's order homotopy theorem [6, 10.11]. Thus, $\omega_{d'}(\overline{S}; i)$ is homotopy equivalent to $\omega_{d'}(\overline{S}; i-1)$. □

References

[1] C.A. Athanasiadis, J.A. De Loera, V. Reiner, and F. Santos, *Fiber polytopes for the projections between cyclic polytopes,* in *Combinatorics of Convex Polytopes* (K. Fukuda and G.M. Ziegler, eds.), European J. Combin., to appear.
[2] E.K. Babson, *A combinatorial flag space*, Ph. D. thesis, MIT, 1993.
[3] L.J. Billera, P. Filliman and B. Sturmfels, *Constructions and complexity of secondary polytopes,* Adv. in Math. **83** (1990), 155–179.
[4] L.J. Billera, M.M. Kapranov, and B. Sturmfels, *Cellular strings on polytopes,* Proc. Amer. Math. Soc. **122** (1994), 549–555.
[5] L.J. Billera and B. Sturmfels, *Fiber polytopes,* Ann. Math. **135** (1992), 527–549.
[6] A. Björner, *Topological methods*, in *Handbook of Combinatorics*, (R.L. Graham, M. Grötschel, and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819–1872.
[7] P.H. Edelman and V. Reiner, *The higher Stasheff-Tamari posets,* Mathematika **43** (1996), 127–154.
[8] P.H. Edelman and V. Reiner, *Visibility complexes and the Baues problem for triangulations in the plane,* Discrete Comput. Geom. **20** (1998), 35–59.
[9] P.H. Edelman, J. Rambau, and V. Reiner, *On subdivision posets of cyclic polytopes,* MSRI Preprint 1997-030.

[10] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[11] J. Rambau, *Polyhedral subdivisions and projections of polytopes*, Berichte aus der Mathematik, Shaker, Aachen, 1996, Dissertation, TU-Berlin.

[12] J. Rambau, *Triangulations of cyclic polytopes and higher Bruhat orders,* Mathematika **44** (1997), 162–194.

[13] J. Rambau and F. Santos, *The generalized Baues problem for cyclic polytopes I*, in "Combinatorics of Convex Polytopes" (K. Fukuda and G.M. Ziegler, eds.), European J. Combin., to appear.

[14] J. Rambau and G.M. Ziegler, *Projections of polytopes and the generalized Baues conjecture,* Discrete Comput. Geom. **16** (1996), 215–237.

[15] V. Reiner, *The generalized Baues problem,* in *New Perspectives in Algebraic Combinatorics*, MSRI Book Series, Cambridge University Press, to appear.

[16] V. Reiner, *On some instances of the generalized Baues problem,* Manuscript, 1998.

[17] B. Sturmfels and G.M. Ziegler, *Extension spaces of oriented matroids,* Discrete Comput. Geom. **10** (1993), 23–45.

[18] G.M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York, 1995.

Christos A. Athanasiadis
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, USA
E-mail address: `athana@math.upenn.edu`

Jörg Rambau
Konrad-Zuse-Zentrum für Informationstechnik
Takustrasse 7
D-14195 Berlin, Germany
E-mail address: `rambau@zib.de`

Francisco Santos
Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria
Santander, E-39071, Spain
E-mail address: `santos@matesco.unican.es`