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REMAINDER TERM IN CHAKALOV–POPOVICIU QUADRATURES OF RADAU AND LOBATTO TYPE AND INFLUENCE FUNCTION

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ABSTRACT. Let f be a given real function defined on $[a, b]$, $-\infty < a < b \leq \infty$. We develop estimates of the remainder term in the quadrature formulas with multiple nodes (Q) below, where $\sigma = \sigma_n = (s_1, s_2, \dots, s_n)$ is a given sequence of nonnegative integers, $p, q \in \mathbb{N}_0$, and $w(t)$ is a given weight function on (a, b). Let $N = 2(\sum_{\nu=1}^{n} s_\nu + n) + p + q$, and denote by $AC^k[a, b]$, $B^k[a, b]$, $C^k[a, b]$ the classes of functions whose the k-th derivative is absolutely continuous, bounded or continuous on $[a, b]$, respectively. An influence function is introduced, its relevant properties are investigated, and in classes of functions $AC^{N-1}[a, b]$, $B^N[a, b]$, $C^N[a, b]$ the error estimates are given. A numerical example is included.

1. Introduction

Let $d\varphi(t)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\varphi(t)$ $(k = 0, 1, \dots)$ exist and are finite, and $\mu_0 > 0$.

A quadrature formula of the form

$$\int_{\mathbb{R}} f(t)\, d\varphi(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.1}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,s)}$, $\tau_\nu = \tau_\nu^{(n,s)}$ $(i = 0, 1, \dots, 2s;\ \nu = 1, \dots, n)$, which is exact for all algebraic polynomials of degree at most $2(s+1)n - 1$, was considered firstly by P. Turán (see [24]), in the case when $d\varphi(t) = dt$ on $[-1, 1]$. The case with a weight

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function $d\varphi(t) = w(t)\,dt$ on $[a,b]$ has been investigated by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, etc. (see the survey paper [11] for references).

The nodes τ_ν in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the integral

$$\Phi \equiv \Phi(a_0, a_1, \ldots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\varphi(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \ \ldots\ + a_1 t + a_0$. In order to minimize Φ we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k \, d\varphi(t) = 0, \quad k = 0, 1, \ \ldots\ , n-1. \tag{1.2}$$

Such polynomials $\pi_n(t)$, which satisfy this new type of orthogonality so called "*power orthogonality*" are known as s-orthogonal (or s-self associated) polynomials with respect to the measure $d\varphi(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

Take now a sequence of nonnegative integers $\sigma = (s_1, s_2, \ldots)$. Consider a generalization of Gauss-Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities

$$\int_{\mathbb{R}} f(t)\, d\varphi(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.3}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$ $(i = 0, 1, \ldots, 2s_\nu;\ \nu = 1, \ldots, n)$. Such formulas were derived independently by Chakalov [2, 3] and Popoviciu [18]. A deep theoretical progress in this subject was made by Stancu (see [23] and references in it).

In this case, it is important to assume that the nodes τ_ν $(= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$\tau_1 < \tau_2 < \cdots < \tau_n, \quad \tau_\nu \in \operatorname{supp}(d\lambda), \tag{1.4}$$

with odd multiplicities $2s_1 + 1,\ 2s_2 + 1,\ \ldots,\ 2s_n + 1$, respectively, in order to have uniqueness of Chakalov-Popoviciu quadrature formula (1.3) (cf. Karlin and Pinkus [8]). Then this quadrature formula has the maximum degree of exactness

$$d_{\max} = 2 \sum_{\nu=1}^{n} s_\nu + 2n - 1$$

if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k \, d\varphi(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{1.5}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov [2], Popoviciu [18], Morelli and Verna [15], and existence and uniqueness subject to (1.4) by Ghizzetti and Ossicini [6].

The conditions (1.5) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbb{N}_0}$,

$$\pi_{n,\sigma}(t)=\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right),\quad \tau_1^{(n,\sigma)}<\tau_2^{(n,\sigma)}<\cdots<\tau_n^{(n,\sigma)},\quad \tau_\nu^{(n,\sigma)}\in\operatorname{supp}(d\lambda),$$

such that

$$\int_{\mathbb{R}}\pi_{k,\sigma}(t)\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right)^{2s_\nu+1}d\varphi(t)=0,\quad k=0,1,\ldots,n-1.$$

These polynomials are called σ-orthogonal polynomials and they correspond to the sequence $\sigma=(s_1,s_2,\ldots)$. If we have $\sigma=(s,s,\ldots)$, the above polynomials reduce to the s-orthogonal polynomials.

In this paper we consider the generalized Chakalov-Popoviciu quadrature formulae

$$\text{(Q)}\qquad \begin{aligned}\int_a^b w(t)f(t)\,dt=&\sum_{i=0}^{p-1}A_{i,0}f^{(i)}(a)+\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}A_{i,\nu}f^{(i)}(\tau_\nu)\\&+\sum_{i=0}^{q-1}A_{i,n+1}f^{(i)}(b)+R(f),\end{aligned}$$

with arbitrary, $\tau_\nu(\nu=1,\ldots,n)$, and fixed, a and (or) b, multiple nodes. A such quadrature formula has maximum degree of exactness $N-1$ if and only if

$$\int_a^b w(t)(1+t)^p(1-t)^q\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}t^k\,dt=0,\quad k=0,1,\ldots,n-1.$$

Recent proofs of the existence and the uniqueness of such quadrature rules have been obtained in [19], [22]. Proofs of convergence of such formulas can be found in [17], [20].

In Section 2 an influence function is considered, its relevant properties are investigated, and in the classes of functions $AC^{N-1}[a,b]$, $B^N[a,b]$, $C^N[a,b]$ the error estimates are given. In order to illustrate the possibility of use these error estimates we give a numerical example.

2. Error estimates for quadrature formulae of Radau and Lobatto type connected to σ-orthogonal polynomials

In [16] (see also [5]) Ossicini, for the Gauss-Turán quadratures (formula (Q) with $p=q=0$, $s_1=\cdots=s_n=s$, or (1.1) with $d\varphi(t)=w(t)\,dt$), and the Gauss-Turán quadratures of Lobatto type (formula (Q) with $s_1=\cdots=s_n=s$

and $p = q = 2s+1$), constructed an influence function, investigated its relevant properties, and in some classes of functions gave error estimates. Recently, we have generalized those results to the formula (Q) with $p = q = 0$ (or (1.3) with $d\varphi(t) = w(t)\,dt$) (see [14]). In this section we will consider the quadrature formula (Q), the general case.

For all undefined notions and notations we refer to [5].

Radau formula. Let

$$\int_a^b w(t)f(t)\,dt = \sum_{i=0}^{p-1} A_{i,0}^R f^{(i)}(a) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^R f^{(i)}(\tau_\nu) + R^R(f), \tag{2.1}$$

$-\infty < a < \infty$, $p \in \mathbb{N}$, with $R^R(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n)+p-1}$, be the generalized Chakalov-Popoviciu quadrature formula of Radau type. With $\mathcal{P}_k$ we denoted the set of all polynomials of degree at most k, $k \in \mathbb{N}_0$. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.1) we have the following theorem:

THEOREM 2.1. *Formula* (2.1) *is valid under the following hypotheses:*

$$\begin{aligned} &w(t) \in L[a,b], \quad f(t) \in AC^{N-1}[a,b], \qquad \text{if } b \text{ is finite},\\ &t^N w(t) \in L[a,\infty), \quad f(t) \in AC_{loc}^{N-1}[a,\infty),\\ &f^{(N)}(t)\int_t^\infty \xi^{N-1} w(\xi)\,d\xi \in L[0,\infty). \end{aligned}$$

The proof is the same as the one of Theorem 4.13.I. in [5, pp. 132–133] and will be omitted.

Consider, for example, the case:

$$p-1 < 2s_1 < 2s_2 < \cdots < 2s_n, \tag{2.2}$$

i.e., $N-p-1 > N-2s_1-2 > \cdots > N-2s_n-2$. Let p be odd, without loss of generality. Then, we have that N is odd. Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}^R$ for the remainder in (2.1) we have (see [5]):

$$R^R(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t)\,dt, \tag{2.3}$$

where the influence-function $\Phi(t)$ is expressed by

$$\Phi(t) = \varphi_{\nu+1}(t) \qquad \text{for} \quad \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0,1,\dots,n;\ \tau_0 = a, \tau_{n+1} = b, \tag{2.4}$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = -w(t)$ (since N is odd), are given by the formulae

$$\begin{aligned} \varphi_\nu(t) = -\int_a^t w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)!}\,d\xi + \sum_{j=1}^{\nu-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-1}}{(N-i-1)!}\\ + \sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-1}}{(N-i-1)!}, \end{aligned} \tag{2.5}$$

where $\nu = 1, \dots, n+1$, and $A_{h,j} = A^R_{h,j}$.

For $\varphi_{n+1}(t)$ we have

$$\varphi_{n+1}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)\,!}\, d\xi. \tag{2.6}$$

From (2.4), (2.5) it follows, differentiating k times (with $0 \le k \le N-1$):

$$\Phi^{(k)}(t) = \varphi_\nu^{(k)}(t) \qquad \text{for} \quad t \in (\tau_{\nu-1}, \tau_\nu), \quad \nu = 1, \dots, n+1, \tag{2.7}$$

where for $l = \nu, \nu-1, \dots, 1$,

$$\begin{aligned}\varphi_\nu^{(k)}(t) = &-\int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi + \sum_{j=1}^{l-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} \\ &+ \sum_{j=l}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} + \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!}\,,\end{aligned}$$

with: (i) $0 \le k \le N - 2s_{\nu-1} - 2$, for $l = \nu$, (ii) $N - 2s_l - 1 \le k \le N - 2s_{l-1} - 2$, for $l = \nu-1, \nu-2, \dots, 1$, where we put $s_0 = (p-1)/2$, and

$$\begin{aligned}\varphi_\nu^{(k)}(t) = &-\int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi + \sum_{j=1}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} \\ &+ \sum_{i=0}^{N-k-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!}\,,\end{aligned} \tag{2.8}$$

for $N - p \le k \le N-1$. (We used the convention $\sum_i^j \cdot \equiv 0$ for $j < i$.)

For the derivatives of $\varphi_{n+1}(t)$ $(t \in (\tau_n, b))$ we can use the following formulas:

$$\varphi_{n+1}^{(k)}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi, \quad t \in (\tau_n, b). \tag{2.9}$$

Now, we can conclude that

$$\begin{aligned}&\Phi^{(k)}(a) = 0, \;\; k = 0, 1, \dots, N-p-1, \\ &\Phi^{(k)}(b) = 0, \;\; k = 0, 1, \dots, N-1,\end{aligned} \tag{2.10}$$

and that the functions $\Phi(t), \Phi'(t), \dots, \Phi^{(n-2_n-2)}(t)$ are continuous in $[a, b]$, since $N - 2s_n - 2 = \min_{1 \le \nu \le n}(N - 2s_\nu - 2)$, while $\Phi^{(N-2s_n-1)}(t), \dots, \Phi^{(N-1)}(t)$ have discontinuities of the first kind at the points $\tau_1, \tau_2, \dots, \tau_n$. From (2.9) we conclude

$$(-1)^k \Phi^{(k)}(t) > 0 \quad \text{for} \quad t \in (\tau_n, b), \; k = 0, 1, \dots, N-1, \tag{2.11}$$

and, particularly, $\Phi(t) > 0$ on (τ_n, b).

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a, b]$. We will prove that the influence function $\Phi(t)$ is positive inside $[a, b]$. We give the proof for the case (2.2). (The general case can be considered in analogous way, see the Lobatto case.) If we identify the function $\Phi(t)$ as a monospline, then the property $\Phi(t) > 0$ on (a, b) is a corollary from the Micchelli estimate [9] of the number of zeros for monosplines with multiplicities (cf. Braess [p. 241, 1]). Our proof is direct and use only Rolle theorem.

We show that $\Phi^{(N-2s_l-2)}(t)$ $(l = 1, 2, \ldots, n)$ has at most $2s_l + 2$ zeros in each interval $[\tau_{\nu-1}, \tau_\nu]$, $\nu = 1, 2, \ldots, n$. In fact, should it have $2s_l + 3$ of them, for the Rolle theorem, $\Phi^{(N-2s_l-1)}(t)$ would have at least $2s_l + 2$ zeros inside $[\tau_{\nu-1}, \tau_\nu]$, $\Phi^{(N-2s_l)}(t)$ would have at least $2s_l + 1$ zeros and so on, until we may conclude that $\Phi^{(N-1)}(t)$ would have at least two zeros inside $[\tau_{\nu-1}, \tau_\nu]$. But this is absurd since from (2.8) there follows that, for $t \in (\tau_{\nu-1}, \tau_\nu)$, we have

$$\Phi^{(N-1)}(t) = \varphi_\nu^{(N-1)}(t) = -\int_a^t w(\xi)\, d\xi + \sum_{j=0}^{\nu-1} A_{0,j}$$

and this function is decreasing (for the hypothesis on $w(t)$).

$\Phi^{(N-p)}(t)$ is continuous in $[a, \tau_1]$ and, we can prove as for $\Phi^{(N-2s_l-2)}(t)$, it has at most p zeros in $[a, \tau_1]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 1, 2, \ldots, n)$ is continuous in $[a, \tau_1]$ and let it have α zeros in $[a, \tau_1]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-p)}(t)$ has at least α zeros in (a, τ_1). Since $\alpha \le p$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 1, 2, \ldots, n)$ has at most p zeros in $[a, \tau_1]$.

$\Phi^{(N-2s_1-2)}(t)$ is continuous in $[a, \tau_2]$ and has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 2, \ldots, n)$ is continuous in $[a, \tau_2]$ and let it have α_1 zeros in $[a, \tau_2]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_1 zeros in (a, τ_2), etc., $\Phi^{(N-2s_1-2)}(t)$ has at least α_1 zeros in (a, τ_2). Since $\alpha_1 \le p + (2s_1 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 2, \ldots, n)$ has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[a, \tau_3]$ and has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 3, \ldots, n)$ is continuous in $[a, \tau_3]$ and let it have α_2 zeros in $[a, \tau_3]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_2 zeros in (a, τ_3), etc., $\Phi^{(N-2s_2-2)}(t)$ has at least α_2 zeros in (a, τ_3). Since $\alpha_2 \le p + (2s_1 + 2) + (2s_2 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 3, \ldots, n)$ has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$.

In analogous way, we conclude that $\Phi^{(N-2s_n-2)}(t)$ is continuous in $[a, \tau_n]$ and has at most

$$p + \sum_{\nu=1}^{n-1} (2s_\nu + 2) = N - 2s_n - 2$$

zeros in $[a, \tau_n]$, and also in (a, b), because of (2.11).

We may then show that $\Phi(t)$ does not vanish inside $[a, b]$ and therefore it is positive, because it is such on (x_n, b). In fact, if $\Phi(t)$ should vanish at one point in

(a,b), using (2.2) and (2.10) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, etc., $\Phi^{(N-2s_n-2)}(t)$ would vanish at least $N-2s_n-1$ times, in contraposition with the preceding deduction, because $N-2s_n-1 \le N-2s_n-2$ gives $1 \le 0$.

So, we proved the theorem:

THEOREM 2.2. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a,b]$, the influence function $\Phi(t)$ defined by (2.4) (together with (2.5) and (2.6)) belongs to the class $C^{N-2s_k-2}[a,b]$, where $N-2s_k-2 = \min_{1\le\nu\le n}(N-2s_\nu-2)$, and it is positive inside $[a,b]$.*

Now, we can estimate the remainder in the formulas of the type (2.1), by using (2.3).

1^0 If $f(t) \in AC^{N-1}[a,b]$ and $a,b \in R$ we have

$$|R(f)| \le \max_{a\le t\le b} \Phi(t)V_{N-1} = \Phi(x_0)V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a,b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a,b) it holds $(\exists x_0 \in (a,b)) \max_{a\le t\le b} \Phi(t) = \Phi(x_0)$.

2^0 If $f^{(N)}(t)$ is bounded in $[a,b]$, i.e., $M_N = \sup_{a\le t<b} \left|f^{(N)}(t)\right|$, $b \le \infty$, then we have

$$|R(f)| \le M_N \int_a^b \Phi(t)\,dt.$$

3^0 If $f \in C^N[a,b]$, $b < \infty$, because $\Phi(t) > 0$ on (a,b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t)\,dt, \quad \xi \in (a,b).$$

Lobatto formula. Let

$$\begin{aligned}\int_a^b w(t)f(t)\,dt &= \sum_{i=0}^{p-1} A_{i,0}^L f^{(i)}(a) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^L f^{(i)}(\tau_\nu) \\ &\quad + \sum_{i=0}^{q-1} A_{i,n+1}^L f^{(i)}(b) + R^L(f),\end{aligned} \tag{2.12}$$

$-\infty < a < b < \infty$, $p,q \in \mathbb{N}$, with $R^L(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n)+p+q-1}$, be the generalized Chakalov-Popoviciu quadrature formula of Lobatto type. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p + q$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.12) we have the following theorem:

THEOREM 2.3. *Formula* (2.12) *is valid under the following hypothesis:*

$$w(t) \in L[a,b], \quad f(t) \in AC^{N-1}[a,b].$$

The proof is the same as one of the theorem 4.13.I. in [5, pp. 132–133] and will be omitted.

Let, for simplicity,

$$p-1 < 2s_1 < 2s_2 < \cdots < 2s_n < q-1, \tag{2.13}$$

i.e.,

$$N-p-1 > N-2s_1-2 > \cdots > N-2s_n-2 > N-q-1,$$

and, let $p+q$ – be even, without loss of generality. Then, N is even.

Assuming already computed the nodes τ_ν and the coefficients $A^L_{i,\nu}$ for the remainder in (2.12) we have (see [5]):

$$R^L(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t)\,dt, \tag{2.14}$$

where the influence-function $\Phi(t)$ is expressed by

$$\Phi(t) = \varphi_{\nu+1}(t) \qquad \text{for} \quad \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0,1,\ldots,n;\ \tau_0 = a, \tau_{n+1} = b, \tag{2.15}$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = w(t)$ (since N is even), are given by the formulae

$$\begin{aligned}\varphi_\nu(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)!}\,d\xi - \sum_{j=1}^{\nu-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-1}}{(N-i-1)!}\\ -\sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-1}}{(N-i-1)!},\end{aligned} \tag{2.16}$$

where $\nu = 1,\ldots,n+1$, and $A_{h,j} = A^L_{h,j}$.

For $\varphi_{n+1}(t)$ we have

$$\varphi_{n+1}(t) = -\int_t^b w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)!}\,d\xi + \sum_{i=0}^{q-1}(-1)^i A_{i,n+1}\frac{(t-b)^{N-i-1}}{(N-i-1)!}. \tag{2.17}$$

From (2.15), (2.16) it follows, differentiating k times (with $0 \le k \le N-1$):

$$\Phi^{(k)}(t) = \varphi_\nu^{(k)}(t) \qquad \text{for} \quad t \in (\tau_{\nu-1}, \tau_\nu), \quad \nu = 1,\ldots,n+1,$$

where for $l = \nu, \nu-1, \ldots, 1$,

$$\varphi_\nu^{(k)}(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\,d\xi - \sum_{j=1}^{l-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!}$$
$$- \sum_{j=l}^{\nu-1}\sum_{i=0}^{N-k-1}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} - \sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!},$$

with

(i) $0 \le k \le N - 2s_{\nu-1} - 2$, for $l = \nu$,

(ii) $N - 2s_l - 1 \le k \le N - 2s_{l-1} - 2$, for $l = \nu-1, \nu-2, \ldots, 1$, where we put $s_0 = (p-1)/2$,

and

$$\varphi_\nu^{(k)}(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\,d\xi - \sum_{j=1}^{\nu-1}\sum_{i=0}^{N-k-1}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!}$$
$$- \sum_{i=0}^{N-k-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!},$$

for $N - p \le k \le N - 1$.

For the derivatives of $\varphi_{n+1}(t)$ $(t \in (\tau_n, b))$ we can use the following formulas:

$$\varphi_{n+1}^{(k)}(t) = -\int_t^b w(\xi)\frac{(t-\xi)^{N-k-1}}{(n-k-1)\,!}\,d\xi + \sum_{h=0}^{I}(-1)^i A_{i,n+1}\frac{(t-b)^{N-i-k-1}}{(N-i-k-1)\,!},$$

with $I = q-1$, for $0 \le k \le N-q-1$; $I = N-k-1$, for $N-q \le k \le N-1$.

Now, we can conclude that

$$\begin{aligned} \Phi^{(k)}(a) &= 0, \ k = 0, 1, \ldots, N-p-1, \\ \Phi^{(k)}(b) &= 0, \ k = 0, 1, \ldots, N-q-1, \end{aligned} \tag{2.18}$$

and that the functions $\Phi(t), \Phi'(t), \ldots, \Phi^{(N-2s_n-2)}(t)$ are continuous in $[a,b]$, since $N - 2s_n - 2 = \min_{1\le\nu\le n}(N - 2s_\nu - 2)$, while $\Phi^{(n-2s_n-1)}(t), \ldots, \Phi^{(n-1)}(t)$ have discontinuities of first kind at the points $\tau_1, \tau_2, \ldots, \tau_n$.

The same conclusions can be derived for an arbitrary case $\sigma = (s_1, s_2, \ldots, s_n)$, $p, q \in \mathbb{N}$, $s_\nu \in \mathbb{N}_0$ $(\nu = 1, 2, \ldots, n)$. So, we have just proved that the influence function $\Phi(t)$ defined by (2.15) (together with (2.16) and (2.17)) belongs to the class $C^{N-2s_k-2}[a,b]$, where $N - 2s_k - 2 = \min\limits_{1\le\nu\le n}(N - 2s_\nu - 2)$.

If we put $f(t) = (t-a)^p(b-t)^q\prod_{\nu=1}^n(t-\tau_\nu)^{2s_\nu+2}$ in (2.14), then

$$(-1)^q N\,!\int_a^b \Phi(t)\,dt = \int_a^b w(t)(t-a)^p(b-t)^q\prod_{\nu=1}^n(t-\tau_\nu)^{2s_\nu+2}\,dt.$$

So, we obtain that

$$\int_a^b \Phi(t)\,dt \begin{cases} < 0, & \text{if } q \text{ is odd,} \\ > 0, & \text{if } q \text{ is even.} \end{cases}$$

Therefore, if $\Phi(t)$ does not vanish in (a,b) it holds a sign on this interval.

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a,b]$. We will prove that the influence function $\Phi(t)$ holds a sign in (a,b).

We will give a proof for a sufficient general case. Then, proceed in analogous way, a proof for any other case can be performed. The consideration will be given in detail.

Let $n=9$ and

$$q-1>2s_1>2s_5>2s_4>2s_8>p-1>2s_9>2s_6>2s_3>2s_7>2s_2,$$

i. e.,

$$N-q-1<N-2s_1-2<\cdots<N-2s_2-2.$$

A point τ_ν, $\nu=2,3,\dots,n-1$, we will call a *point of partition* of $[a,b]$ if for it holds $s_{\nu-1}\le s_\nu>s_{\nu+1}$, i. e., $N-2s_{\nu-1}-2\ge N-2s_\nu-2<N-2s_{\nu+1}-2$. τ_1 is the *point of partition* of $[a,b]$ if for it holds $p-1\le s_1>s_2$. τ_n is the *point of partition* of $[a,b]$ if for it holds $s_{n-1}\le s_n>q-1$. Denote by I the *index set* whose elements are the indices of the points of partition of $[a,b]$. It is clearly that $I\subset\{1,2,\dots,n\}$. Therefore, in our case, the points of partition of $[a,b]$ are τ_1,τ_5,τ_8, and $I=\{1,5,8\}$. $[a,b]$ by the points of partition we divide into the *intervals of partition*, in our case $[a,\tau_1],[\tau_1,\tau_5],[\tau_5,\tau_8],[\tau_8,b]$, on which we consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, $\nu\in I$. It is clearly that $[a,b]$ can be represented as the union of the intervals of partition.

For $\nu\in I$, order in the decreasing sequence the values $N-2s_\nu-2$, and consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively . Therefore, in our case we consider $\Phi^{(N-2s_8-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, $\Phi^{(N-2s_1-2)}(t)$, respectively.

a) Firstly, consider $\Phi^{(N-2s_8-2)}(t)$, which is continuous in $[\tau_5,b]=[\tau_5,\tau_8]\cup[\tau_8,b]$.

a.1) Firstly, consider $[\tau_5,\tau_8]$. For $\nu\in\{5,6,7,8\}$ (the indices of the nodes belong to $[\tau_5,\tau_8]$), order in the decreasing sequence the values $N-2s_\nu-2$ so that the last be one which correspond to the point of partition τ_8, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_7-2)}(t)$, $\Phi^{(N-2s_6-2)}(t)$,$\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_7-2)}(t)$ is continuous in $[\tau_6,\tau_8]=[\tau_6,\tau_7]\cup[\tau_7,\tau_8]$ and has at most $(2s_7+2)+(2s_7+2)$ zeros in it.

$\Phi^{(N-2s_6-2)}(t)$ is continuous in $[\tau_5,\tau_8]=[\tau_5,\tau_6]\cup[\tau_6,\tau_8]$. Let $\Phi^{(N-2s_6-2)}(t)$ have β_6 zeros in $[\tau_6,\tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_6-1)}(t)$ has at least β_6-1 zeros in (τ_6,τ_8), etc., $\Phi^{(N-2s_7-2)}(t)$ has at least $\beta_6-(2s_6-2s_7)$ zeros in (τ_6,τ_8). Therefore, we have $\beta_6-(2s_6-2s_7)\le(2s_7+2)+(2s_7+2)$, i. e., $\beta_6\le(2s_6+2)+(2s_7+2)$.

Therefore, $\Phi^{(N-2s_6-2)}(t)$ has at most $(2s_6+2)+(2s_6+2)+(2s_7+2)$ zeros in $[\tau_5,\tau_8]$.

Let $\Phi^{(N-2s_8-2)}(t)$ have α_8 zeros in $[\tau_5, \tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\alpha_8 - 1$ zeros in (τ_5, τ_8), etc., $\Phi^{(N-2s_6-2)}(t)$ has at least $\alpha_8 - (2s_8 - 2s_6)$ zeros in (τ_5, τ_8). Therefore, we have $\alpha_8 - (2s_8 - 2s_6) \leq (2s_6 + 2) + (2s_6 + 2) + (2s_7 + 2)$, i. e., $\alpha_8 \leq (2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2)$.

a.2) Consider $[\tau_8, b]$. For $\nu \in \{8, 9\}$ (the indices of the nodes belong to $[\tau_8, b]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_8, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_9-2)}(t)$, $\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_9-2)}(t)$ is continuous in $[\tau_8, b] = [\tau_8, \tau_9] \cup [\tau_9, b]$ and has at most $(2s_9 + 2) + (2s_9 + 2)$ zeros in it.

Let $\Phi^{(N-2s_8-2)}(t)$ have β_8 zeros in $[\tau_8, b]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\beta_8 - 1$ zeros in (τ_8, b), etc., $\Phi^{(N-2s_9-2)}(t)$ has at least $\beta_8 - (2s_8 - 2s_9)$ zeros in (τ_8, b). Therefore, we have $\beta_8 - (2s_8 - 2s_9) \leq (2s_9 + 2) + (2s_9 + 2)$, i. e., $\beta_8 \leq (2s_8 + 2) + (2s_9 + 2)$.

Therefore, using a.1), a.2), we conclude that $\Phi^{(N-2s_8-2)}(t)$ has at most $(2s_8 + 2) + \sum_{\nu=6}^{9}(2s_\nu + 2)$ zeros in $[\tau_5, b]$.

b) Now, consider $\Phi^{(N-2s_5-2)}(t)$, which is continuous in $[\tau_1, b] = [\tau_1, \tau_5] \cup [\tau_5, b]$.

b.1) Consider $[\tau_1, \tau_5]$. For $\nu \in \{1, 2, 3, 4, 5\}$ (the indices of the nodes belong to $[\tau_1, \tau_5]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_5, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_2-2)}(t)$, $\Phi^{(N-2s_3-2)}(t)$, $\Phi^{(N-2s_4-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, respectively.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[\tau_1, \tau_3] = [\tau_1, \tau_2] \cup [\tau_2, \tau_3]$ and has at most $(2s_2 + 2) + (2s_2 + 2)$ zeros in it.

$\Phi^{(N-2s_3-2)}(t)$ is continuous in $[\tau_1, \tau_4] = [\tau_1, \tau_3] \cup [\tau_3, \tau_4]$.

Let $\Phi^{(N-2s_3-2)}(t)$ have α_3 zeros in $[\tau_1, \tau_3]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_3-1)}(t)$ has at least $\alpha_3 - 1$ zeros in (τ_1, τ_3), etc., $\Phi^{(N-2s_2-2)}(t)$ has at least $\alpha_3 - (2s_3 - 2s_2)$ zeros in (τ_1, τ_3). Therefore, we have $\alpha_3 - (2s_3 - 2s_2) \leq (2s_2 + 2) + (2s_2 + 2)$, i. e., $\alpha_3 \leq (2s_2 + 2) + (2s_3 + 2)$.

Therefore, $\Phi^{(N-2s_3-2)}(t)$ has at most $(2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$ zeros in $[\tau_1, \tau_4]$.

$\Phi^{(N-2s_4-2)}(t)$ is continuous in $[\tau_1, \tau_5] = [\tau_1, \tau_4] \cup [\tau_4, \tau_5]$.

Let $\Phi^{(N-2s_4-2)}(t)$ have α_4 zeros in $[\tau_1, \tau_4]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_4-1)}(t)$ has at least $\alpha_4 - 1$ zeros in (τ_1, τ_4), etc., $\Phi^{(N-2s_3-2)}(t)$ has at least $\alpha_4 - (2s_4 - 2s_3)$ zeros in (τ_1, τ_4). Therefore, we have $\alpha_4 - (2s_4 - 2s_3) \leq (2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$, i. e., $\alpha_4 \leq (2s_2 + 2) + (2s_3 + 2) + (2s_4 + 2)$.

Therefore, $\Phi^{(N-2s_4-2)}(t)$ has at most $\sum_{\nu=2}^{4}(2s_\nu + 2) + (2s_4 + 2)$ zeros in $[\tau_1, \tau_5]$.

Finally, let $\Phi^{(N-2s_5-2)}(t)$ has α_5 zeros in $[\tau_1, \tau_5]$. As above, we conclude that $\alpha_5 \leq \sum_{\nu=2}^{5}(2s_\nu + 2)$.

b.2) Let $\Phi^{(N-2s_5-2)}(t)$ have β_5 zeros in $[\tau_5, b]$. In analogous way as above, by using the conclusions from a), we conclude that $\beta_5 \le \sum_{\nu=5}^{9}(2s_\nu + 2)$.

Therefore, $\Phi^{(N-2s_5-2)}(t)$ has at most $\sum_{\nu=2}^{9}(2s_\nu+2)+(2s_5+2)$ zeros in $[\tau_1, b]$.

c) Finally, consider $\Phi^{(N-2s_1-2)}(t)$, which is continuous in $[a,b] = [a,\tau_1]\cup[\tau_1,b]$.

c.1) Consider $[a,\tau_1]$. $\Phi^{(N-p)}(t)$ has at most p zeros in $[a,\tau_1]$. Let $\Phi^{(N-2s_1-2)}(t)$ have α_1 zeros in $[a,\tau_1]$. Then, using (2.18) (for the point a) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_1-1)}(t)$ has at least α_1 zeros in (a,τ_1), etc., $\Phi^{(N-p)}(t)$ has at least α_1 zeros in (a,τ_1). Therefore, $\alpha_1 \le p$.

c.2) By using the conclusions for $\Phi^{(N-2s_5-2)}(t)$, from b), in analogous way as above, we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $\sum_{\nu=1}^{9}(2s_\nu+2)$ zeros in $[\tau_1,b]$.

Therefore, on the basis of c.1), c.2), we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $p+\sum_{\nu=1}^{9}(2s_\nu+2) = N-q$ zeros in $[a,b]$.

We may then show that $\Phi(t)$ does not vanish inside $[a,b]$ and therefore holds a sign in it. In fact, if $\Phi(t)$ should vanish at one point in (a,b), using (2.18) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, $\Phi^{(N-q-1)}(t)$ would vanish at least $N-q$ times, $\Phi^{(N-q)}(t)$ would vanish at least $N-q+1$ times, etc., $\Phi^{(N-2s_1-2)}(t)$ would vanish at least $N-q+1$ times, in contraposition with the preceding deduction, because $N-q+1 \le N-q$ gives $1 \le 0$.

On the basis of the upper considerations we have just proved the following statement:

THEOREM 2.4. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a,b]$, the influence function $\Phi(t)$ defined by* (2.15) (*together with* (2.16) *and* (2.17)) *belongs to the class $C^{n-2s_k-2}[a,b]$, where $N-2s_k-2 = \min_{1\le\nu\le n}(N-2s_\nu-2)$, and one holds a sign inside $[a,b]$.*

Now, we can estimate the remainder in the formulas of the type (2.12), by using (2.14).

1^0 If $f(t) \in AC^{N-1}[a,b]$ we have

$$|R(f)| \le \max_{a\le t\le b} |\Phi(t)|\, V_{N-1} = |\Phi(x_0)|\, V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a,b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a,b) it holds $(\exists x_0 \in (a,b))\ \max\limits_{a\le t\le b} \Phi(t) = \Phi(x_0)$.

2^0 If $f^{(N)}(t)$ is bounded in $[a,b]$, i.e., $M_N = \sup\limits_{a\le t\le b} \left|f^{(N)}(t)\right|$, then we have

$$|R(f)| \le M_N \int_a^b |\Phi(t)|\, dt.$$

3^0 If $f \in C^N[a,b]$, because $\Phi(t)$ holds a sign on (a,b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t)\, dt, \quad \xi \in (a,b).$$

Example. An iterative process for computing the coefficients of s-orthogonal polynomials in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [25]. He applied his process to the Legendre case. When n and s increase, the process becomes numerically unstable.

In [10] (see also [4]) a stable procedure for numerical construction of s-orthogonal polynomials with respect to $d\varphi(t)$ on $\mathbb{R}$ is given.

Recently, a simple and numerically stable procedure for construction of σ-orthogonal polynomials is proposed by Milovanović and Spalević [14].

A stable numerical procedure for calculating the coefficients $A_{i,\nu}$ in (1.1) was given by Gautschi and Milovanović [4]. Some alternative methods were proposed by Stroud and Stancu [23], Golub and Kautsky [7], and Milovanović and Spalević [12] (see also [21]). A generalization of methods, for the weights, from [4, 12] to the general case when $s_\nu \in \mathbb{N}_0$, $\nu = 1, \ldots, n$, was derived recently by Milovanović and Spalević [13].

Finally, a method for calculating the nodes and the coefficients in the generalized Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type, by using the results from [13, 14], has been proposed in [22]. We use that method, for calculating in this example, in order to tabulate the corresponding influence function. Consider the Legendre case with $w(t) = 1$ on $[-1, 1]$. Let $p = q = 1$ and $\sigma = (1, 0, 1)$ in (2.12). Therefore, we have a symmetric task. The results show that the nodes of the corresponding quadrature are symmetrically distributed with respect to the origin, namely,

$$\tau_1 = -\tau_3 = -0.66772435790692, \quad \tau_2 = 0.$$

The coefficients of the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type are:

$$A_{0,1} = A_{0,3} = 6.68946557387391(-01), \quad A_{1,1} = -A_{1,3} = 2.90757109134605(-02),$$
$$A_{2,1} = A_{2,3} = 8.27917955975223(-03), \quad A_{0,2} = 5.47406124470793(-01),$$
$$A_{0,0} = A_{0,4} = 5.7350380377213(-02).$$

The results show that the influence function is even (see Table), and

$$\max_{-1\le t\le 1} |\Phi(t)| = -\Phi(0) = 7.06(-12).$$

TABLE

t	∓ 1	$\pm\tau_1$	∓ 0.5	∓ 0.4	∓ 0.3	∓ 0.2	∓ 0.1
$\Phi(t)$	0	$-4.05(-15)$	$-1.94(-13)$	$-8.23(-13)$	$-2.23(-12)$	$-4.31(-12)$	$-6.25(-12)$

Therefore, these results can be use in estimations of the remainder given above for the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type.

All computations were done using FORTRAN in double precision arithmetics. The numbers in parentheses denoted the decimal exponents.

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