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RELATIONSHIP BETWEEN MATUSZEWSKA-ORLICZ, SEMENOV AND SIMONENKO INDICES OF φ-FUNCTIONS

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ABSTRACT. We obtain a relationship between Matuszewska-Orlicz indices, Semenov indices and Simonenko indices. The main results are:

$$2\nu_\Phi^i\mu_\Psi^i = 1 = 2\nu_\Psi^i\mu_\Phi^i, \quad 2^{-1/\alpha_\Phi^i} \leqslant \nu_\Phi^i \leqslant \mu_\Phi^i \leqslant 2^{-1/\beta_\Phi^i},$$

$$\frac{1}{2(1-\nu_\Phi^i)} \leqslant p_\Phi^i \leqslant q_\Phi^i \leqslant \frac{\mu_\Phi^i}{1-\mu_\Phi^i}.$$

1. On Matuszewska-Orlicz Indices of φ-functions

A φ-function Φ is defined to be even, continuous, increasing on $[0,+\infty)$, satisfying $\Phi(0)=0$, $\Phi(\infty)=\infty$. Further, a φ-function $\Phi(u)$ is called an N-function if Φ is convex, satisfying $\lim\limits_{u\to 0}\frac{\Phi(u)}{u}=0$, $\lim\limits_{u\to\infty}\frac{\Phi(u)}{u}=\infty$. The concept of indices of φ-functions has become a powerful tool in the study of Orlicz spaces, particularly in the interpolation and extrapolation theories(see [**2**] and [**8**]),as well as in the geometric theory of Orlicz spaces (see [**11, 14**]). Maligranda [**8**] systematically studied certain types of (especially Matuszewska-Orlicz) indices of both φ-functions and N-functions and obtained a group of properties. Recently, Fiorenza and Krbec [**3**] studied a formula for the Boyd indices which is tightly connected with Matuszewska-Orlicz indices. In this paper we shall establish the relationship between Matuszewska-Orlicz indices and Semenov indices of φ-functions and make further investigation of the relationship between Semenov indices and Simonenko indices of N-functions. Formulae for Semenov indices are shown. Calculation of Simonenko indices for some classical N-functions is also discussed.

The definitions and notations in this paper will follow mainly Maligranda [**8**].

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For a φ-function Φ, define

$$M_\infty(t,\Phi)=\limsup_{u\to\infty}\frac{\Phi(tu)}{\Phi(u)},\quad M_0(t,\Phi)=\limsup_{u\to0+}\frac{\Phi(tu)}{\Phi(u)},\quad M_a(t,\Phi)=\sup_{u>0}\frac{\Phi(tu)}{\Phi(u)}.$$

All the functions above are non-decreasing submultiplicative, and are equal to 1 at the point 1. Then the limits

$$\alpha_\Phi^i=\lim_{t\to0+}\frac{\ln M_i(t,\Phi)}{\ln t}=\sup_{0<t<1}\frac{\ln M_i(t,\Phi)}{\ln t},$$
$$\beta_\Phi^i=\lim_{t\to\infty}\frac{\ln M_i(t,\Phi)}{\ln t}=\inf_{t>1}\frac{\ln M_i(t,\Phi)}{\ln t},$$

exist for $i=\infty,0,a$. These numbers are called Matuszewska–Orlicz indices and were introduced in 1960 [**9**]. The following propositions were given by Maligranda [**8**] and will be used for the main results of this paper.

PROPOSITION 1. [**8**] *If φ-functions $\Phi_1\sim\Phi_2$ at ∞ (at the point 0 or for all u), i.e., there exist $C_i>0$ $(1\leqslant i\leqslant4)$ and $u_0>0$ such that $C_1\Phi_1(C_2u)\leqslant\Phi_2(u)\leqslant C_3\Phi_1(C_4u)$ for $u\geqslant u_0$ (for $u\leqslant u_0$ or for all $u>0$, respectively), then*

$$\alpha_{\Phi_1}^i=\alpha_{\Phi_2}^i;\quad\beta_{\Phi_1}^i=\beta_{\Phi_2}^i.\qquad(i=\infty,0,a)\tag{1}$$

PROPOSITION 2. [**8**] *Let Φ^{-1} be the inverse function of a φ-function Φ, then we have*

$$\alpha_{\Phi^{-1}}^i=\frac{1}{\beta_\Phi^i},\quad\beta_{\Phi^{-1}}^i=\frac{1}{\alpha_\Phi^i}.\qquad(i=\infty,0,a)\tag{2}$$

PROPOSITION 3. [**6, 8**] *Let Φ,Ψ be a pair of complementary N-functions, then*

$$\frac{1}{\alpha_\Phi^i}+\frac{1}{\beta_\Psi^i}=1=\frac{1}{\alpha_\Psi^i}+\frac{1}{\beta_\Phi^i}.\qquad(i=\infty,0,a)\tag{3}$$

PROPOSITION 4. [**8**] *Let Φ be an N-function; then*

$$p_\Phi^i\leqslant\alpha_\Phi^i\leqslant\beta_\Phi^i\leqslant q_\Phi^i.\qquad(i=\infty,0,a)\tag{4}$$

where p_Φ^i and q_Φ^i $(i=\infty,0,a)$ are Simonenko indices [**13**], *i.e.,*

$$p_\Phi^\infty=\liminf_{u\to\infty}\frac{u\phi(u)}{\Phi(u)},\qquad q_\Phi^\infty=\limsup_{u\to\infty}\frac{u\phi(u)}{\Phi(u)},$$
$$p_\Phi^0=\liminf_{u\to0}\frac{u\phi(u)}{\Phi(u)},\qquad q_\Phi^0=\limsup_{u\to0}\frac{u\phi(u)}{\Phi(u)},$$
$$p_\Phi^a=\inf_{u>0}\frac{u\phi(u)}{\Phi(u)},\qquad q_\Phi^a=\sup_{u>0}\frac{u\phi(u)}{\Phi(u)}.$$

Proposition 4 reveals the relationship between Simonenko indices and Matuszewska–Orlicz indices for N-functions. In 1967, Semenov [**12**] introduced another type of indices on φ-functions which were greatly used by Rao and Ren [**11**] and the author [**14**] to estimate the geometric constants of Orlicz spaces:

$$\nu_\Phi^\infty=\liminf_{u\to\infty}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},\qquad\mu_\Phi^\infty=\limsup_{u\to\infty}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$

$$\nu_\Phi^0=\liminf_{u\to 0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},\qquad \mu_\Phi^0=\limsup_{u\to 0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},$$
$$\nu_\Phi^a=\inf_{u>0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)},\qquad \mu_\Phi^a=\sup_{u>0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

The relationship between Semenov indices and Matuszewska-Orlicz indices for φ-functions is as follows.

THEOREM 1. *For a φ-function $\Phi(u)$ we have*

$$2^{-1/\alpha_\Phi^i}\leqslant \nu_\Phi^i\leqslant \mu_\Phi^i\leqslant 2^{-1/\beta_\Phi^i}.\qquad (i=\infty,0,a) \tag{5}$$

PROOF. We first prove that $\mu_\Phi^\infty\leqslant 2^{-1/\beta_\Phi^\infty}$. By Proposition 2 and the definition of $\alpha_{\Phi^{-1}}^\infty$ (note that $\Phi^{-1}(u)$ is a φ-function),

$$\frac{\ln M_\infty(\frac12,\Phi^{-1})}{\ln\frac12}\leqslant \alpha_{\Phi^{-1}}^\infty=\frac{1}{\beta_\Phi^\infty},$$

which implies $\ln M_\infty(\frac12,\Phi^{-1})\geqslant \ln 2^{-1/\beta_\Phi^\infty}$, i.e., $M_\infty(\frac12,\Phi^{-1})\geqslant 2^{-1/\beta_\Phi^\infty}$. Then by the definition of $M_\infty(\frac12,\Phi^{-1})$, for any given $\varepsilon>0$, there exists a $v_0>0$, such that

$$\frac{\Phi^{-1}(\frac12 v)}{\Phi^{-1}(v)}<2^{-1/\beta_\Phi^\infty}+\varepsilon$$

for $v>v_0$. Let $v=2u$; then for any $u\geqslant u_0=v_0/2$, we have

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}<2^{-1/\beta_\Phi^\infty}+\varepsilon.$$

Take the upper limit for $u\to\infty$, one gets $\mu_\Phi^\infty\leqslant 2^{-1/\beta_\Phi^\infty}+\varepsilon$, and hence $\mu_\Phi^\infty\leqslant 2^{-1/\beta_\Phi^\infty}$ since ε is arbitrary.

Next we show that $\nu_\Phi^\infty\geqslant 2^{-1/\alpha_\Phi^\infty}$. Also by Proposition 2 and the definition of $\beta_{\Phi^{-1}}^\infty$ we have

$$\frac{\ln M_\infty(2,\Phi^{-1})}{\ln 2}\leqslant \beta_{\Phi^{-1}}^\infty=\frac{1}{\alpha_\Phi^\infty}.$$

It follows that $\ln M_\infty(2,\Phi^{-1})\leqslant \ln 2^{1/\alpha_\Phi^\infty}$, i.e., $M_\infty(2,\Phi^{-1})\leqslant 2^{1/\alpha_\Phi^\infty}$. By the definition of $M_\infty(2,\Phi^{-1})$, for any $\varepsilon>0$, there is a $u_0>0$, such that

$$\frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}<2^{1/\alpha_\Phi^\infty}+\varepsilon$$

for $u\geqslant u_0$, that is

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}>\frac{1}{2^{1/\alpha_\Phi^\infty}+\varepsilon}.$$

Take the lower limit for $u\to\infty$ and since $\varepsilon>0$ is arbitrary, one gets that $\nu_\Phi^\infty\geqslant 2^{-1/\alpha_\Phi^\infty}$.

The other inequalities cab be proved analogously. The proof is completed. □

COROLLARY 1. *Let $\Phi(u)$ be an N-function; then*

$$2^{-1/p_\Phi^i}\leqslant 2^{-1/\alpha_\Phi^i}\leqslant \nu_\Phi^i\leqslant \mu_\Phi^i\leqslant 2^{-1/\beta_\Phi^i}\leqslant 2^{-1/q_\Phi^i}.\qquad (i=\infty,0,a) \tag{6}$$

The proof can be deduced from Proposition 4 and Theorem 1.

REMARK 1. For an N-function Φ, Corollary 1 links the above three types of indices with inequalities.

COROLLARY 2. *Let* Φ *be an* N*-function. If* $\Phi \sim |u|^p$ *at* ∞ (*at the point* 0 *or for all* u), *then* $\nu_\Phi^\infty = \mu_\Phi^\infty = 2^{-1/p}$ $(\nu_\Phi^0 = \mu_\Phi^0 = 2^{-1/p}, \nu_\Phi^a = \mu_\Phi^a = 2^{-1/p}$, *respectively*).

PROOF. In view of Proposition 3, one has $\alpha_\Phi^a = \beta_\Phi^a = p$ when $\Phi \sim |u|^p$. $\square$

REMARK 2. Generally speaking, to calculate ν_Φ^i and μ_Φ^i for $i = \infty, 0, a$, one must first find the inverse function of $\Phi(u)$, which is usually not practical. By Corollary 2 we can easily obtain ν_Φ^i and μ_Φ^i from the equivalent functions of Φ. For example, for the N-function $\Phi(u) = e^{|u|} - |u| - 1$, we have $\nu_\Phi^0 = \mu_\Phi^0 = 2^{-1/2}$ since $\Phi \sim |u|^2$ at the point 0.

2. On Simonenko and Semenov Indices

Recall the equivalents between the Simonenko indices (parallel to Proposition 3):

PROPOSITION 5. [**6, 10**] *Let* Φ, Ψ *be a pair of complementary* N*-functions; then*

$$\frac{1}{p_\Phi^i} + \frac{1}{q_\Psi^i} = 1 = \frac{1}{p_\Psi^i} + \frac{1}{q_\Phi^i}. \qquad (i = \infty, 0, a) \tag{7}$$

Now we show the equivalents for Semenov indices:

THEOREM 2. *Let* $\Phi(u), \Psi(v)$ *be a pair of complementary* N*-functions; then*

$$2\nu_\Phi^i \mu_\Psi^i = 1 = 2\nu_\Psi^i \mu_\Phi^i. \qquad (i = \infty, 0, a) \tag{8}$$

PROOF. We only show that $2\nu_\Phi^0 \mu_\Psi^0 = 1$ since the other equalities can be obtained analogously. In fact, for any $0 < \varepsilon < 1/2$, by the definition of ν_Φ^0, there exists a $u_0 = u_0(\varepsilon) > 0$, such that

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} > \nu_\Phi^0 - \varepsilon,$$

i.e., $u > \Phi[(\nu_\Phi^0 - \varepsilon)\Phi^{-1}(2u)]$ for $u \leqslant u_0$. Put $t = \Phi^{-1}(2u)$, or $u = \frac{1}{2}\Phi(t)$, then

$$\Phi(t) > 2\Phi[(\nu_\Phi^0 - \varepsilon)t]$$

for $t \leqslant t_0 = \Phi^{-1}(2u_0)$. By the properties of complementary N-functions (cf. [**4**, pp. 11 and 14]), there is an $s_0 > 0$, such that

$$\Psi(s) < 2\Psi\left(\frac{s}{2(\nu_\Phi^0 - \varepsilon)}\right)$$

for $s \geqslant s_0$, i.e., $\Psi^{-1}(\frac{\Psi(s)}{2}) < \frac{s}{2(\nu_\Phi^0 - \varepsilon)}$. Denote by $v = \frac{1}{2}\Psi(s)$, then

$$\frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)} < \frac{1}{2(\nu_\Phi^0 - \varepsilon)}$$

for $v \leqslant v_0 = \frac{1}{2}\Psi(s_0)$. Take the upper limit for $v \to 0$, one has $\mu_\Psi^0 \leqslant \frac{1}{2(\nu_\Phi^0 - \varepsilon)}$, namely, $2(\nu_\Phi^0 - \varepsilon)\mu_\Psi^0 \leqslant 1$, and hence $2\nu_\Phi^0 \mu_\Psi^0 \leqslant 1$, since ε is arbitrary.

On the other hand, by the definition of μ_Ψ^0, for any $\varepsilon > 0$, there is a $v_0 = v_0(\varepsilon) > 0$, such that

$$\frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)} < \mu_\Psi^0 + \varepsilon$$

for $v \leqslant v_0$, i.e., $v < \Psi[(\mu_\Psi + \varepsilon)\Psi^{-1}(2v)]$. Let $s = \Psi^{-1}(2v)$; then $\Psi(s) < 2\Psi[(\mu_\Psi^0 + \varepsilon)s]$ for $s \leqslant s_0 = \Psi^{-1}(2v_0)$. Thus there exists a $t_0 > 0$ such that

$$\Phi(t) > 2\Phi\Big(\frac{t}{2(\mu_\Psi^0 + \varepsilon)}\Big) \quad (t \leqslant t_0).$$

Therefore,

$$\Phi^{-1}\Big(\frac{\Phi(t)}{2}\Big) > \frac{t}{2(\mu_\Psi^0 + \varepsilon)}.$$

Put $u = \frac{1}{2}\Phi(t)$, so that

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} > \frac{1}{2(\mu_\Psi^0 + \varepsilon)}$$

for $u \leqslant u_0 = \frac{1}{2}\Phi(t_0)$. Take the lower limit; one has that $\nu_\Phi^0 \geqslant \frac{1}{2(\mu_\Psi^0+\varepsilon)}$, i.e., $2\nu_\Phi^0(\mu_\Psi^0 + \varepsilon) \geqslant 1$. It follows that $2\nu_\Phi^0\mu_\Psi^0 \geqslant 1$ by the arbitrariness of ε. The proof is finished. $\square$

Corollary 1 indicates that the indices p_Φ^i, q_Φ^i are "farthest" to each other among the three types of indices. The following result shows that they are estimated by the "nearest" pair ν_Φ^i, μ_Φ^i.

THEOREM 3. *Let Φ be an N-function; then*

$$\frac{1}{2(1-\nu_\Phi^i)} \leqslant p_\Phi^i \leqslant q_\Phi^i \leqslant \frac{\mu_\Phi^i}{1-\mu_\Phi^i}. \qquad (i = \infty, 0, a) \tag{9}$$

PROOF. We only show the case for $i = \infty$, i.e.,

$$\frac{1}{2(1-\nu_\Phi^\infty)} \leqslant p_\Phi^\infty \leqslant q_\Phi^\infty \leqslant \frac{\mu_\Phi^\infty}{1-\mu_\Phi^\infty}.$$

If $\mu_\Phi^\infty = 1$, then $\frac{\mu_\Phi^\infty}{1-\mu_\Phi^\infty} = \infty$ and the inequality $q_\Phi^\infty \leqslant \frac{\mu_\Phi^\infty}{1-\mu_\Phi^\infty}$ holds obviously.

Let $\mu_\Phi^\infty < 1$. By the definition of μ_Φ^∞, given $0 < \varepsilon < 1 - \mu_\Phi^\infty$, there exists a $u_0 = u_0(\varepsilon) > 0$, such that

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} < \mu_\Phi^\infty + \varepsilon$$

i.e., $u < \Phi[(\mu_\Phi^\infty + \varepsilon)\Phi^{-1}(2u)]$ for $u \geqslant u_0$,. Let $t = \Phi^{-1}(2u)$; then

$$\Phi(t) < 2\Phi\,[(\mu_\Phi^\infty + \varepsilon)t]$$

for $t \geqslant t_0 = \Phi^{-1}(2u_0)$. Define $(\mu_\Phi^\infty + \varepsilon)t = s$; we have

$$\Phi\Big(\frac{s}{\mu_\Phi^\infty + \varepsilon}\Big) < 2\Phi(s)$$

for $s \geqslant s_0 = (\mu_\Phi^\infty + \varepsilon)t_0$. Since

$$\Phi\Big(\frac{s}{\mu_\Phi^\infty + \varepsilon}\Big) = \int_0^s \phi(r)dr + \int_s^{s/(\mu_\Phi^\infty+\varepsilon)} \Phi(r)dr \geqslant \Phi(s) + \phi(s)\Big(\frac{s}{\mu_\Phi^\infty + \varepsilon} - s\Big),$$

we deduce that

$$2 > \frac{\Phi\left(\frac{s}{\mu_\Phi^\infty+\varepsilon}\right)}{\Phi(s)} \geqslant 1 + \frac{\left(\frac{1}{\mu_\Phi^\infty+\varepsilon}-1\right)s\phi(s)}{\Phi(s)},$$

that is,

$$\frac{s\phi(s)}{\Phi(s)} \leqslant \frac{1}{\frac{1}{\mu_\Phi^\infty+\varepsilon}-1},$$

and hence

$$q_\Phi^\infty \leqslant \frac{1}{\frac{1}{\mu_\Phi^\infty+\varepsilon}-1} = \frac{\mu_\Phi^\infty+\varepsilon}{1-(\mu_\Phi^\infty+\varepsilon)}$$

by taking the upper limit for $s \to \infty$. Therefore, $q_\Phi^\infty \leqslant \frac{\mu_\Phi^\infty}{1-\mu_\Phi^\infty}$ since ε is arbitrary.

By the arguments above and Theorem 2 as well as Proposition 5, one has

$$p_\Phi^\infty = \frac{1}{1-\frac{1}{q_\Psi^\infty}} \geqslant \frac{1}{1-\frac{1-\mu_\Psi^\infty}{\mu_\Psi^\infty}} = \frac{1}{2-\frac{1}{\mu_\Psi^\infty}} = \frac{1}{2-2\nu_\Phi^\infty} = \frac{1}{2(1-\nu_\Phi^\infty)}.$$

Consequently, the theorem is proved. □

COROLLARY 3. *If Φ is an N-function, then*

$$p_\Phi^i = 1 \Longleftrightarrow \nu_\Phi^i = 1/2, \qquad q_\Phi^i = \infty \Longleftrightarrow \mu_\Phi^i = 1. \qquad (i = \infty, 0, a) \tag{10}$$

Corollary 3 is directly derived from Theorem 3 and Corollary 1, independent of $\triangle_2$ or ∇_2 conditions of N-functions (see [**10**]).

3. Examples

EXAMPLE 1. [**7, 8**] Let

$$\Phi(u) = |u|^p\Big(1 + \frac{1}{k}\sin(p\ln|u|)\Big), \qquad k > \sqrt{2}, \quad k(p-1) - \sqrt{(2p-1)^2+1} > 0.$$

Then $\Phi(u)$ is an N-function. Since

$$\frac{u\Phi'(u)}{\Phi(u)} = p + p\frac{\cos(p\ln|u|)}{k+\sin(p\ln|u|)},$$

we know that

$$\min\frac{u\Phi'(u)}{\Phi(u)} = p\Big(1 - \frac{1}{\sqrt{k^2-1}}\Big), \qquad \max\frac{u\Phi'(u)}{\Phi(u)} = p\Big(1 + \frac{1}{\sqrt{k^2-1}}\Big),$$

so that

$$p_\Phi^i = p\Big(1 - \frac{1}{\sqrt{k^2-1}}\Big), \qquad q_\Phi^i = p\Big(1 + \frac{1}{\sqrt{k^2-1}}\Big)$$

for $i = \infty, 0, a$. However, observing that $\Phi(u) \sim u^p$ for all u, we have by Proposition 2 that

$$\alpha_\Phi^i = \beta_\Phi^i = p, \qquad (i = \infty, 0, a),$$

and hence we deduce from Theorem 2 that

$$\nu_\Phi^i = \mu_\Phi^i = 2^{-1/p} \qquad (i = \infty, 0, a).$$

REMARK 3. Example 1 implies that for an N-function $\Phi(u)$, when a pair of Semenov indices is equal, the Simonenko indices need not be equal, and that for a pair of equivalent N-functions Φ_1 and Φ_2 their Semenov indices $p^i_{\Phi_1}$ and $p^i_{\Phi_2}$ ($q^i_{\Phi_1}$ and $q^i_{\Phi_2}$) (for $i = \infty, 0, a$) need not be equal although their Matuszewska–Orlicz indices must be the same (see Proposition 1).

EXAMPLE 2. [**5, 8**] Suppose for $|u| \geqslant 1+\delta$ with $\delta > 0$, the N-function M can be expressed as:

$$M(u) = |u|^{p+k\sin\ln\ln|u|}, \qquad k>0, \quad k > 1+\sqrt{2}k.$$

It is easy to check that $M(u)$ is the main part of an N-function. We show that

$$p^\infty_M = p-\sqrt{2}k, \qquad q^\infty_M = p+\sqrt{2}k. \tag{11}$$

$$\nu^\infty_M = 2^{-1/p-\sqrt{2}k}, \qquad \mu^\infty_M = 2^{-1/p+\sqrt{2}k}. \tag{12}$$

Indeed, observing that

$$p(u) = M'(u) = u^{p-1+k\sin\ln\ln u}\cdot[p+k\sin\ln\ln u + k\cos\ln\ln u]$$

for $u>0$, one gets

$$\frac{up(u)}{M(u)} = p + k\,[k\sin\ln\ln u + k\cos\ln\ln u] = p+\sqrt{2}k\sin(\ln\ln u + \pi/4).$$

Therefore, we have by the definition

$$p^\infty_M = p-\sqrt{2}k, \qquad q^\infty_M = p+\sqrt{2}k.$$

To calculate μ^∞_M, we need to find a sequence $\{u_n\}\to\infty$ and a proper constant $0<C<1$ such that

$$\frac{M^{-1}(u_n)}{M^{-1}(2u_n)} > C. \tag{13}$$

For the inequality

$$\frac{M^{-1}(u)}{M^{-1}(2u)} > C,$$

let $M^{-1}(u) = t$; then $u = M(t)$, and hence $M(t/C) > 2M(t)$, i.e.,

$$\left(\frac{t}{C}\right)^{p+k\sin\ln\ln\frac{t}{C}} > 2t^{p+k\sin\ln\ln t},$$

or equivalently,

$$t^{k(\sin\ln\ln\frac{t}{C}-\sin\ln\ln t)} > 2C^{p+k\sin\ln\ln\frac{t}{C}}. \tag{14}$$

The left-hand side of the above inequality is

$$\begin{aligned} L(t) &= t^{2k\cdot\cos\frac{1}{2}(\ln\ln\frac{t}{C}+\ln\ln t)\cdot\sin\frac{1}{2}(\ln\ln\frac{t}{C}-\ln\ln t)} \\ &= \left(t^{2k\cdot\sin\frac{1}{2}(\ln\ln\frac{t}{C}-\ln\ln t)}\right)^{\cos\frac{1}{2}(\ln\ln\frac{t}{C}+\ln\ln t)}. \end{aligned}$$

Note that $\ln C < 0$, and $\ln t \to +\infty$ $(t \to \infty)$, and one has that

$$\begin{aligned}
&\lim_{t\to\infty} t^{2k\cdot\sin\left[\frac{1}{2}(\ln\ln\frac{t}{C}-\ln\ln t)\right]} = \lim_{t\to\infty} e^{2k\ln t\cdot\sin\left[\frac{1}{2}(\ln\ln\frac{t}{C}-\ln\ln t)\right]} \\
= &\lim_{t\to\infty} e^{2k\ln t\cdot\sin\left[\frac{1}{2}\ln\left(1+(-\frac{\ln C}{\ln t})\right)\right]} = \lim_{s\to 0} e^{2k\ln t\cdot\frac{1}{s}\sin\frac{1}{2}\ln[1+(-\ln C\cdot s)]} \\
= &\ e^{k\cdot(-\ln C)} = e^{\ln C^{-k}} = C^{-k},
\end{aligned}$$

and that

$$\lim_{t\to\infty}\frac{\frac{1}{2}(\ln\ln\frac{t}{C}+\ln\ln t)}{\ln\ln\frac{t}{C}} = \lim_{t\to\infty}\frac{1+\frac{\ln\ln t}{\ln(\ln t-\ln C)}}{2} = 1.$$

Therefore, given $\varepsilon > 0$, there is a t_0, such that the left-hand side $L(t)$ of (14) satisfies

$$\left(C^{-k}\right)^{\cos\ln\ln\frac{t}{C}-\varepsilon} < L(t) < \left(C^{-k}\right)^{\cos\ln\ln\frac{t}{C}+\varepsilon}$$

for $t \geqslant t_0$. In order that (14) holds, it suffices that

$$(C^{-k})^{\cos\ln\ln\frac{t}{C}-\varepsilon} > 2C^{p+k\sin\ln\ln\frac{t}{C}},$$

i.e.,

$$C^{p+k\sin\ln\ln\frac{t}{C}+\cos\ln\ln\frac{t}{C}-\varepsilon} < 1/2,$$

and hence,

$$C^{p+\sqrt{2}k\sin\left(\ln\ln\frac{t}{C}+\frac{\pi}{4}\right)-\varepsilon} < 1/2.$$

Thus, we obtain a sequence $\{t_n\} \to \infty$ (and hence a corresponding sequence $u_n \to \infty$), such that

$$\ln\ln\frac{t_n}{C}+\frac{\pi}{4} = 2k\pi+\frac{\pi}{2}, \qquad (k \in \mathbb{N}).$$

Then $C^{p+\sqrt{2}k-\varepsilon} < 1/2$, so that

$$C < 2^{-1/(p+\sqrt{2}k-\varepsilon)}. \tag{15}$$

It follows that

$$\mu_M^\infty \geqslant 2^{-1/(p+\sqrt{2}k-\varepsilon)}.$$

(Otherwise, if $\mu_M^\infty < 2^{-1/(p+\sqrt{2}k-\varepsilon)}$, then there exists an $\varepsilon_0 > 0$ such that $\mu_M^\infty+\varepsilon_0 < 2^{-1/(p+\sqrt{2}k-\varepsilon)}$ and $\frac{M^{-1}(u)}{M^{-1}(2u)} < \mu_M^\infty+\varepsilon_0$ for all $u \geqslant u_0$ with some sufficiently large u_0. This contradicts (13) since $\mu_M^\infty+\varepsilon_0$ satisfies (15) and can be taken as some C.) Thus, we have $\mu_M^\infty \geqslant 2^{-1/(p+\sqrt{2}k}$ since ε is arbitrary.

On the other hand, it is clear from Corollary 1 and (11) that

$$\mu_M^\infty \leqslant 2^{-1/q_M^\infty} = 2^{-1/(p+\sqrt{2}k)}.$$

Finally, we obtain that

$$\mu_M^\infty = 2^{-1/(p+\sqrt{2}k)},$$

and analogously,

$$\nu_M^\infty = 2^{-1/(p-\sqrt{2}k)}.$$

There are a few counterexamples showing that a pair of Semenov indices can be indeed different. Example 2 indicates that the gape between ν_M^i and μ_M^i can be any interval in the whole interval $(1/2, 1)$ as soon as we choose properly the parameters p and k.

EXAMPLE 3. Krasnoselskiĭ and Rutickiĭ [**4**, p. 25] give an example that $M(u)$ is determined by its derivative

$$p(u) = \begin{cases} u, & \text{if } u \in [0,1) \\ k!, & \text{if } u \in [(k-1)!, k!) \qquad (k = 2, 3, \cdots) \end{cases}$$

Then $p_M^\infty = 1$, $q_M^\infty = +\infty$, and hence $\nu_M^\infty = 1/2$, $\mu_M^\infty = 1$.

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