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# ON GENERALIZED INEQUALITIES OF HEINZ, HALMOS AND BERNSTEIN

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ABSTRACT. We present generalized inequalities of Heinz, Halmos and Bernstein. It is proved that they are all equivalent to one another. In fact they are all equivalent to the Cauchy–Schwarz inequality. Equality condition and the bound of inequality are established.

In what follows the capital letters mean bounded linear operators on a Hilbert space H. The identity operator is denoted by $I$. $T = U|T|$ is the polar decomposition of $T$ with $U$ the partial isometry, and $|T|$ the positive square root of the positive operator $T^*T$. A basic well-known property about the polar decomposition of $T$ is that equality $|T^*|^r = U|T|^rU^*$ holds for any positive real number $r$, and $U^*U = I$ [2].

Recently several authors have been interested in inequalities of Heinz, Halmos and Bernstein [1,2,4,5,6,7], just mentioning a few. In this article we present generalizations of these inequalities, and prove that they have one thing in common, i.e., they are all equivalent to the Cauchy–Schwarz inequality. We also consider equality condition and the bound of inequality. We shall begin with the next lemma. It is a basic tool characterizing the Cauchy–Schwarz inequality from which we derive our main result.

LEMMA. *For $e, x, y \in H$ with $\|e\| = 1$ the following statements are equivalent.*

*(i) $|(x,y)| \leqslant \|x\|\,\|y\|$ (Cauchy–Schwarz inequality). Equality holds if and only if $x = \alpha y$ for suitable $\alpha$. Moreover,*

$$\|x\|^2\|y\|^2 - |(x,y)|^2 \leqslant \frac{\|x\|^2\|ry-x\|^2}{r^2}$$

*for any real number $r \neq 0$.*

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*(ii) If $(x,e) = 0$, then $|(y,e)|^2\|x\|^2 \leqslant \|x\|^2\|y\|^2 - |(x,y)|^2$. Equality holds if and only if $x = \alpha[y - (y,e)e]$ for suitable $\alpha$. Moreover,*

$$\|x\|^2\|y\|^2 - |(x,y)|^2 - |(y,e)|^2\|x\|^2 \leqslant \frac{\|x\|^2\|ry - r(y,e)e - x\|^2}{r^2}$$

*for any real number $r \neq 0$.*

PROOF. We show the bound in (i) first. Define a function of any real number $r \neq 0$ by $f(r) = \|x\|^2\|ry - x\|^2 - r^2[\|y\|^2\|x\|^2 - |(x,y)|^2]$. Then, by expanding the right-hand side,

$$f(r) = r^2|(x,y)|^2 - 2r\,\mathrm{Re}(x,y)\|x\|^2 + \|x\|^4 \geqslant [r\,\mathrm{Re}(x,y) - \|x\|^2]^2 \geqslant 0,$$

since $\mathrm{Re}\,\beta \leqslant |\beta|$, and we have the required bound in (i).

(i) $\Rightarrow$ (ii). Let $a = y - (y,e)e$. Then $\|a\|^2 = \|y\|^2 - |(y,e)|^2$, and $(x,a) = (x,y)$ as $(x,e) = 0$. Now, all we have to do is substituting above relations into the Cauchy–Schwarz inequality $|(x,a)|^2 \leqslant \|x\|^2\|a\|^2$. Equality condition follows from (i) under this Cauchy–Schwarz inequality, and the bound of the inequality (ii) is obtained from that of (i) by replacing $y$ by $a$ as above.

(ii) $\Rightarrow$ (i). Clearly, $0 \leqslant \|x\|^2\|y\|^2 - |(x,y)|^2$ by (ii). The bound of (i) is due to that of (ii) for a suitable $e$ such that $(y,e) = 0$.

Incidentally, for any nonzero vector $x$, there exists a unit vector $e$ such that $(x,e) = 0$. Indeed, we may take $e = z/\|z\|$, where $z = y - (y,x)x/\|x\|^2$ and $y$ is any nonzero vector. □

Now, we are ready for our main result.

THEOREM. *For $e, x, y \in H$ with $\|e\| = 1$ the following assertions are equivalent.*

(1) *The Cauchy–Schwarz inequality.*

(2) *Generalized Heinz inequality: Let $T = U|T|$ be the polar decomposition of $T$. For $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geqslant 1$, we have*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 \leqslant \|\,|T|^\alpha x\|^2[\|\,|T^*|^\beta y\|^2 - |(|T^*|^\beta y, e)|^2] \text{ if } (U|T|^\alpha x, e) = 0.$$

(3) *Generalized Halmos inequality: If $A$, $B$ and $C$ are operators such that $A$, $A^2B$ and $A^2C$ are all selfadjoint, then $|(ABx, ACy)| \leqslant r(B)r(C)\|Ax\|\,\|Ay\|$, where $r(E)$ denotes the spectral radius of $E$.*

(4) *Generalized Bernstein inequality: If $e_\lambda$ is a unit eigenvector corresponding to an eigenvalue $\lambda$ of an operator $S$, then*

$$|(y,e_\lambda)|^2 \leqslant \frac{\|y\|^2\|S^*x\|^2 - |(y,S^*x)|^2}{\|(S^* - \overline{\lambda}I)x\|^2} \text{ for } S^*x \neq \overline{\lambda}x.$$

PROOF. (1) $\Leftrightarrow$ (2). Replace $x$ and $y$ in (ii) of Lemma by $U|T|^\alpha x$ and $|T^*|^\beta y$, respectively, and use the well-known property of the polar decomposition of $T$ to simplify the inequality. Indeed, $|(U|T|^\alpha x, |T^*|^\beta y)| = |(T|T|^{\alpha+\beta-1}x, y)|$. Also, $\|U|T|^\alpha x\| = \|\,|T|^\alpha x\|$, and (2) follows. Conversely, (1) follows by letting $T = I$ in (2).

(1) $\Leftrightarrow$ (3). Recall that the Cauchy–Schwarz inequality for a positive operator $E$ is $|(Ex,y)|^2 \leqslant (Ex,x)(Ey,y)$, which is known to be equivalent to (i) in Lemma 1. Clearly $A^2 = A^*A$ is positive, and it follows by (1) that

$$\begin{aligned}|(ABx, ACy)|^2 \leqslant (A^2Bx, Bx)(A^2Cy, Cy) &= (B^*A^2Bx, x)(C^*A^2Cy, y)\\ &= (A^2B^2x, x)(A^2C^2y, y).\end{aligned}$$

The last equality is by the selfadjointness of $A^2B$ and $A^2C$. In fact, by induction we have $(B^*)^iA^2B^i = A^2B^{2i}$ and $(C^*)^iA^2C^i = A^2C^{2i}$ for $i = 1, 2, \ldots$. If we continue the process, i.e., square both sides of the above inequality successively with simplification, and proceed by induction. Then we conclude that the power inequality

$$(\natural)\qquad |(ABx, ACy)|^{2^n} \leqslant (A^2B^{2^n}x, x)(A^2x, x)^{2^{n-1}-1}(A^2C^{2^n}y, y)(A^2y, y)^{2^{n-1}-1}$$

holds, which yields

$$|(ABx, ACy)| \leqslant \|A^2\|^{\frac{2}{2^n}}\|B^{2^n}\|^{\frac{1}{2^n}}\|C^{2^n}\|^{\frac{1}{2^n}}\|x\|^{\frac{2}{2^n}}\|y\|^{\frac{2}{2^n}}(A^2x, x)^{\frac{1}{2}-\frac{1}{2^n}}(A^2y, y)^{\frac{1}{2}-\frac{1}{2^n}}.$$

Next, take the $2^n$-th root of both sides in above and pass to the limit as $n \to \infty$, and note that $r(E) = \lim_n \|E^n\|^{1/n}$. Then we arrive at (3). Conversely, (1) follows by letting $A = B = C = I$ in (3).

(1) $\Leftrightarrow$ (4). Notice that $((S^* - \overline{\lambda}I)x, e_\lambda) = (S^*x, e_\lambda) - \overline{\lambda}(x, e_\lambda) = 0$ as $Se_\lambda = \lambda e_\lambda$. It follows that replacing $x$ by $(S^* - \overline{\lambda}I)x$ and $e$ by $e_\lambda$ in (ii) of Lemma yield

$$|(y, e_\lambda)|^2 \leqslant \frac{\|y\|^2\|(S^* - \overline{\lambda}I)x\|^2 - |(y, (S^* - \overline{\lambda}I)x)|^2}{\|(S^* - \overline{\lambda}I)x\|^2} = \frac{\|y\|^2\|S^*x\|^2 - |(y, S^*x)|^2}{\|(S^* - \overline{\lambda}I)x\|^2}.$$

Equality above is due to a known equality

$$\|y - \delta x\|^2\|x\|^2 - |(y - \delta x, x)|^2 = \|x\|^2\|y\|^2 - |(y, x)|^2$$

for any $x, y \in H$ and any scalar $\delta$ [9, Lemma]. Conversely, rewrite inequality (4) as follows.

$$|(y, e_\lambda)|^2 \leqslant \|y\|^2 - \frac{|(y, (S^* - \overline{\lambda}I)x)|^2}{\|(S^* - \overline{\lambda}I)x\|^2} \leqslant \|y\|^2.$$

If $S = I$ in particular, then $\lambda = 1$. So, we may let $e_1 = x/\|x\|$ in above and obtain $|(x, y)| \leqslant \|x\|\,\|y\|$, which proves (1). $\square$

REMARKS. (1) According to [2] the Heinz inequality is the inequality $|(Tx, y)| \leqslant \|\,|T|^\alpha x\|\,\|\,|T^*|^{1-\alpha}y\|$ for any $x, y \in H$ and $\alpha \in [0, 1]$, which is a special case of (2) in Theorem. In fact, $|(T|T|^{\alpha+\beta-1}x, y)|^2 \leqslant \|\,|T|^\alpha x\|^2\|\,|T^*|^\beta y\|^2$ by (2) in Theorem. Thus the Heinz inequality follows by letting $\alpha + \beta = 1$. Independently, the Heinz inequality can be proved by our substituting method as in above, i.e., replace $x$ by $U|T|^\alpha x$ and $y$ by $|T^*|^{1-\alpha}y$ in (i) of Lemma. The equality holds if and only if $U|T|^\alpha x = \delta|T^*|^{1-\alpha}y$ for suitable $\delta$, which is more precise then the one in [2]. Moreover, the bound of the Heinz inequality is

$$\|\,|T|^\alpha x\|^2\|\,|T^*|^{1-\alpha}y\|^2 - |(Tx, y)|^2 \leqslant \frac{\|\,|T|^\alpha x\|^2\|\gamma|T^*|^{1-\alpha}y - U|T|^\alpha x\|^2}{\gamma^2}$$

for any real number $r \neq 0$ by (i) in Lemma.

(2) To prove the inequality (♮) above we may proceed by induction on $n$. Indeed, if $n = 1$, then $|(ABx, ACy)|^2 \leqslant (A^2B^2x, x)(A^2C^2y, y)$ as was shown in above. Now,

$$|(ABx, ACy)|^{2^{n+1}} \leqslant (A^2B^{2^n}x, x)^2(A^2x, x)^{2^n-2}(A^2C^{2^n}y, y)^2(A^2y, y)^{2^n-2}.$$

Simplify the right-hand side of the inequality above, it becomes

$$(A^2B^{2^{n+1}}x, x)(A^2x, x)^{2^n-1}(A^2C^{2^{n+1}}y, y)(A^2y, y)^{2^n-1},$$

and the process is completed.

(3) Recall Reid's inequality [11]: If $A$ is positive and $AB$ is selfadjoint, then $|(ABx, x)| \leqslant \|B\|(Ax, x)$ for every $x \in H$. The inequality was sharpened by Halmos [4, p. 51 and 244] in that he has $r(B)$ instead of $\|B\|$. We shall show that this is a special case of (3) in Theorem. Let in particular $A$ be positive and let $y = x$ and $C = I$ in (3) of Theorem, and then replace $A$ by $A^{1/2}$, the unique square root of positive $A$. Then clearly we obtain the Halmos inequality $|(ABx, x)| \leqslant r(B)(Ax, x)$ for every $x \in H$, where $AB$ is selfadjoint. A different type of generalization of Halmos' inequality can be found in [6].

(4) It was proved in [8, Theorem 1 and Corollary 2] that a generalized Reid's inequality is equivalent to Furuta's inequality [3], and, in particular, Reid's inequality is equivalent to Löwner's inequality, i.e., if $A \geqslant B \geqslant 0$, then $A^{1/2} \geqslant B^{1/2}$ [10].

(5) Let us show that the Bernstein inequality [1] is a special case of (4) in Theorem. Let $y = x$ in (4) of Theorem and if $e_\lambda$ is a unit eigenvector corresponding to an eigenvalue $\lambda$ of a selfadjoint operator $S$ (so that $\lambda$ is a real number), then, from (4) in Theorem,

$$|(x, e_\lambda)|^2 \leqslant \frac{\|x\|^2\|Sx\|^2 - |(x, Sx)|^2}{\|(S - \lambda I)x\|^2} \text{ for } Sx \neq \lambda x,$$

which is precisely the Bernstein inequality. We may find the bound as follows. From the proof of Theorem and the bound in (i) of Lemma we see that

$$|(x, e_\lambda)|^2 \leqslant \frac{\|x\|^2\|(S - \lambda I)x\|^2 - |(x, (S - \lambda I)x)|^2}{\|(S - \lambda I)x\|^2} \leqslant \frac{\|rx - (S - \lambda I)x\|^2}{r^2} = \frac{\|(S - cI)x\|^2}{(\lambda - c)^2},$$

where $c = \lambda + r$ and $\gamma \neq 0$. Equality condition and the bound of inequality also appeared in [1]. Notice that our proof of the generalized Bernstein inequality in Theorem is cosiderably shorter then the proof of the Bernstein inequality in [1].

Now we are ready to consider equality conditions and bounds of (2) and (4) in Theorem.

COROLLARY 1. *Let $x, y \in H$ and $e$ be a unit vector.*

(1) *Under the same assumptions as in Theorem, equality in* (2) *of Theorem holds if and only if $|T|^\alpha x = \delta[|T^*|^\beta y - (|T^*|^\beta y, e)e]$ for suitable $\delta$. Moreover, for any real number $r \neq 0$,*

$$\begin{aligned}\| |T|^\alpha x\|^2 |(|T^*|^\beta y, e)|^2 &\leqslant \| |T|^\alpha x\|^2 \| |T^*|^\beta y\|^2 - |(T|T|^{\alpha+\beta-1}x, y)|^2 \\ &\leqslant \frac{\| |T|^\alpha x\|^2 \|\gamma |T^*|^\beta y - U|T|^\alpha x\|^2}{\gamma^2}.\end{aligned}$$

(2) *Under the same assumptions as in Theorem, equality in* (4) *of Theorem holds if and only if* $(S^* - \overline{\lambda} I)x = \alpha[y - (y, e_\lambda)e_\lambda]$ *for suitable* $\alpha$. *Moreover, for any real number* $r \neq 0$,

$$\begin{aligned}|(y, e_\lambda)|^2 \leqslant \frac{\|y\|^2 \|S^* x\|^2 - |(y, S^* x)|^2}{\|(S^* - \overline{\lambda} I)x\|^2} &= \frac{\|y\|^2 \|(S^* - \overline{\lambda} I)x\|^2 - |(y, (S^* - \overline{\lambda} I)x)|^2}{\|(S^* - \overline{\lambda} I)x\|^2} \\ &\leqslant \frac{\|ry - (S^* - \overline{\lambda} I)x\|^2}{r^2}.\end{aligned}$$

PROOF. (1) Equality condition is due to (ii) in Lemma and replacement of $x$ by $|T|^\alpha x$, and $y$ by $|T^*|^\alpha y$ (this is what we did in the proof of (1) $\Rightarrow$ (2) in Theorem). The bound is due to (i) in Lemma.

(2) Replacing $x$ by $(S^* - \overline{\lambda} I)x$ and $e$ by $e_\lambda$ in (ii) of Lemma (the same replacement in the proof of (1) $\Rightarrow$ (4) in Theorem) to obtain equality condition, and the bound is obtained by (i) in Lemma. □

Finally, let us show a different type of generalization of the Heinz inequality, which involves three vectors instead of two. Notice first that if $x$ in the Cauchy–Schwarz inequality is replaced by the vector $u = 2(z, x)x - \|x\|^2 z$, $z \in H$, then $\|u\| = \|x\|^2 \|z\|$. Indeed,

$$\begin{aligned}\|u\|^2 &= (2(z, x)x - \|x\|^2 z, 2(z, x)x - \|x\|^2 z) \\ &= 4|(z, x)|^2 \|x\|^2 - 2|(z, x)|^2 \|x\|^2 - 2|(z, x)|^2 \|x\|^2 + \|x\|^4 \|z\|^2 = \|x\|^4 \|z\|^2.\end{aligned}$$

It follows that

$$(*) \qquad 2|(z, x)(x, y)| \leqslant \|x\|^2 [\|y\| \, \|z\| + |(z, y)|].$$

The inequality is obviously equivalent to the Cauchy–Schwarz inequality.

COROLLARY 2. *For* $x, y, z \in H$ *and* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \geqslant 1$, *the Cauchy–Schwarz inequality is equivalent to a generalized Heinz inequality*

$$2|(|T|^{2\alpha} z, x)(T|T|^{\alpha+\beta-1}x, y)| \leqslant \| |T|^\alpha x\|^2 [\| |T^*|^\beta y\| \, \| |T|^\alpha z\| + |(T|T|^{\alpha+\beta-1}z, y)|].$$

PROOF. Assume the Cauchy–Schwarz inequality, i.e., the inequality (*). Replacing $x$ by $U|T|^\alpha x$, $y$ by $|T^*|^\beta y$, $z$ by $U|T|^\alpha z$ in (*), and a straightforward simplification yield the required inequality above. Conversely, let $z = x$ and $T = I$ in the inequality above. Then the Cauchy–Schwarz inequality follows easily. □

## References

[1] H. J. Bernstein, *An inequality for selfadjont operators in a Hilbert space*, Proc. Amer. Math. Soc. **100** (1987), 319–321.

[2] T. Furuta, *A simplified proof of Heinz inequality and scrutiny of its equality*, Proc. Amer. Math. Soc. **97** (1986), 751–753.

[3] T. Furuta, *$A \geqslant B \geqslant 0$ assures $(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc. **101** (1987), 85–88.
[4] P. R. Halmos, *Hilbert Space Problem Book*, Van Nostrand, Princeton, 1967.
[5] C.-S. Lin, *Heinz's inequality and Bernstein's inequality*, Proc. Amer. Math. Soc. **125** (1997), 2319–2325.
[6] C.-S. Lin, *On Halmos' sharpening of Reid's inequality*, C. R. Math. Rep. Acad. Sci. Canada **20** (1998), 62–64.
[7] C.-S. Lin, *On inequalities of Heinz and Kato, and Furuta for linear operators*, Math. Japonica **50** (1999), 463–468.
[8] C.-S. Lin, *Inequalities of Reid type and Furuta*, Proc. Amer. Math. Soc. **129** (2001), 855–859.
[9] C.-S. Lin, *Operator versions of inequalities and equalities on a Hilbert space*, Lin. Algebra Appl. **268** (1998), 365–374.
[10] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177–216.
[11] W. T. Reid, *Symmetrizable completely continuous linear transformations in Hilbert space*, Duke Math. J. **18** (1951), 41–56.

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