PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(89) (2004), 3–8

ON SOLUTIONS OF THE BELTRAMI EQUATION. II

Melkana A. Brakalova and James A. Jenkins

ABSTRACT. We study the existence of solutions of the generalized Beltrami equation $f_{\bar{z}} = \mu(z) f_z$, $\|\mu(z)\|_\infty = 1$, in a plane domain Δ, under general conditions that include previously known results.

1. Introduction

Let $\mu(z)$ be a measurable complex valued function. In our previous paper [2] we treated the question of existence and uniqueness of solutions for the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{1}$$

assuming that $|\mu(z)|$ satisfies a subexponential integrability condition. In the present paper we treat the existence problem under general conditions which include previous results.

2. Main results

Let $h(x)$ be a convex, increasing function defined on $[1, \infty)$ such that $h(x) \geqslant C_\lambda x^\lambda$ for any $\lambda > 1$ with $C_\lambda > 0$. From now on we will assume also that

$$\int_1^\infty \frac{1}{t h^{-1}(t)} dt = \infty. \tag{2}$$

MAIN THEOREM. *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\iint_B h\left(\frac{1}{1 - |\mu|}\right) dA < \Phi_B. \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 30C62.

Then there exists an ACL homeomorphism $f(z)$ of Δ into the plane, which satisfies the Beltrami equation a.e., with partials f_z and $f_{\bar{z}}$, locally in L^q, for $0 < q < 2$. The partials are also distributional derivatives. The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$, and has partials g_w and $g_{\bar{w}}$ locally in L^2.

THEOREM A. (the case of the plane) *If Δ is the plane and if, in addition to* (3), *$\mu(z)$ satisfies*

$$\iint_{\{|z|<R\}} \frac{1}{1-|\mu|} dA = O(R^2), \quad R \to \infty.$$

then there exists an ACL homeomorphism f which maps the plane onto itself with all the properties listed in the Main Theorem.

3. Auxiliary Results and an Equivalent Statement

Let $h(x)$ be the function defined in Section 2. Denote by $\theta(x) = \ln(h(x))$ for x greater than some constant $c \geqslant 1$, such that $h(c) > e$. $\theta(x)$ is a positive increasing function in $[\ln h(c), \infty)$. Next we show that the following conditions

$$(4) \qquad \int_{c_1}^{\infty} \frac{dx}{xh^{-1}(x)} = \infty, \qquad\qquad (5) \qquad \int_{c_2}^{\infty} \frac{\theta(x)}{x^2} dx = \infty.$$

hold simultaneously, where c_1 and c_2 are suitable constants. The result can be stated as:

LEMMA 1. *Conditions* (4) *and* (5) *are equivalent.*

Proof. Make a change of variables in $\int_{c_1}^{\infty} \frac{dx}{xh^{-1}(x)}$ by using the substitution $y = \ln(x)$. Then the last integral becomes $\int_{c^*}^{\infty} \frac{dx}{\theta^{-1}(x)}$, where $c^* = \ln c_1$. Since $\frac{1}{\theta^{-1}(x)}$ and $\theta\left(\frac{1}{x}\right)$ are inverses of each other, it follows that $\int_{c^*}^{\infty} \frac{dx}{\theta^{-1}(x)}$ is divergent iff $\int_{0}^{c_*} \theta\left(\frac{1}{x}\right) dx$ is for some suitable constant $c_* < 1$. After another substitution $y = \frac{1}{x}$ we obtain that the divergence of the last integral is equivalent to the divergence of $\int_{c_3}^{\infty} \frac{\theta(x)}{x^2} dx$, where $c_3 = \frac{1}{c_*}$.

From the auxiliary results above follows a statement equivalent to the Main Theorem.

THEOREM B. *Let Δ be a plane domain, $\mu(z)$ a measurable function defined a.e. in Δ, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant Φ_B such that*

$$\iint_B \exp\left(\theta\left(\frac{1}{1-|\mu|}\right)\right) dA < \Phi_B.$$

If

$$\int_1^\infty \frac{\theta(x)}{x^2} dt = \infty,$$

there exists an ACL homeomorphism $f(z)$ of Δ into the plane, which satisfies the Beltrami equation a.e., with partials f_z and $f_{\bar{z}}$, locally in L^q, for $0 < q < 2$. The partials are also distributional derivatives . The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$, and has partials g_w and $g_{\bar{w}}$ locally in L^2.

4. Construction of the solution $f(z)$

Here we assume that $\mu(z)$ satisfies condition (3), with $h(z)$ satisfying (2). In Δ we define μ_n, $n = 1, 2, \ldots$, so that

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leqslant 1 - 1/n \\ 0, & \text{if } |\mu(z)| > 1 - 1/n. \end{cases}$$

From the theory of quasiconformal mappings we know that there exist q.c. mappings f_n, $n = 1, 2, \ldots$, of Δ into the plane with complex dilatations μ_n, $n = 1, 2, \ldots$.

Let z_0 be a fixed point in the plane. For $r_2 > r_1 > 0$ denote by A the circular ring $A = \{z : r_1 < |z - z_0| < r_2\}$, and by $M_n(r_1, r_2)$ the module of its image under f_n.

Proposition 1. *For any point z_0 and circular ring $A = \{r_1 < |z - z_0| < r_2\}$, the module $M_n(r_1, r_2)$ of the image of A under f_n tends uniformly to ∞ as $r_1 \to 0$.*

Proof. The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation μ_n, where $\mu_n = \mu_n(z) = \mu_n(z_0 + re^{i\theta})$, as follows (see [4]):

$$M_n(r_1, r_2) \geqslant \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1 - |\mu_n|^2} d\theta} \frac{dr}{r}.$$

Using this we obtain:

$$M_n(r_1, r_2) \geqslant \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{1}{1 - |\mu|} d\theta} \frac{dr}{r}.$$

For any z_0 in a compact subset T of the plane containing the disc $|z - z_0| < r_2$

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} h\left(\frac{1}{1 - |\mu|}\right) d\theta \frac{dr}{r} \leqslant C,$$

where C depends only on the compact subset T and the choice of r_2.

Now we have

$$r^2 \int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right) d\theta < \frac{2C}{\log \dfrac{r_2}{r_1}}$$

on a set E of logarithmic measure $\dfrac{1}{2}\log\dfrac{r_2}{r_1}$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right) d\theta < \frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}} \text{ on } E.$$

Using the convexity of $h(x)$, we have

$$h\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta\right) < \frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}} \quad \text{on } E$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-|\mu|} d\theta < h^{-1}\left(\frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}}\right) \quad \text{on } E.$$

From the estimates of the module and monotonicity properties of $h(x)$ we have

$$M_n(r_1, r_2) \geqslant \frac{1}{8\pi}\int_{r_1}^{r_2} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log \frac{r_2}{r_1}}\right)} \frac{dr}{r} \geqslant \frac{1}{8\pi}\int_{r_1}^{\sqrt{r_1 r_2}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log \frac{r_2}{r_1}}\right)} \frac{dr}{r}.$$

Now we consider a monotonically decreasing sequence $\{s_k\}_{k=1}^{\infty}$ of positive numbers tending to 0 such that each interval $[s_{k+1}, s_k]$ has the same logarithmic length, where $\dfrac{s_k}{s_{k+1}} = c$. By a ring decomposition we mean a family of rings $r_1^{(j)} < |z - z_0| < r_2^{(j)}$ with $r_2^{(j+1)} \leqslant r_1^{(j)}$ and $r_1^{(j)}$ and $r_2^{(j)} \to 0$ as $j \to \infty$.

We take two ring decompositions with

$$r_1^{(j)} = s_{2j+1}, r_2^{(j)} = s_{2j-1}$$

$$\hat{r}_1^{(j)} = s_{2j+2}, \hat{r}_2^{(j)} = s_{2j}.$$

Now

$$\sum_{j=1}^{\infty} M_n\left(r_1^{(j)}, r_2^{(j)}\right) \geqslant \frac{1}{8\pi}\sum_{j=1}^{\infty}\int_{s_{2j+1}}^{s_{2j}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)} \frac{dr}{r},$$

while

$$\sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi}\sum_{j=1}^{\infty}\int_{s_{2j+2}}^{s_{2j+1}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)} \frac{dr}{r},$$

so

$$\sum_{j=1}^{\infty} M_n\left(r_1^{(j)}, r_2^{(j)}\right) + \sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi}\int_0^{s_1} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)}\frac{dr}{r}.$$

Making the change of variables $t = \dfrac{C}{\pi r^2 \log c}$ this last term becomes equal to $\dfrac{1}{8\pi}\int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)}dt$, with a well defined lower limit. Thus at least one of the ring decompositions has module sum bounded below by $\dfrac{1}{16\pi}\int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)}dt$ and therefore approaches ∞ uniformly with respect to n and z_0. From the superadditivity property of the module it follows that $\lim_{r_1\to 0} M_n(r_1, r_2) = \infty$, uniformly with respect to z_0 and n.

From now on we shalll assume that the quasiconformal mappings $\{f_n(z)\}$, $n = 1, 2\ldots$ have two fixed points a_1 and a_2, with $d = |a_2 - a_1|$. The following proposition was proved in [2]:

Proposition 2. *If* $\lim_{r_1\to 0} M_n(r_1, r_2) = \infty$, *uniformly with respect to* z_0 *and* n, *then the family of quasiconformal mappings* $\{f_n(z)\}$, $n = 1, 2, \ldots$, *is uniformly equicontinuous on each compact subset* T *of* Δ.

Thus from this proposition and the Arzela-Ascoli's theorem follows:

Proposition 3. *For the sequence* $\{f_n(z)\}$ *there exists a subsequence of functions, which converges uniformly to a function* $f(z)$ *on compact subsets.*

We follow the statements in [2] to show the properties of $f(z)$ outlined in the Main Theorem.

5. $f(z)$ is a homeomorphism

In the same manner as in [2], one can prove that:

Proposition 4. *The function* $f(z)$ *constructed in Proposition 3 is a homeomorphism of* Δ *into the plane.*

6. Differentiability properties of $f(z)$

In the same manner as in [2], one can prove that:

Proposition 5. *The function f(z) is ACL.*

Proposition 6. *The partials* f_z *and* $f_{\bar{z}}$ *of* $f(z)$ *are in* L^q *on compact subsets of* Δ *for every* $q < 2$.

Thus $f(z)$ has generalized L^q-derivatives according to the terminology introduced in [3].

7. $f(z)$ satisfies the Beltrami equation

Using the same methods as in [2], one can prove that:

PROPOSITION 7. *The function $f(z)$ satisfies the Beltrami equation.*

8. The inverse function $g(w)$ of $f(z)$

In the same manner as in [2], one can prove that:

PROPOSITION 8. *The function g is ACL and g_w and $g_{\bar{w}}$ are locally in L^2.*

So far we have proved the Main Theorem and Theorem B.

9. The case of mapping the plane onto itself

In the same manner as in [2], one can prove that

PROPOSITION 9. *If*

$$\iint_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2) \qquad \text{as } R \to \infty,$$

then $f_n(z)$ converges uniformly to ∞, as $z \to \infty$.

This proposition and the rest of the results imply Theorem A. This concludes the proofs of the Main Theorem, Theorem A and Theorem B.

References

1. L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, 1966.
2. M. Brakalova and J. A. Jenkins, *On solutions of the Beltrami equation*, J. Anal. Math. **76** (1998), 67–92.
3. O. Lehto and K. Virtanen, *Quasiconformal Mappings in the Plane*, Springer Verlag, 1973.
4. E. Reich and H. Walczak, *On the behavior of quasiconformal mappings at a point*, Trans. Amer. Math. Soc. **117** (1965), 338–351.

INSTITUTE FOR MATHEMATICAL SCIENCES, SUNY, STONY BROOK, NY 11794-3660, USA
melkana@math.sunysb.edu, or

MATHEMATICS DEPARTMENT FORDHAM UNIVERSITY ROSE HILL CAMPUS 441 EAST FORDHAM RD, BRONX, NY 10458
brakalova@fordham.edu

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, CAMPUS BOX 1146, ST. LOUIS, MO 63130-4899, USA (James A. Jenkins)