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THREE-SPACE-PROBLEM FOR INDUCTIVELY (SEMI)-REFLEXIVE LOCALLY CONVEX SPACES

Stojan Radenović and Zoran Kadelburg

Abstract. Three-space-stability of inductively (semi)-reflexive and some related classes of locally convex spaces is considered. It is shown that inductively (semi)-reflexive spaces behave more regularly than (semi)-reflexive spaces in that sense.

Let (E, t) be a Hausdorff locally convex space (l.c.s.) with the topological dual E'; there exist several topologies on E' (the weak topology $\sigma(E', E)$, the topology $\kappa(E', E)$ of uniform convergence on compact and absolutely convex sets, Mackey topology $\tau(E', E)$, the strong topology $b(E', E)$ and others). The so-called inductive topology TE' on E' was introduced in [**3**] and [**5**] as the inductive-limit topology of the Banach spaces E'_{V°, where V runs through a zero-neighborhood basis of (E, t) formed by closed and absolutely convex sets. Here $E'_{V^\circ} = \bigcup_{n\in\mathbb{N}} nV^\circ$ is equipped with the norm having V° as the unit ball. The zero-neighborhood basis of TE' is formed by all absolutely convex subsets of E' that absorb all t-equicontinuous subsets. This topology is the strongest locally convex topology on E' for which all t-equicontinuous subsets are bounded. Particularly, it is finer than the strong topology $b(E', E)$.

Obviously, (E', TE') is an ultrabornological l.c.s. While the weak, Mackey and strong topologies depend only on the dual pair $\langle E, E'\rangle$, topology TE' depends on the topology t. E.g., the topology corresponded in this way to the weak topology $\sigma(E, E')$ is the strongest locally convex topology $\tau(E', E'^*)$, i.e., $T(E, \sigma(E, E'))' = \tau(E', E'^*)$.

It was defined in [**3**], resp. [**5**], resp. [**1**] that an l.c.s. (E, t) is *inductively semi-reflexive* (resp. *b-reflexive*, resp. *with the property HC*) if the topology TE' is compatible with the duality $\langle E, E'\rangle$, i.e., if $TE' = \tau(E', E)$; in other words if $(E', TE')' = E$ (algebraically). If, moreover, $T(TE')' = t$, then (E, t) is called *inductively reflexive*.

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By the previous remark, $(E, \sigma(E, E'))$ is inductively semi-reflexive if and only if E is finite-dimensional.

W. Roelcke and S. Dierolf showed in [**10**, Ex. 1.5] that neither of the properties "being semi-reflexive" and "being reflexive" of l.c.s.'s is three-space-stable, i.e., there exists a non-semi-reflexive space E having a closed subspace F such that both F and E/F are reflexive. We shall prove here that inductively (semi)-reflexive spaces behave more regularly, i.e, that the properties "being inductively semi-reflexive" and "being inductively reflexive" are three-space-stable. This will also be a result better than the one obtained in [**7**, Prop. 3.2].

Terminology that is not defined here explicitly is taken from [**9**].

THEOREM 1. *If the outer terms F and E/F of the short exact sequence*

$$0 \to F \xrightarrow{i} E \xrightarrow{q} E/F \to 0 \tag{1}$$

of l.c.s.'s are inductively semi-reflexive, then the middle term E is inductively semi-reflexive, too.

In order to prove the theorem we state two lemmas which may be of interest on their own.

LEMMA 1. *If F is a closed subspace of an l.c.s. (E, t), then the quotient topology TE'/F° of the topology TE' is equal to the topology TF', i.e., $TE'/F^\circ = TF'$.*

PROOF. First we prove that $TF' \geqslant TE'/F^\circ$. TF' is the strongest locally convex topology on F' such that all $t|F$-equicontinuous subsets of F' are bounded. So, it is enough to prove that all $t|F$-equicontinuous subsets of F' are TE'/F°-bounded. Let $A \subset F'$ be a $t|F$-equicontinuous subset, i.e., $A = i'(B)$, where $B \subset E'$ is t-equicontinuous. Then, A is TE'/F°-bounded since B is TE'-bounded. So, $TF' \geqslant TE'/F^\circ$.

Conversely, let us prove that $TF' \leqslant TE'/F^\circ$. Let W be a TF'-neighborhood of zero, so that W absorbs all $t|F$-equicontinuous subsets of F'. Then $(i')^{-1}(W)$ absorbs all t-equicontinuous subsets of E', and so $(i')^{-1}(W)$ is a TE'-neighborhood of zero. Hence, W is a TE'/F°-neighborhood of zero and so $TF' \leqslant TE'/F^\circ$ is proved. □

Note that the strong topology $b(E', E)$ does not possess the mentioned property of topology TE'.

LEMMA 2. *If F is a closed subspace of an l.c.s. (E, t), then $TE'|F^\circ \leqslant TF^\circ$ on $F^\circ \simeq (E/F)'$.*

PROOF. Let V be a $TE'|F^\circ$-neighborhood of zero. Then there exists a TE'-neighborhood of zero U such that $V \supset U \cap F^\circ$. Since each t/F-equicontinuous subset of F° is also a t-equicontinuous subset of E' (indeed, if $A \subset F^\circ$ is t/F-equicontinuous, then $A \subset (U + F)^\circ \subset U^\circ \cap F^\circ \subset U^\circ$, for some t-neighborhood of zero U), we have that $U \cap F^\circ$ absorbs all t/F-equicontinuous subsets of F°. This means that $V \cap F^\circ$ absorbs all t/F-equicontinuous subsets of F° and so it is a TF°-neighborhood of zero. Thus, $TE'|F^\circ \leqslant TF^\circ$ is proved. □

Note that there exist examples when $TE'|F^\circ < TF^\circ$. E.g., let (E,t) be an ultrabornological space and $(F,t|F)$ its subspace that is not ultrabornological (such examples exist). Then $TE = t$ and $TF > TE|F = t|F$, where TE and TF are the associated utrabornological topologies on E, F, respectively.

Proof of Theorem 1. The following relations among topologies in the space E'/F° are valid:

$$\tau(E'/F^\circ, F) = b(E'/F^\circ, F) = TF' = TE'/F^\circ \geqslant b(E', E)/F^\circ \geqslant b(E'/F^\circ, F).$$

The first and the second equality follow from the inductive semi-reflexivity of the subspace F; the third follows from Lemma 1; the last two inequalities are obvious. In the subspace F° we have:

$$\tau(F^\circ, E/F) = b(F^\circ, E/F) = TF^\circ \geqslant TE'|F^\circ \geqslant \sigma(E', E)|F^\circ = \sigma(F^\circ, E/F)$$

by inductive semi-reflexivity of the quotient E/F and Lemma 2. Hence the following sequence

$$0 \to (F^\circ, TE'|F^\circ) \xrightarrow{q'} (E', TE') \xrightarrow{i'} (F', TF' = TE'/F^\circ) \to 0$$

is exact (both algebraically and topologically). Denote by E_1'' the topological dual of the space (E', TE'). Then the sequence

$$0 \to F \to E_1'' \to E/F \to 0$$

is algebraically exact. It remains to prove the inclusion $E_1'' \subset E$.

Let $x'' \in E_1'' = (E', TE')'$. The restriction $x''|F^\circ$ to the subspace F° is $TE'|F^\circ$-continuous by Lemma 2, hence $x''|F^\circ \in E/F$ (the space E/F is inductively semi-reflexive). So, there exists $x_1 \in E$ such that

$$x''(x') = x'(x_1) \quad \text{for each } x' \in F^\circ.$$

Hence, $x'' - x_1$ is a continuous linear form on the space (E', TE') which vanishes on F°, and so $x'' - x_1 \in U^\circ$ for a TE'-neighborhood of zero U. Further, this means that $x'' - x_1$ is a bounded linear form on $U + F^\circ$ (and so, by Lemma 1, on a TF'-neighborhood of zero in the space (F', TF')). So, there exists $x_2 \in F$ such that $(x'' - x_1)(x') = x_2(x')$ for each $x' \in F'$, i.e., $x'' = x_1 + x_2 \in E + F \subset E + E = E$, which finishes the proof. □

Following [**3**], we shall call an l.c.s. (E,t) *strongly distinguished* if each $\sigma(\cdot, E')$-bounded subset A of $(E', TE')'$ is contained in the $\sigma(\cdot, E')$-closure of a t-bounded subset B of E (here, $\sigma(\cdot, E')$ stands for the weak topology in $(E', TE')'$). Using the associated Schwartz topology, it was proved in [**3**, Prop. 3.2] that the space (E,t) is strongly distinguished if and only if $b(E', E) = TE'$. We give a direct proof.

Proposition 1. *An l.c.s.* (E,t) *is strongly distinguished if and only if* $b(E', E) = TE'$.

Proof. Since the dual space E' with the topology TE' is ultrabornological, and so barrelled, the equality $b(E', E) = TE'$ implies that the space (E,t) is distinguished in the classical (Grothendieck) sense. Hence, the bidual E'' of the space E is equal to the topological dual $(E', TE')'$ of the space (E', TE') and so for each

$\sigma(E'', E')$-bounded subset A of E'' there exists a t-bounded subset B of E such that A is contained in the $\sigma(E'', E')$-closure of B. By the definition, it means that (E, t) is strongly distinguished.

Conversely, let (E, t) be a strongly distinguished space and let V be a closed and absolutely convex TE'-neighborhood of zero. Then the polar V° (corresponding to the duality $\langle E', (E', TE')' = E_1''\rangle$) is a $\sigma(E_1'', E')$-bounded, closed and absolutely convex subsets of E_1''. By the assumption, there exists a t-bounded subset B of E such that A is contained in the weak closure $B^{\circ\circ}$ of B. It follows that $V = V^{\circ\circ} \supset B^\circ$. Hence, V is a neighborhood of zero in the space (E', TE'), and so $b(E', E) = TE'$. □

In the sequel we prove propositions on the three-space-stability of strongly distinguished and inductively reflexive spaces. First we state a dual property of inductively reflexive spaces.

PROPOSITION 2. *Let (E, t) be an l.c.s. and consider the following properties:*

(a) (E, t) *is inductively reflexive (i.e., inductively semi-reflexive and ultrabornological);*
(b) $(E, \tau(E, E'))$ *is inductively reflexive;*
(c) $(E', \tau(E', E))$ *is inductively reflexive.*

Then, (a) *implies* (b) *and* (b) *is equivalent to* (c).

PROOF. Proof can be deduced from the following observations. If an l.c.s. (E, t) is inductively semi-reflexive (with $\sigma(E, E') \leqslant t \leqslant \tau(E, E')$), then $(E', \tau(E', E))$ is an ultrabornological space; conversely, if the space $(E', \tau(E', E))$ is ultrabornological, then $(E, \tau(E, E'))$ is inductively semi-reflexive. Dually, if (E', t') is inductively semi-reflexive (with $\sigma(E', E) \leqslant t' \leqslant \tau(E', E)$), then $(E, \tau(E, E'))$ is ultrabornological; conversely, if $(E, \tau(E, E'))$ is ultrabornological, then $(E', \tau(E', E))$ is inductively semi-reflexive. □

THEOREM 2. *If the quotient map $q : E \to E/F$ lifts bounded sets with closure and if the closed subspace F and the corresponding quotient E/F are strongly distinguished, then the space E has the same property.*

PROOF. Recall that the mapping q is said to lift bounded sets with closure if for each bounded set $B \subset E/F$ there exists a bounded set $A \subset E$ such that $B \subset \overline{q(A)}$. We shall prove that under this assumption the topologies $b(E', E)$ and TE' coincide both on the subspace F° and on the quotient E/F; according to [**6**, Lemma 1] it will follow that they coincide on E', i.e., that the space E is strongly distinguished.

On the space F° we have that:

$$b(F^\circ, E/F) = b(E', E)|F^\circ \leqslant TE'|F^\circ \leqslant TF^\circ = b(F^\circ, E/F).$$

The first equality follows from the assumption about lifting of bounded sets, and last one because the space E/F is strongly distinguished. The first inequality is obvious and the second follows from Lemma 2. Therefore, $b(E', E)|F^\circ = TE'|F^\circ$.

On the space F' we have that:

$$b(F',F) = TF' = TE'/F^\circ \geqslant b(E',E)/F^\circ \geqslant b(F',F),$$

hence $TE'/F^\circ = b(E',E)/F^\circ$. The first equality follows since the space F is strongly distinguished, and the second from Lemma 1. The last two inequalities are clear. □

Since the notions of "distinguished" and "strongly distinguished" spaces coincide for Fréchet spaces, the example from [**4**] shows that the "lifting" condition cannot be omitted in the previous Theorem. In other words, without the lifting assumption the property of "being strongly distinguished" is not three-space-stable.

By an old result from [**8**], "being a reflexive space" is a three-space-stable property in the class of Banach spaces. This is no longer the case for arbitrary locally convex spaces as the mentioned example 1.5 from [**10**] shows. However, for inductively reflexive spaces we have

Theorem 3. *If the outer terms F and E/F of the short exact sequence* (1) *of l.c.s.'s are inductively reflexive, then the middle term E is inductively reflexive, too.*

Proof. According to Theorem 1, the space E is inductively semi-reflexive; it is also barrelled (barrelledness is three-space-stable by [**10**, Th. 2.6]). We have to prove that E is bornological, i.e. ultrabornological since it is complete [**3**, Th. 1.7].

Note that each topology ξ on the dual E' of an l.c.s. (E,t) satisfying $\kappa(E',E) \leqslant \xi \leqslant \tau(E',E)$ gives in E the same topology TE and this topology is not weaker than t. Particularly, the space (E,t) is ultrabornological if and only if $t = TE$.

On the other hand, by [**9**, Lemma 24.21], if (E,t) is a complete l.c.s., then $\kappa(E',E)$ is the finest locally convex topology on E' which coincides with the weak topology $\sigma(E',E)$ on t-equicontinuous subsets of E'. Consequently, $\kappa(E',E)|F^\circ = \kappa(F^\circ, E/F)$.

Consider now the sequence

$$0 \to (F^\circ, \kappa(E',E)|F^\circ) \xrightarrow{q'} (E', \kappa(E',E)) \xrightarrow{i'} (F', \kappa(E',E)/F^\circ) \to 0. \tag{2}$$

By the previous remark, outer terms in the sequence (2) are strongly distinguished, and since the transposed mapping i' lifts bounded sets with closure (can be checked directly), according to Theorem 2 the middle term $(E', \kappa(E',E))$ is strongly distinguished, too. This means that $T(E',\kappa(E',E))' = TE = b(E,E')$ and since the topology TE on E is ultrabornological, we obtain that the space E is inductively reflexive. □

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Mašinski fakultet
11000 Beograd
Serbia
radens@beotel.yu

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Matematički fakultet 11000 Beograd
Serbia
kadelbur@matf.bg.ac.yu