ON THE DISTRIBUTION OF
M-TUPLES OF B-NUMBERS

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ABSTRACT. In the classical sense, the set B consists of all integers which can be written as a sum of two perfect squares. In other words, these are the values attained by norms of integral ideals over the Gaussian field $\mathbb{Q}(i)$. G. J. Rieger (1965) and T. Cochrane, R. E. Dressler (1987) established bounds for the number of pairs $(n, n+h)$, resp., triples $(n, n+1, n+2)$ of B-numbers up to a large real parameter x. The present article generalizes these investigations into two directions: The result obtained deals with arbitrary M-tuples of arithmetic progressions of positive integers, excluding the trivial case that one of them is a constant multiple of one of the others. Furthermore, the estimate applies to the case of an arbitrary normal extension K of the rational field instead of $\mathbb{Q}(i)$.

1. Introduction. Already E. Landau's in his classic monograph [4] provided a proof of the result that the set B of all positive integers which can be written as a sum of two squares of integers is distributed fairly regularly: It satisfies the asymptotic formula

$$(1.1) \qquad \sum_{1 \leqslant n \leqslant x,\ n \in B} 1 \sim \frac{c\,x}{\sqrt{\log x}} \qquad (c > 0).$$

Almost six decades later, G. J. Rieger [9] was the first to deal with the question of "B-twins": How frequently does it happen that both n and $n+1$ belong to the

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set B? A bit more general, he was able to show that, for any positive integer h and large real x,

$$(1.2) \qquad \sum_{\substack{1 \leqslant n \leqslant x \\ n \in B,\ n+h \in B}} 1 \ \ll \ \prod_{\substack{p \mid h \\ p \equiv 3 \ (\text{mod } 4)}} \left(1 + \frac{1}{p} \right) \frac{x}{\log x} \, .$$

Later on, C. Hooley [2] and K.-H. Indlekofer [3], independently and at about the same time, showed that this bound is essentially best possible. In 1987, T. Cochrane and R. E. Dressler [1] extended the question to triples of B-numbers. Replacing Rieger's sieve technique by a more recent variant of Selberg's method, they succeeded in proving that

$$(1.3) \qquad \sum_{\substack{1 \leqslant n \leqslant x \\ n \in B,\ n+1 \in B,\ n+2 \in B}} 1 \ \ll \ \frac{x}{(\log x)^{3/2}} \, .$$

2. Statement of result. In this article we intend to generalize these estimates in two different directions: Firstly, instead of pairs or triples we consider M-tuples of arithmetic progressions $(a_m\, n + b_m)$, $m = 1, \ldots, M \geqslant 2$, where $a_m \in \mathbb{Z}^+$, $b_m \in \mathbb{Z}$ throughout. Secondly, we deal with an arbitrary number field K which is supposed to be a normal extension of the rationals of degree $[K : \mathbb{Q}] = N \geqslant 2$. Denoting by $\mathcal{O}_K$ the ring of algebraic integers in K, we put

$$\boldsymbol{b}_K(n) := \begin{cases} 1 & \text{if there exists an integral ideal } \mathfrak{A} \text{ in } \mathcal{O}_K \text{ of norm } \mathcal{N}(\mathfrak{A}) = n, \\ 0 & \text{else.} \end{cases}$$

Our target is then the estimation of the sum

$$(2.1) \qquad S(x) = S(a_1, b_1, \ldots, a_M, b_M;\, x) := \sum_{1 \leqslant n \leqslant x} \prod_{m=1}^{M} \boldsymbol{b}_K(a_m\, n + b_m) \, .$$

Of course, the classic case reported in section 1 is contained in this, by the special choice $K = \mathbb{Q}(i)$, the Gaussian field.

THEOREM. *Suppose that* $(a_m, b_m) \in \mathbb{Z}^+ \times \mathbb{Z}$ *for* $m = 1, \ldots, M$, *and, furthermore,*

$$\prod_{\substack{m,k=1 \\ m \neq k}}^{M} (a_m b_k - a_k b_m) \ \neq \ 0 \, .$$

Then, for large real x,

$$S(a_1, b_1, \ldots, a_M, b_M;\, x) \ll \gamma(a_1, b_1, \ldots, a_M, b_M) \, \frac{x}{(\log x)^{M(1-1/N)}} \, ,$$

with

$$\gamma(a_1, b_1, \ldots, a_M, b_M) = \prod_{p \in \mathbb{P}'} \left(1 + \frac{M}{p} \right) \, ,$$

the finite set of primes $\mathbb{P}' = \mathbb{P}'(a_1, b_1, \ldots, a_M, b_M)$ *to be defined below in* (4.6). *The* $\ll$*-constant depends on* M *and the field* K, *but not on* $a_1, b_1, \ldots, a_M, b_M$.

3. Some auxiliary results. *Notation.* Variables of summation automatically range over all integers satisfying the conditions indicated. p denotes rational primes throughout, and $\mathbb{P}$ is the set of all rational primes. $\mathfrak{P}$ stands for prime ideals in $\mathcal{O}_K$. For any subset $\mathbb{P}^\circ \subseteq \mathbb{P}$, we denote by $\mathcal{D}(\mathbb{P}^\circ)$ the set of all positive integers whose prime divisors all belong to $\mathbb{P}^\circ$. The constants implied in the symbols $O(\cdot)$, $\ll$, $\gg$, etc., may depend throughout on the field K and on M, but not on $a_1, b_1, \ldots, a_M, b_M$.

LEMMA 1. *For each prime power p^α, $\alpha \geqslant 1$, let $\overline{\Omega}(p^\alpha)$ be a set of distinct residue classes $\overline{\mathfrak{c}}$ modulo p^α. Define further*

$$\Omega(p^\alpha) = \left\{ n \in \mathbb{Z}^+ : n \in \bigcup_{\overline{\mathfrak{c}} \in \overline{\Omega}(p^\alpha)} \overline{\mathfrak{c}} \right\},$$

and let

$$\theta(p^\alpha) := 1 - \sum_{j=1}^{\alpha} \frac{\#\overline{\Omega}(p^j)}{p^j} > 0, \qquad \theta(1) := 1.$$

Suppose that $\Omega(p^\alpha) \cap \Omega(p^{\alpha'}) = \emptyset$ for all primes p and positive integers $\alpha \neq \alpha'$. For real $x > 0$, let finally

$$A(x) = \left\{ n \in \mathbb{Z}^+ : n \leqslant x \ \& \ n \notin \bigcup_{p \in \mathbb{P}, \, \alpha \in \mathbb{Z}^+} \Omega(p^\alpha) \right\}.$$

Then, for arbitrary real $Y > 1$,

$$\#A(x) \leqslant \frac{x + Y^2}{V_Y},$$

where

$$V_Y := \sum_{0 < d < Y} \ \prod_{p^\alpha \, \| \, d} \left(\frac{1}{\theta(p^\alpha)} - \frac{1}{\theta(p^{\alpha-1})} \right).$$

Proof. This is a deep sieve theorem due to A. Selberg [10]. It can be found in Y. Motohashi [5, p. 11], and also in T. Cochrane and R. E. Dressler [1].

LEMMA 2. *Let $(c_n)_{n \in \mathbb{Z}^+}$ be a sequence of nonnegative reals, and suppose that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} c_n \, n^{-s}$$

converges for $\mathrm{Re}(s) > 1$. Assume further that, for some real constants A and $\beta > 0$,

$$f(s) = (A + o(1))(s-1)^{-\beta},$$

as $s \to 1+$. Then, for $x \to \infty$,

$$\sum_{1 \leqslant n \leqslant x} \frac{c_n}{n} = \left(\frac{A}{\Gamma(1+\beta)} + o(1) \right) (\log x)^\beta.$$

Proof. This is a standard Tauberian theorem. For the present formulation, cf. Cochrane and Dressler [1, Lemma B].

4. Proof of the Theorem. We recall the decomposition laws in a normal extension K over $\mathbb{Q}$ of degree $N \geqslant 2$ (cf. W. Narkiewicz [6, Theorem 7.10]): Every rational prime p which does not divide the field discriminant $\mathrm{disc}(K)$ belongs to one of the classes

$$\mathbb{P}_r = \left\{ p \in \mathbb{P} : (p) = \mathfrak{P}_1 \cdots \mathfrak{P}_{N/r}, \; \mathcal{N}(\mathfrak{P}_1) = \cdots = \mathcal{N}(\mathfrak{P}_{N/r}) = p^r \right\},$$

where r ranges over the divisors of N, and $\mathfrak{P}_1, \ldots, \mathfrak{P}_{N/r}$ are distinct. As an easy consequence, if $p \in \mathbb{P}_r$, $\alpha \in \mathbb{Z}^+$,

$$(4.1) \qquad\qquad \boldsymbol{b}_K(p^\alpha) = \begin{cases} 1, & \text{if } r \mid \alpha, \\ 0, & \text{else.} \end{cases}$$

In order to apply Lemma 1, we need a bit of preparation. Let

$$\mathbb{P}_r^* = \left\{ p \in \mathbb{P}_r : p \nmid \prod_{m=1}^{M} a_m \prod_{\substack{m,k=1 \\ m \neq k}}^{M} (a_m b_k - a_k b_m), \; p \neq M - 1 \right\},$$

$$\mathbb{P}^* = \bigcup_{r \mid N, \, r > 1} \mathbb{P}_r^*.$$

Then we choose

$$\overline{\Omega}(p^\alpha) := \bigcup_{m=1}^{M} \left\{ \overline{a_m}^{\,(-1)} \overline{(jp^{\alpha-1} - b_m)} : j = 1, \ldots, p-1 \right\},$$

if $p \in \mathbb{P}_r^*$ and $r \nmid (\alpha - 1)$, while $\overline{\Omega}(p^\alpha) := \emptyset$ in all other cases. Here $\overline{\cdot}$ denotes residue classes modulo p^α, in particular $\overline{a_m}^{\,(-1)}$ is the class which satisfies $\overline{a_m}\,\overline{a_m}^{\,(-1)} = \overline{1}$ mod p^α. We summarize the relevant properties of these sets $\overline{\Omega}(p^\alpha)$, and of the corresponding sets $\Omega(p^\alpha)$ (see Lemma 1), as follows.

PROPOSITION. *Suppose throughout that $p \in \mathbb{P}^*$ and $\alpha \in \mathbb{Z}^+$.*
(i) *If $p \in \mathbb{P}_r^*$, $r \nmid (\alpha - 1)$, then $\overline{\Omega}(p^\alpha)$ contains exactly $M(p-1)$ elements.*
(ii) *If a positive integer k lies in some $\Omega(p^\alpha)$, it follows that there exists an $m \in \{1, \ldots, M\}$ such that $p^{\alpha-1} \parallel (a_m k + b_m)$.*
(iii) *It is impossible that there exist $m, n \in \{1, \ldots, M\}$, $m \neq n$, and a positive integer k, such that some $p \in \mathbb{P}^*$ divides both $a_m k + b_m$ and $a_n k + b_n$.*
(iv) *If $k \in \Omega(p^\alpha)$, it follows that*

$$p^{\alpha-1} \parallel \prod_{m=1}^{M} (a_m k + b_m).$$

Consequently, $\Omega(p^\alpha) \cap \Omega(p^{\alpha'}) = \emptyset$ for any positive integers $\alpha \neq \alpha'$.

(v) *If $k \in \Omega(p^\alpha)$, then*

$$\prod_{m=1}^{M} \boldsymbol{b}_K(a_m\, k + b_m) = 0\,.$$

As a consequence,

$$S(x) \leqslant \#A(x)\,,$$

where $S(x)$ and $A(x)$ have been defined in (2.1) and Lemma 1, respectively.

Proof of the Proposition. (i) Assume that two of these residue classes would be equal, say, $\overline{a_m}^{(-1)}\overline{(u\,p^{\alpha-1} - b_m)}$ and $\overline{a_n}^{(-1)}\overline{(v\,p^{\alpha-1} - b_n)}$, where $u, v \in \{1, \ldots, p-1\}$, $m, n \in \{1, \ldots, M\}$. Multiplying by $\overline{a_m}\,\overline{a_n}$, we could conclude that

$$a_n(u\,p^{\alpha-1} - b_m) \equiv a_m(v\,p^{\alpha-1} - b_n) \qquad \mathrm{mod}\ \ p^\alpha\,,$$

or, equivalently, that

(4.2) $$(a_n\,u - a_m\,v)p^{\alpha-1} \equiv a_n\,b_m - a_m\,b_n \qquad \mathrm{mod}\ \ p^\alpha\,.$$

Hence $p\,|\,(a_n\,b_m - a_m\,b_n)$, which is only possible if $m = n$. This in turn simplifies (4.2) to

$$a_m(u - v)p^{\alpha-1} \equiv 0 \qquad \mathrm{mod}\ \ p^\alpha\,,$$

thus also $u = v$. $\qquad\qquad\square$

(ii) If $k \in \Omega(p^\alpha)$, there exist $j \in \{1, \ldots, p-1\}$, $m \in \{1, \ldots, M\}$, and an integer q, such that

$$a_m\,k = j\,p^{\alpha-1} - b_m + q\,p^\alpha\,.$$

From this the assertion is obvious. $\qquad\qquad\square$

(iii) Assuming the contrary, we would infer that p divides

$$(a_m\,k + b_m)b_n - (a_n\,k + b_n)b_m = (a_m\,b_n - a_n\,b_m)k\,,$$

hence $p\,|\,k$, thus p divides also b_m and b_n, which contradicts $p \in \mathbb{P}^*$. $\qquad\qquad\square$

(iv) This is immediate from (ii) and (iii). $\qquad\qquad\square$

(v) By (ii), $p^{\alpha-1} \,\|\, (a_m\,k + b_m)$ for some $m \in \{1, \ldots, M\}$. Recalling that $r \nmid (\alpha-1)$ (otherwise $\Omega(p^\alpha)$ would be empty), along with (4.1) and the multiplicativity of $\boldsymbol{b}_K(\cdot)$, it is clear that $\boldsymbol{b}_K(a_m k + b_m) = 0$. The last inequality is obvious from the relevant definitions. $\qquad\qquad\square$

We are now ready to apply Lemma 1. Choosing $Y = \sqrt{x}$ and appealing to part (v) of the Proposition, we see that

(4.3) $$S(x) \leqslant \frac{2x}{V_Y}\,.$$

To derive a lower bound for V_Y, observe that $\overline{\Omega}(p) = \emptyset$ for every prime p, and $\#\overline{\Omega}(p^2) = M(p-1)$ for each $p \in \mathbb{P}^*$. Further, if $p \notin \mathbb{P}^*$, then $\overline{\Omega}(p^j) = \emptyset$ throughout. Therefore, if $p \in \mathbb{P}^*$, then $\theta(p) = 1$ and

$$\theta(p^2) = 1 - \frac{1}{p^2} \#\overline{\Omega}(p^2) = 1 - \frac{M(p-1)}{p^2},$$

hence

(4.4) $$\frac{1}{\theta(p^2)} - \frac{1}{\theta(p)} = \frac{M(p-1)}{p^2 - M(p-1)} \geqslant \frac{M}{p}.$$

Furthermore, for any $\alpha > 2$,

$$\theta(p^\alpha) \geqslant 1 - M(p-1) \sum_{2 \leqslant j \leqslant \alpha} p^{-j} > 1 - \frac{M}{p} \geqslant 0,$$

since $M(p-1) \leqslant p^2 - 1$ according to clause (i) of the Proposition, and $M = p+1$ is impossible for $p \in \mathbb{P}^*$. Thus actually $\theta(p^\alpha) > 0$ for all primes p and all $\alpha \in \mathbb{Z}^+$. Thus all the terms in the sum V_Y are nonnegative, and restricting the summation to the set

$$Q := \left\{ d = d_1^2 : \ d_1 \in \mathbb{Z}^+, \ \mu(d_1) \neq 0, \ d_1 \in \mathcal{D}(\mathbb{P}^*) \right\},$$

we conclude by (4.4) that

(4.5)
$$V_Y \geqslant \sum_{0 < d < Y, \, d \in Q} \prod_{p \mid d} \left(\frac{1}{\theta(p^2)} - \frac{1}{\theta(p)} \right) \geqslant \sum_{0 < d_1 < \sqrt{Y}, \, d_1 \in \mathcal{D}(\mathbb{P}^*)} \mu^2(d_1) \prod_{p \mid d_1} \frac{M}{p} =$$
$$= \sum_{0 < d_1 < \sqrt{Y}, \, d_1 \in \mathcal{D}(\mathbb{P}^*)} \mu^2(d_1) \frac{M^{\omega(d_1)}}{d_1},$$

where $\omega(d_1)$ denotes the number of primes dividing d_1. Our next step is to take care of the primes excluded in the construction of $\mathbb{P}^*$. We define
(4.6)
$$\mathbb{P}' = \mathbb{P}'(a_1, b_1, \ldots, a_M, b_M) := \left\{ p \in \bigcup_{\substack{r \mid N \\ r > 1}} \mathbb{P}_r : p \mid \prod_{m=1}^{M} a_m \prod_{\substack{m, k = 1 \\ m \neq k}}^{M} (a_m b_k - a_k b_m) \right\}$$

and

(4.7) $$\gamma = \gamma(a_1, b_1, \ldots, a_M, b_M) := \prod_{p \in \mathbb{P}'} \left(1 + \frac{M}{p} \right) = \sum_{k_1 \in \mathcal{D}(\mathbb{P}')} \mu^2(k_1) \frac{M^{\omega(k_1)}}{k_1}.$$

Putting finally

$$\mathbb{P}_\cup := \bigcup_{r \mid N, \, r > 1} \mathbb{P}_r,$$

we readily infer from (4.5) and (4.7) that

$$(4.8) \qquad \gamma V_Y \gg \sum_{k < \sqrt{Y},\, k \in \mathcal{D}(\mathbb{P}_\cup)} \mu^2(k) \frac{M^{\omega(k)}}{k}\,.$$

We shall estimate this latter sum by the corresponding generating function

$$f(s) := \sum_{k \in \mathcal{D}(\mathbb{P}_\cup)} \mu^2(k) \frac{M^{\omega(k)}}{k^s} = \prod_{p \in \mathbb{P}_\cup} \left(1 + \frac{M}{p^s}\right) \qquad (\mathrm{Re}(s) > 1)\,,$$

applying Lemma 2. By $h_1(s), h_2(s), \ldots$ we will denote functions which are holomorphic and bounded, both from above and away from zero, in every half-plane $\mathrm{Re}(s) \geqslant \sigma_0 > 1/2$. We first observe that

$$(4.9) \qquad f(s) = h_1(s) \prod_{p \in \mathbb{P}_\cup} \left(1 - p^{-s}\right)^{-M} \qquad (\mathrm{Re}(s) > 1)\,.$$

This follows by a standard argument which can be found exposed neatly in G. Tenenbaum [11, p. 200]. The next step is to consider the Euler product of the Dedekind zeta-function $\zeta_K(s)$: For $\mathrm{Re}(s) > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{P}} \left(1 - \mathcal{N}(\mathfrak{P})^{-s}\right)^{-1} = h_2(s) \prod_{r \mid N} \left(\prod_{p \in \mathbb{P}_r} \left(1 - p^{-rs}\right)^{-N/r}\right) =$$

$$= h_3(s) \prod_{p \in \mathbb{P}_1} \left(1 - p^{-s}\right)^{-N}\,.$$

Therefore,

$$\frac{\left(\zeta(s)\right)^M}{\left(\zeta_K(s)\right)^{M/N}} = h_4(s) \prod_{p \in \mathbb{P}_\cup} \left(1 - p^{-s}\right)^{-M} \qquad (\mathrm{Re}(s) > 1)\,.$$

Comparing this with (4.9), we arrive at

$$f(s) = h_5(s) \frac{\left(\zeta(s)\right)^M}{\left(\zeta_K(s)\right)^{M/N}}\,.$$

From this it is evident that, as $s \to 1+$,

$$f(s) \sim h_5(1)\, \rho_K^{-M/N}\, (s-1)^{-M+M/N}\,,$$

where ρ_K denotes the residue of $\zeta_K(s)$ at $s = 1$. Lemma 2 now immediately implies that

$$\sum_{k < \sqrt{Y},\, k \in \mathcal{D}(\mathbb{P}_\cup)} \mu^2(k) \frac{M^{\omega(k)}}{k} \gg (\log Y)^{M-M/N} \gg (\log x)^{M-M/N}\,,$$

in view of our earlier choice $Y = \sqrt{x}$. Combing this with (4.3) and (4.8), we complete the proof of our Theorem. $\qquad\qquad\square$

5. Concluding remarks. 1. Taking more care and imposing special conditions on the numbers $a_1, b_1, \ldots, a_M, b_M$, one could improve slightly on the factor γ in our estimate. (Observe that Rieger's bound (1.2) is in fact a bit sharper than our general result.) But it is easy to see that γ is rather small anyway: By elementary facts about the Euler totient function (see K. Prachar [8, p. 24–28]),

$$\gamma(a_1, b_1, \ldots, a_M, b_M) \ll \prod_{p \in \mathbb{P}'} \left(1 - \frac{1}{p}\right)^{-M} \ll (\log \log x)^M,$$

under the very mild restriction that, for some constant $c > 0$,

$$\max_{m=1,\ldots,M} \big(a_m, |b_m|\big) \ll \exp((\log x)^c).$$

2. As far as the asymptotics (1.1) is concerned, the generalization to an arbitrary normal extension K of $\mathbb{Q}$ can be found in W. Narkiewicz' monograph [6, p. 361, Prop. 7.11], where it is attributed to E. Wirsing. For this question, the case of non-normal extensions K has been dealt with by R. W. K. Odoni [7]. It may be interesting to extend our present problem to the non-normal case as well. We might return to this at a later occasion.

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