PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 81(95) (2007), 95–102 DOI 102298/PIM0795095R

ON REGULAR ANTI-CONGRUENCE IN ANTI-ORDERED SEMIGROUPS

Daniel Abraham Romano

Abstract. For an anti-congruence q we say that it is regular anti-congruence on semigroup $(S, =, \neq, \cdot, \alpha)$ ordered under anti-order α if there exists an anti-order θ on S/q such that the natural epimorphism is a reverse isotone homomorphism of semigroups. Anti-congruence q is regular if there exists a quasi-antiorder σ on S under α such that $q = \sigma \cup \sigma^{-1}$. Besides, for regular anti-congruence q on S, a construction of the maximal quasi-antiorder relation under α with respect to q is shown.

1. Introduction and preliminaries

This short investigation in Bishop's Constructive Algebra is a continuation of [9] and [10]. Bishop's Constructive Mathematics is developed on Constructive Logic [11] – logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in the Constructive Logic the 'Double Negation Law' $P \Leftrightarrow \neg\neg P$ does not hold, but the following implication $P \Rightarrow \neg\neg P$ holds even in the Minimal Logic. We have to note that 'the crazy axiom' $\neg P \Rightarrow (P \Rightarrow Q)$ is included in the Constructive Logic. In the Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting, that in the Constructive Logic the following deduction principle $A \vee B, A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. Bishop's Constructive Mathematics is consistent with the Classical Mathematics.

Relational structure $(S, =, \neq)$, where the relation $\neq$ is a binary relation on S, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Rightarrow y \neq x,\ x \neq z \Rightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Rightarrow x \neq z$$

we call *set*. Following Heyting, the relation $\neq$ is called *apartness*. A relation q on S is a *coequality relation* on S if and only if it is consistent, symmetric and cotransitive [6]–[8]: $q \subseteq \neq$, $q^{-1} = q$, $q \subseteq q * q$, where $*$ is the *filled product* between relations [5]. Let β be a consistent relation on S. We put ${}^{1}\beta = \beta$ and $({}^{n}\beta) = \beta * \cdots * \beta$

2000 *Mathematics Subject Classification*: Primary 03F65; Secondary 06F05, 20M10.

Key words and phrases: Constructive mathematics, semigroup with apartness, anti-ordered semigroup, anti-congruence, regular anti-congruence, quasi-antiorder.

Supported by the Ministry of sciences and technology of the Republic of Srpska, Bosnia and Herzegovina.

(n factors, $n \in \mathbf{N}$). Then the relation $c(\beta) = \bigcap_{n\in N}({}^n\beta)$, the *cotransitive fulfillment of* β, is the maximal consistent and cotransitive relation on S under β [5].

Let $(S, =, \neq, \cdot)$ be a semigroup with an apartness [6]–[8]. As in [8], a coequality relation q on S is *anti-congruence* if and only if it is compatible with the semigroup operation in the following sense

$$(\forall x, y, z \in S)\big(\big((xz, yz) \in q \Rightarrow (x, y) \in q\big) \wedge \big((zx, zy) \in q \Rightarrow (x, y) \in q\big)\big).$$

A relation α on S is an *anti-order* [9] on S if and only if $\alpha \subseteq \neq$, $\alpha \subseteq \alpha * \alpha$, $\neq \subseteq \alpha \cup \alpha^{-1}$ (linearity) and

$$(\forall x, y, z \in S)\big(\big((xz, yz) \in \alpha \Rightarrow (x, y) \in \alpha\big) \wedge \big((zx, zy) \in \alpha \Rightarrow (x, y) \in \alpha\big)\big).$$

A relation τ on S is a *quasi-antiorder* [9] on S if $\tau \subseteq \neq$, $\tau \subseteq \tau * \tau$ and

$$(\forall x, y, z \in S)\big(\big((xz, yz) \in \tau \Rightarrow (x, y) \in \tau\big) \wedge \big((zx, zy) \in \tau \Rightarrow (x, y) \in \tau\big)\big).$$

Let x be an element of S and A a subset of S. We denote $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in S : x \bowtie A\}$. If τ is a quasi-antiorder on S, then the relation $q = \tau \cup \tau^{-1}$ is an anti-congruence on S. Firstly, the relation $q^C = \{(x, y) \in S \times S : (x, y) \bowtie q\}$ is a congruence on S compatible with q, in the following sense

$$(\forall a, b, c \in S)\big((a, b) \in q^C \wedge (b, c) \in q \Rightarrow (a, c) \in q\big).$$

We can construct the semigroup $S/(q^C, q) = \{aq^C : a \in S\}$ with

$$aq^C = bq^C \Leftrightarrow (a, b) \bowtie q, \quad aq^C \neq bq^C \Leftrightarrow (a, b) \in q, \quad aq^C \cdot bq^C = (ab)q^C.$$

We can also construct the semigroup $S/q = \{aq : a \in S\}$ with

$$aq = bq \Leftrightarrow (a, b) \bowtie q, \quad aq \neq bq \Leftrightarrow (a, b) \in q, \quad aq \cdot bq = (ab)q.$$

It is easy to check that $S/(q^C, q) \cong S/q$. The mapping $\pi : S \to S/q$, defined by $\pi(a) = aq$ for any $a \in S$, is a strongly extensional epimorphism. Secondly, note that the relation α^C is an order relation on set $(S, \neg \neq, \neq)$. If the relation $\neg\alpha$ is an order relation on set $(S, =, \neq)$, then, as for example in [1] when the apartness is tight, $\neg \neq \subseteq =$ [7], the relation α is called *excise relation* on S. (The notion of anti-order relation is more general than notion of excise relation.)

For a given anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$ it is essential to know if there exists an anti-congruence q on S such that S/q be an anti-ordered semigroup. This plays an important role for studying the structure of anti-ordered semigroups. The following question is natural: If $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and q an anti-congruence on S, is then the set S/q an anti-ordered semigroup? A possible anti-order on S/q could be the relation Θ on S/q defined by the anti-order α on S, $\Theta = \{(xq, yq) \in S/q \times S/q : (x, y) \in \alpha\}$. But is not an anti-order, in general. The following question arises: Is there any anti-congruence q on S for which S/q is an anti-ordered semigroup? The concept of quasi-antiorder relation was defined in [9]. According to [9], if $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and σ a quasi-antiorder on S, then the relation q on S, defined by $q = \sigma \cup \sigma^{-1}$, is an anti-congruence on S and the set S/q is an anti-ordered semigroup under anti-order Θ defined by $(xq, yq) \in \Theta \Leftrightarrow (x, y) \in \sigma$. So, according to the results in

[9], each quasi-antiorder σ on an ordered semigroup S under anti-order α induces an anti-congruence $q = \sigma \cup \sigma^{-1}$ on S such that S/q is an ordered semigroup under anti-order Θ. In [10] we proved that the converse of this statement also holds. If $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and q anti-congruence on S and if there exists an anti-order relation Θ_1 on S/q such that $(S/q, =_1, \neq_1, \circ, \Theta_1)$ is an ordered semigroup under anti-order Θ_1, then there exists a quasi-antiorder τ on S such that $q = \tau \cup \tau^{-1}$ and $\Theta_1 = \Theta$. So, each anti-congruence q on a semigroup $(S, =, \neq, \cdot, \alpha)$ such that S/q is an anti-ordered semigroup induces a quasi-antiorder on S. This was the motivation for introduction of a new notion. For that we need the following notion: Let f be a strongly extensional mapping of anti-ordered sets from $(X, =, \neq, \alpha)$ into $(Y, =, \neq, \beta)$. For f we say that it is *reverse isotone* if

$$(\forall a, b \in X)\big((f(a), f(b)) \in \beta \Rightarrow (a, b) \in \alpha\big)$$

holds. An anti-congruence q on S is called *regular* if there is an anti-order "θ_1" on S/q satisfying the following conditions:

(1) $(S/q, =_1, \neq_1, , \theta_1)$ is an anti-ordered semigroup;
(2) The mapping $\pi : S \ni a \mapsto aq \in S/q$ is an anti-order reverse isotone epimorphism.

We call the anti-order "θ_1" on S/q a *regular anti-order* with respect to a regular anti-congruence q on S and the anti-order α.

It is obvious that the regular anti-order on S/q with respect to a regular anti-congruence q and to the anti-order α on S is in general not unique. The following questions now naturally arise: Does there exist the maximal regular anti-order on S/q with respect to a regular anti-congruence q on S? Are all anti-congruences on anti-ordered semigroups regular? In this note, we give a partial answer on the questions above. In Theorem 1 and Corollary 2 we give necessary and sufficient conditions for anti-congruence on an anti-ordered semigroup to be regular. In Theorem 3 we give a construction of the maximal quasi-antiorder on anti-ordered semigroup S induced by a regular anti-congruence q on S.

For the necessary undefined notions, the reader is referred to books [2]–[4], [11] and to papers [5]–[8].

LEMMA 0. *Let τ be a quasi-antiorder on set $(S, =, \neq)$. Then $x\tau$ (τx) is a strongly extensional subset of S, such that $x \bowtie x\tau$ $(x \bowtie \tau x)$, for each $x \in S$. Also, the implication $(x, z) \in \tau \Rightarrow x\tau \cup \tau z = S$ holds for each x, z of S.*

PROOF. From $\tau \subseteq \neq$ it follows $x \bowtie x\tau$. Let $y \in x\tau$ and let z be an arbitrary element of S. Then, $(x, y) \in \tau$ and $(x, z) \in \tau \vee (z, y) \in \tau$. So, we have $z \in x\tau \vee y \neq z$. Therefore, $x\tau$ is a strongly extensional subset of S such that $x \bowtie x\tau$.

The proof that τx is a strongly extensional subset of S such that $x \bowtie \tau x$ is analogous. Besides, the following implication $(x, z) \in \tau \Rightarrow x\tau \cup \tau z = S$ holds for each x, z of S. Indeed, if $(x, z) \in \tau$ and y is an arbitrary element of S, then $(\forall y \in S)\big((x, y) \in \tau \vee (y, z) \in \tau\big)$. Thus, $S = x\tau \cup \tau z$. □

2. Regular anti-congruences

In order to obtain the relationship between regular anti-congruence and quasi-antiorder on S, the following theorem is essential.

THEOREM 1. *Let $(S,=,\neq,\cdot,\alpha)$ be an anti-ordered semigroup, q an anti-congruence on S. The following statements are equivalent:*

(1) *q is regular.*
(2) *There exists a quasi-antiorder σ on S, such that $q=\sigma\cup\sigma^{-1}$.*

PROOF. (2) $\Rightarrow$ (1). By Lemma 1 in [9], since $q=\sigma\cup\sigma^{-1}$, we have that the quotient semigroup $(S/q,=_1,\neq_1,\cdot)$ is an anti-ordered semigroup with respect to the anti-order θ defined as $(qx,qy)\in\theta\Leftrightarrow(x,y)\in\sigma$. If $x,y\in S$ and $(qx,qy)\in\theta$, then $(x,y)\in\sigma\subseteq\alpha$. By definition, q is regular.

(1) $\Rightarrow$ (2). Let q be a regular anti-congruence. Then there exists an anti-order relation θ on the quotient semigroup $(S/q,=_1,\neq_1,\cdot)$ such that $(S/q,=_1,\neq_1,\cdot,\theta)$ is an anti-ordered semigroup, and $\pi:S\to S/q$ is a strongly extensional reverse isotone homomorphism of anti-ordered semigroups. Let $\sigma=\{(x,y)\in S\times S:(qx,qy)\in\theta\}$. By [10], σ is a quasi-antiorder on S and it is easy to check that $q=\sigma\cup\sigma^{-1}$. $\square$

COROLLARY 2. *Let $(S,=,\neq,\cdot,\alpha)$ be an anti-ordered semigroup, q an anti-congruence on S. The following statements are equivalent:*

(1) *q is regular;*
(2) *There exists an anti-ordered semigroup $(T,=,\neq,\cdot,\theta)$ and a strongly extensional reverse isotone homomorphism $\varphi:S\to T$ such that $q=\{(a,b)\in S\times S:\varphi(a)\neq\varphi(b)\}$.*

PROOF. (1) $\Rightarrow$ (2). Let q be regular. Then there exists an anti-order relation θ on the semigroup S/q such that the natural epimorphism $\pi:S\to S/q$ is a reverse isotone mapping. Then, by [10], there exists a quasi-antiorder σ on S such that $q=\sigma\cup\sigma^{-1}$. So, there exists an anti-ordered semigroup $T=(S/q,=_1,\neq_1,\cdot)$ under θ and a strongly extensional reverse isotone homomorphism $\pi:S\to T$ such that $\sigma=\{(a,b)\in S\times S:(\pi(a),\pi(b))\in\theta\}$. Further on, we have $q=\{(a,b)\in S\times S:\pi(a)\neq_1\pi(b)\}$. In fact,

$$(a,b)\in q\Leftrightarrow(a,b)\in\sigma\vee(a,b)\in\sigma^{-1}\Leftrightarrow(\pi(a),\pi(b))\in\theta\vee(\pi(a),\pi(b))\in\theta^{-1}$$
$$(\pi(a),\pi(b))\in\theta\cup\theta^{-1}=\neq_1\Leftrightarrow\pi(a)\neq_1\pi(b).$$

(2) $\Rightarrow$ (1). Let T be an anti-ordered semigroup under an anti-order θ and $\varphi:S\to T$ a strongly extensional reverse isotone homomorphism such that $q=\{(a,b)\in S\times S:\varphi(a)\neq\varphi(b)\}$. Since θ is an anti-order relation on T, then $\neq=\theta\cup\theta^{-1}$ holds. Thus, by Theorem in [10], the relation σ on S, defined by $(a,b)\in\sigma$ if and only if $(\varphi(a),\varphi(b))\in\theta$, is a quasi-antiorder relation on S. On the other hand, $q=\sigma\cup\sigma^{-1}$. In fact, if (a,b) is an arbitrary element of q, then

$$\begin{aligned}\varphi(a)\neq\varphi(b)\Leftrightarrow(\varphi(a),\varphi(b))\in\theta\vee(\varphi(b),\varphi(a))\in\theta&\Leftrightarrow(a,b)\in\sigma\vee(b,a)\in\sigma\\&\Leftrightarrow(a,b)\in\sigma\cup\sigma^{-1}=q\end{aligned}$$

Let $(\varphi(a), \varphi(b)) \in \theta$. Then $(a, b) \in \sigma \subseteq \alpha$. By Theorem 1, q is regular anti-congruence on S. $\square$

Recall that, by Lemma 0, any class aq of anti-congruence q, generated by the element $a \in S$, is a strongly extensional subset of S. Besides, we have the following assertion, which is crucial in characterization of regular anti-congruences on an anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$: If q is a regular anti-congruence on an anti-ordered semigroup S, then for every $q-$class aq in S we have

$$((x, y) \bowtie \alpha \wedge (y, z) \bowtie \alpha \wedge x, z \in aq) \Rightarrow y \in aq$$

for any $x, y, z, a \in S$. If q is a regular anti-congruence on a semigroup S, then there exists an anti-order relation θ on S/q such that the natural epimorphism $\pi : S \to S/q$ is a strongly extensive reverse isotone homomorphism. Besides, there exists a quasi-antiorder σ under α, defined by $(x, y) \in \sigma \Leftrightarrow (xq, yq) \in \theta$ such that $\sigma \cup \sigma^{-1} = q$. Let t be an arbitrary element of aq. Then $(a, t) \in q = \sigma \cup \sigma^{-1}$. Thus $(a, t) \in \sigma$ or $(t, a) \in \sigma$. Hence, we have

$$(a, t) \in \sigma \Rightarrow (a, x) \in \sigma \subseteq q \vee (x, y) \in \sigma \subseteq \alpha \vee (y, t) \in \sigma \subseteq q \subseteq \neq \Rightarrow t \neq y;$$
$$(t, a) \in \sigma \Rightarrow (t, y) \in \sigma \subseteq \neq \vee (y, z) \in \sigma \subseteq \alpha \vee (z, a) \in \sigma \subseteq q \Rightarrow t \neq y.$$

So, in both cases, we have that $t \in aq \Rightarrow t \neq y$. Therefore, $y \bowtie aq$. We have

$$((x, y) \bowtie \alpha \wedge (y, z) \bowtie \alpha \wedge y \in aq) \Rightarrow x \in aq \vee z \in aq$$

for any $x, y, a \in S$. Indeed, if $x, y, z, a \in S$ such that $(x, y) \bowtie \alpha$ and $(y, z) \bowtie \alpha$ and $x \in aq$, then $(a, y) \in q = \sigma \cup \sigma^{-1} \Rightarrow ((a, y) \in \sigma \vee (y, a) \in \sigma)$. Thus, we have

$$\big((a, y) \in \sigma \vee (y, a) \in \sigma\big) \Rightarrow$$
$$\big((a, x) \in \sigma \subseteq q \vee (x, y) \in \sigma \subseteq \alpha\big) \vee \big((y, z) \in \sigma \subseteq \alpha \vee (z, a) \in \sigma \subseteq q\big) \Rightarrow x \in aq \vee z \in aq.$$

It is not known whether the condition given above on q-classes is sufficient for regularity of an anti-congruence on an anti-ordered semigroup.

Example. We consider the anti-ordered set $S = \{a, b, c, d, e, f\}$ under the relation $\alpha = S \times S \setminus \{(a, a), (a, d), (a, e), (b, b), (b, e), (c, c), (c, b), (c, e), (d, d), (d, e), (e, e), (f, f), (f, a), (f, b), (f, c), (f, d), (f, e)\}$. Define a coequality relation q on S by $q = S \times S \setminus \{(a, a), (b, b), (b, c), (b, d), (c, c), (c, b), (c, d), (d, d), (d, c), (d, b), (e, e), (f, f)\}$. Then

$$S/q = \big\{aq = \{b, c, d, e, f\},\ bq = \{a, e, f\},\ cq = \{a, e, f\},$$
$$dq = \{a, e, f\},\ eq = \{a, b, c, d, f\},\ fq = \{a, b, c, d, e\}\big\}.$$

The following relation is an anti-order relation on S/q

$$\theta_1 = \wp(S) \times \wp(S) \setminus \{(\{f\}, \{f\}), (\{f\}, \{a\}), (\{f\}, \{b, d, c\}), (\{f\}, \{e\}), (\{a\}, \{a\}),$$
$$(\{a\}, \{b, d, c\}), (\{a\}, \{e\}), (\{b, d, c\}, \{b, d, c\}), (\{b, d, c\}, \{e\}), (\{e\}, \{e\})\}.$$

Then $(S/q, =_1, \neq_1, \theta_1)$ is an anti-ordered set, q is a regular coequality on S. If in S we define the internal operation by the table below, then the set S is an anti-ordered semigroup, q is an anti-congruence on S. It is easy to check that q is a regular anti-congruence on the semigroup S. The proof of these facts is straightforward.

$\cdot$	a	b	c	d	e	f
a	d	d	d	d	d	a
b	e	e	e	e	e	b
c	d	d	d	d	d	c
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	d	d	d	d	d	f

REMARK. If σ is a quasi-antiorder on S, then $q = \sigma \cup \sigma^{-1}$ is the minimal regular anti-congruence on S which contains σ. In fact, if ρ is a regular anti-congruence on S containing σ, then $\rho = \rho \cup \rho^{-1} \supseteq \sigma \cup \sigma^{-1} = q$.

Let q be a regular anti-congruence on an anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$. Then there exists anti-order θ on S/q such that the natural homomorphism $\pi : S \to S/q$ is reverse isotone. Hence, by [10], there exists a quasi-antiorder σ under α such that $q = \sigma \cup \sigma^{-1}$ and $\theta = \{(aq, bq) \in S/q \times S/q : (a, b) \in \sigma\}$. In the following theorem we show that there exists such maximal quasi-antiorder τ under α and we give a construction of that relation.

THEOREM 3. *Let q be a regular anti-congruence on an anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$. Then there exists the maximal quasi-antiorder relation τ under α such that $q = \tau \cup \tau^{-1}$ and $\theta \subseteq \{(aq, bq) \in S/q \times S/q : (a, b) \in \tau\}$. This relation is exactly the following relation $c(q \cap \alpha) = \bigcap_{n\in N} \big({}^{n}(q \cap \alpha)\big)$.*

PROOF. (1) It is clear that $c(\alpha \cap q) \subseteq \alpha \cap q \subseteq q \subseteq \neq$, $c(q \cap \alpha) \subseteq \alpha$, and that the relation $c(\alpha \cap q)$ is cotransitive [4]. In fact, for cotransitivness we need to prove that $(a, c) \in c(q \cap \alpha) \Rightarrow (\forall b \in S)\big((a, b) \in c(q \cap \alpha) \vee (b, c) \in c(q \cap \alpha)\big)$, i.e., to prove that

$$(a, c) \in c(q \cap \alpha) \Rightarrow \\ (\forall b \in S)\Big((\forall i \in N)\big((a, b) \in ({}^{i}(q \cap \alpha))\big) \vee (\forall j \in N)\big((b, c) \in ({}^{j}(q \cap \alpha))\big)\Big).$$

First, we have

$$\begin{aligned}(a, c) \in c(q \cap \alpha) &\Rightarrow (a, c) \in ({}^{2}(q \cap \alpha)) = (q \cap \alpha) * (q \cap \alpha) \\ &\Rightarrow (\forall b \in S)((a, b) \in (q \cap \alpha) \vee (b, c) \in (q \cap \alpha)).\end{aligned}$$

Second, suppose that the following implication holds for some $n \geqslant 2$

$$(a, c) \in c(q \cap \alpha) \Rightarrow (\forall b \in S)\big((a, b) \in \big({}^{n}(q \cap \alpha)\big) \vee (b, c) \in \big({}^{n}(q \cap \alpha)\big)\big).$$

Thus, since the filled product "$*$" is associative, we have

$$\begin{aligned}(a, c) \in c(q \cap \alpha) &\Rightarrow (a, c) \in \big({}^{2(n+1)}(q \cap \alpha)\big) = \big({}^{(n+1)}(q \cap \alpha)\big) * \big({}^{(n+1)}(q \cap \alpha)\big) \\ &\Leftrightarrow (\forall b \in S)((a, b) \in \big({}^{(n+1)}(q \cap \alpha)\big) \vee (b, c) \in \big({}^{(n+1)}(q \cap \alpha)\big)).\end{aligned}$$

Therefore, for any natural number n, by induction, we have

$$(a, c) \in c(q \cap \alpha) \Rightarrow \\ (\forall b \in S)\Big((\forall i \leqslant n)\big((a, b) \in ({}^{i}(q \cap \alpha))\big) \vee (\forall j \leqslant n)\big((b, c) \in ({}^{j}(q \cap \alpha))\big)\Big).$$

Hence $(a,c) \in c(q \cap \alpha) \Rightarrow (\forall b \in S)\big((a,b) \in c(q \cap \alpha) \vee (b,c) \in c(q \cap \alpha)\big)$.

(2) Further on, let a, b, x be arbitrary elements of S such that $(ax, bx) \in c(\alpha \cap q)$. Then $(ax, bx) \in q$ and, by compatibility of α and q in S, we have $(a,b) \in q$. Suppose that the implication $(ax, bx) \in c(\alpha \cap q) \Rightarrow (a,b) \in \big({}^{n}(\alpha \cap q)\big)$, $n \in N$, holds for any $a, b, x \in S$. Then, from $(ax, bx) \in c(\alpha \cap q) \in \big({}^{n+1}(\alpha \cap q)\big) = \big({}^{n}(\alpha \cap q)\big) * (\alpha \cap q)$ it follows

$$\begin{aligned}(ax, bx) \in c(\alpha \cap q) \subseteq \big({}^{n+1}(\alpha \cap q)\big) &= \big({}^{n}(\alpha \cap q)\big) * (\alpha \cap q)\\ &\Rightarrow (\forall y \in S)((ax, yx) \in \big({}^{n}(\alpha \cap q)\big) \vee (yx, bx) \in (\alpha \cap q))\\ &\Rightarrow (\forall y \in S)((a, y) \in \big({}^{n}(\alpha \cap q)\big) \vee (y, b) \in (\alpha \cap q))\\ &\Rightarrow (a,b) \in \big({}^{n+1}(\alpha \cap q)\big).\end{aligned}$$

So, by induction, we have $(ax, bx) \in c(\alpha \cap q) \Rightarrow (a,b) \in c(\alpha \cap q)$.

The other implication $(xa, xb) \in c(\alpha \cap q) \Rightarrow (a,b) \in c(\alpha \cap q)$ can be proved analogously. Therefore, the relation $c(\alpha \cap q)$ is compatible with the semigroup operation in S.

(3) Let σ be a quasi-antiorder relation under α such that $q = \sigma \cup \sigma^{-1}$ and $\theta = \{(aq, bq) \in S/q \times S/q : (a,b) \in \sigma\}$. Then $\sigma \subseteq \alpha \cap q$ and $\sigma = c(\sigma) \subseteq c(\alpha \cap q)$ because, by Lemma 0.4.2 in [7], the cotransitive fulfillment satisfies the implication $\sigma \subseteq \alpha \cap q \Rightarrow c(\sigma) \subseteq c(\alpha \cap q)$ and, in addition, since σ is a cotransitive relation, $\sigma = c(\sigma)$.

So, the relation $\tau = c(\alpha \cap q)$ is the maximal quasi-antiorder under α such that $q = \tau \cup \tau^{-1}$. Indeed, $q = \sigma \cup \sigma^{-1} \subseteq \tau \cup \tau^{-1} \subseteq q$. Besides, it is clear that the relation $\Theta = \{(aq, bq) \in S/q \times S/q : (a,b) \in \tau\}$ is an anti-order relation on S/q such that $\theta \subseteq \Theta$. □

Corollary 4. *Let q be a regular anti-congruence on an anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$. Then there exists the maximal anti-order relation on S/q. This relation is exactly the following relation $\{(aq, bq) \in S/q \times S/q : (a,b) \in c(q \cap \alpha)\}$.*

Proof. Let θ_1 be a regular anti-congruence on S/q with respect to α. Then there exists a quasi-antiorder σ on S such that $q = \sigma \cup \sigma^{-1}$ and $\theta_1 = \{(aq, bq) \in S/q \times S/q : (a,b) \in \sigma\}$. Since $c(\alpha \cap q)$ is the maximal quasi-antiorder with respect to α, then $\sigma \subseteq \tau$ holds. Thus, we have $\theta_1 \subseteq \{(aq, bq) \in S/q \times S/q : (a,b) \in c(\alpha \cap q)\}$. So, there exists the maximal regular anti-congruence with respect to α. □

References

[1] M. A. Baroni, *Constructive order completeness*, UCDMS 2004/12 (www.math.canterbury.ac.nz/php/research/reports/)

[2] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York 1967.

[3] D. S. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987

[4] R. Mines, F. Richman and W. Ruitenburg, *A Course of Constructive Algebra*, Springer-Verlag, New York, 1988

[5] D. A. Romano, *On Construction of Maximal Coequality Relation and its Applications*; in: R. Tošić and Z. Budimac (eds.), *Proc. 8th Internat. Conf. Logic and Computers Sciences "LIRA '97", Novi Sad*, Institute of Mathematics, Novi Sad 1997, 225–230

[6] D. A. Romano, *A Left Compatible Coequality Relation on Semigroup with Apartness*, Novi Sad J. Math 29(2) (1999), 221–234

[7] D. A. Romano, *Some relations and subsets of semigroup with apartness generated by the principal consistent subset*, Univ. Beograd, Publ. Elektroteh. Fak. Ser. Mat. 13(2002), 7–25

[8] D. A. Romano, *A note on a family of quasi-antiorder on semigroup*, Kragujevac J. Math. 27 (2005), 11–18

[9] D. A. Romano, *A note on quasi-antiorder in semigroup*, Novi Sad J. Math. 37(1) (2007), 3–8

[10] D. A. Romano, *Inducing a quasi-antiorder by anticongruence on semigroup*, Novi Sad J. Math., (To appear)

[11] A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics, An Introduction*, North-Holland, Amsterdam, 1988

Prirodno-matematički fakultet
78000 Banja Luka
Srpska
Bosnia and Herzegovina
bato49@hotmail.com