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ON THREE CONJECTURES INVOLVING THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS

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ABSTRACT. We study the signless Laplacian spectral radius of graphs and prove three conjectures of Cvetković, Rowlinson, and Simić [*Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math., Nouv. Sér. 81(95) (2007), 11–27].

1. Introduction

In this paper, we consider only simple connected graphs and follow the notation of [**1**]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *adjacency matrix of* G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if two vertices i and j are adjacent in G and $a_{ij} = 0$ otherwise. The characteristic polynomial of G is just $P_G(x) = \det(xI - A(G))$. Let $D(G)$ be the diagonal degree matrix of G. We call the matrix $L(G) = D(G) - A(G)$ the *Laplacian matrix* of G, and the matrix $Q(G) = D(G) + A(G)$ the *signless Laplacian matrix* or *Q-matrix* of G. We denote the largest eigenvalues of $A(G)$, $L(G)$, $Q(G)$ by $\rho(G)$, $\lambda(G)$, $\mu(G)$, respectively, and call them the adjacency spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius (or the Q-spectral radius) of G, respectively.

The study of the signless Laplacian spectral radius has recently attracted researchers' attention. In [**10**], Fan et al. studied the signless Laplacian spectral radius of bicyclic graphs with fixed order. In [**9**], the authors discussed the smallest eigenvalue of $Q(G)$ as a parameter reflecting the nonbipartiteness of the graph G. In [**7**], the authors studied the smallest signless Laplacian eigenvalue of non-bipartite graphs. In [**11**], the extremal graphs with maximal signless Laplacian spectral radius and fixed diameter were studied. More information about the signless Laplacian can be found in [**2**], [**3**], [**5**], [**6**]. For more information about the spectral radius of graphs, the reader can refer to [**4**].

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In [**5**], the authors proposed the following conjectures and in this paper we confirm that they are true.

Theorem 1.1. [**5**, Conjecture 6] *Let G be a connected graph of order $n \geqslant 4$. Then*

$$\mu(G) - \frac{4m}{n} \leqslant n - 4 + \frac{4}{n}.$$

Equality holds if and only if $G = K_{1,n-1}$.

Theorem 1.2. [**5**, Conjecture 7] *Let G be a connected graph of order $n \geqslant 5$. Then*

$$\mu(G) - \frac{2m}{n} \leqslant n - 1.$$

Equality holds if and only if $G = K_n$.

Theorem 1.3. [**5**, Conjecture 10] *Let G be a connected graph of order $n \geqslant 4$. Then*

$$\mu(G) - \lambda(G) \leqslant n - 2.$$

Equality holds if and only if $G = K_n$.

2. Lemmas and results

Let G be a connected graph. The degree of u in G is denoted by d_u, the average degree of u, denoted by m_u, satisfies $d_u m_u = \sum_{uv \in E} d_v$, where $E = E(G)$.

Lemma 2.1. [**8**] *Let G be a graph with n vertices, and m edges. Then*

$$max\{d_v + m_v \mid v \in V(G)\} \leqslant \frac{2m}{n-1} + n - 2,$$

with equality if and only if $K_{1,n-1} \subseteq G$ or $G = K_{n-1} \cup K_1$.

Lemma 2.2. [**12**] *Let $M = (m_{ij})$ be an $n \times n$ irreducible nonnegative matrix with spectral radius $\rho(M)$, and let $R_i(M)$ be the ith row sum of M, i.e., $R_i(M) = \sum_{j=i}^{n} m_{ij}$. Then*

$$\min\{R_i(M) \mid \leqslant i \leqslant n\} \leqslant \rho(M) \leqslant \max\{R_i(M) \mid 1 \leqslant i \leqslant n\}.$$

Moreover, if the row sums of M are not all equal, then both above inequalities are strict.

Lemma 2.3. *Let G be a connected graph. Then $\mu(G) \leqslant max\{d_v + m_v \mid v \in V(G)\}$, with equality holding if and only if G is either semiregular bipartite or regular.*

Proof. We consider the matrix $K = D^{-1}QD$, where the row sum corresponding to the vertex u is $d_u + m_u$. From Lemma 2.2, we obtain the required upper bound for $\mu(G)$.

If equality holds, then by Lemma 2.2, for a neighbor v of u, $d_u + m_u = d_v + m_v$, thus $\sum_{uv \in E}(d_u + m_u) = \sum_{uv \in E}(d_v + m_v)$, that is $d_u^2 + d_u m_u = d_u m_u + \sum_{uv \in E} m_v$. So we get

$$d_u^2 = \sum_{uv \in E} m_v.$$

Suppose d_u is the maximum degree. Then $d_v m_v = \sum_{wv\in E} d_w \leqslant d_u d_v$, whence $m_v \leqslant d_u$ for all $v \in V(G)$. Since $d_u^2 = \sum_{uv\in E} m_v \leqslant d_u^2$, we have for any edge uv, $d_u = m_v$, and $d_u d_v = d_v m_v$, that is, $\sum_{wv\in E}(d_u - d_w) = 0$. Since d_u is the maximum degree, we have $d_u = d_w$ whenever there exists a vertex v such that $uv, vw \in E$.

If G does not contain odd cycles, then G is bipartite. Suppose $V = S \cup T$ is a bipartion and $u \in T$; then $v \in S$, $w \in T$. This implies that the vertices in T have the same degree. Similarly, the vertices in S also have the same degree. So G is semiregular bipartite.

If G contains odd cycles, then G must be regular.

The converse is easy to check. □

LEMMA 2.4. *Let G be a connected graph with n vertices and m edges. Then*

$$\mu(G) \leqslant \frac{2m}{n-1} + n - 2,$$

with equality if and only if G is $K_{1,n-1}$ or K_n.

PROOF. By Lemmas 2.1 and 2.3, we can get the result. Note that $K_{1,n-1}$ is the only semiregular bipartite graph and K_n is the only regular graph that arises in the case of equality. □

Now we can present the proof of the main results of this paper.

PROOF OF THEOREM 1.1. By Lemma 2.4, we have

$$\begin{aligned}\mu(G) - \frac{4m}{n} &\leqslant \frac{2m}{n-1} + n - 2 - \frac{4m}{n} = (n-2)\Big(1 - \frac{2m}{n(n-1)}\Big)\\ &\leqslant (n-2)\Big(1 - \frac{2(n-1)}{n(n-1)}\Big) = n - 4 + \frac{4}{n}.\end{aligned}$$

The last inequality holds since G is connected and so has at least $n-1$ edges. The equality case is easy to see from Lemma 2.4. □

PROOF OF THEOREM 1.2. By Lemma 2.4, we have

$$\mu(G) - \frac{2m}{n} \leqslant \frac{2m}{n-1} + n - 2 - \frac{2m}{n} = \frac{2m}{n(n-1)} + n - 2 \leqslant 1 + n - 2 = n - 1.$$

The last inequality holds since G has at most $\frac{1}{2}n(n-1)$ edges. When this bound is attained, $G = K_n$. □

PROOF OF THEOREM 1.3. Note that the sum of all the Laplacian eigenvalues is $2m$, so we have $(n-1)\lambda(G) \geqslant 2m$ and hence $\lambda(G) \geqslant \frac{2m}{n-1}$. By Lemma 2.4, we have

$$\mu(G) - \lambda(G) \leqslant \frac{2m}{n-1} + n - 2 - \frac{2m}{n-1} = n - 2.$$

The equality case is easy to see from Lemma 2.4. □

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