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ON A NEW CONVERSE OF JENSEN'S INEQUALITY

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Abstract. We give another global upper bound for Jensen's discrete inequality which is better than already existing ones. For instance, we determine a new converses for generalized A–G and G–H inequalities.

1. Introduction

Throughout this paper $\tilde{x} = \{x_i\}$ represents a finite sequence of real numbers belonging to a fixed closed interval $I = [a, b]$, $a < b$ and $\tilde{p} = \{p_i\}$, $\sum p_i = 1$ is a sequence of positive weights associated with $\tilde{x}$.

If f is a convex function on I, then the well-known Jensen's inequality [1], [4] asserts that:

$$0 \leqslant \sum p_i f(x_i) - f\Big(\sum p_i x_i\Big).$$

There are many important inequalities which are particular cases of Jensen's inequality such as the weighted A–G–H inequality, Cauchy's inequality, the Ky Fan and Hölder inequalities, etc.

One can see that the lower bound zero is of global nature since it does not depend on $\tilde{p}$ and $\tilde{x}$, but only on f and the interval I, whereupon f is convex.

An upper global bound (i.e., depending in f and I only) for Jensen's inequality was given by Dragomir [3],

Theorem 1. *If f is a differentiable convex mapping on I, then we have*

$$\sum p_i f(x_i) - f\Big(\sum p_i x_i\Big) \leqslant \frac{1}{4}(b-a)(f'(b) - f'(a)) := D_f(a, b). \tag{1}$$

There is a number of papers where inequality (1) is utilized in applications concerning some parts of Analysis, Numerical Analysis, Information Theory etc (cf. [1], [2], [3]).

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In [5] we obtained an upper global bound without differentiability restriction on f. Namely, we proved the following

Theorem 2. *If $\tilde{p}$, $\tilde{x}$ are defined as above, we have that*

$$\sum p_i f(x_i) - f\Big(\sum p_i x_i\Big) \leqslant f(a) + f(b) - 2f\Big(\frac{a+b}{2}\Big) := S_f(a,b), \tag{2}$$

for any f that is convex over $I := [a,b]$.

In many cases the bound $S_f(a,b)$ is better than $D_f(a,b)$. For instance, for $f(x) = -x^s$, $0 < s < 1$; $f(x) = x^s$, $s > 1$; $I \subset R^+$, we have that $S_f(a,b) \leqslant D_f(a,b)$, for each $s \in (0,1) \cup (1,2) \cup (3,+\infty)$.

In this article we establish another global bound $T_f(a,b)$ for Jensen's inequality, which is better than both of aforementioned bounds $D_f(a,b)$ and $S_f(a,b)$.

Finally, we determine $T_f(a,b)$ in the case of a generalized A–G inequality as a combination of some well-known classical means. As a consequence, new global upper bounds for A–G and G–H inequalities are established.

2. Results

Our main result is contained in the following

Theorem 3. *Let f, $\tilde{p}$, $\tilde{x}$ be defined as above and $p, q > 0$, $p + q = 1$. Then*

$$\sum p_i f(x_i) - f\Big(\sum p_i x_i\Big) \leqslant \max_p \big[pf(a) + qf(b) - f(pa+qb)\big] := T_f(a,b). \tag{3}$$

For example,

$$T_{x^2}(a,b) = \max_p \big(pa^2 + qb^2 - (pa+qb)^2\big) = \max_p \big(pq(b-a)^2\big) = \frac{1}{4}(b-a)^2,$$

and we obtain at once the well-known pre-Grüss inequality

$$\sum p_i x_i^2 - \Big(\sum p_i x_i\Big)^2 \leqslant \frac{1}{4}(b-a)^2,$$

with $1/4$ as the best possible constant.

Remark 2.1. It is easy to see that $g(p) := pf(a)+(1-p)f(b)-f(pa+(1-p)b)$ is concave for $0 \leqslant p \leqslant 1$ with $g(0) = g(1) = 0$. Hence, there exists the unique positive $\max_p g(p) = T_f(a,b)$.

The next theorem approves that inequality (3) is stronger than (1) or (2).

Theorem 4. *Let $\tilde{I}$ be the domain of a convex function f and $I := [a,b] \subset \tilde{I}$. Then*

$$\text{(i)} \quad T_f(a,b) \leqslant D_f(a,b); \qquad \text{(ii)} \quad T_f(a,b) \leqslant S_f(a,b),$$

for each $I \subset \tilde{I}$.

The following well known A-G inequality [4], asserts that $A(\tilde{p},\tilde{x}) \geqslant G(\tilde{p},\tilde{x})$, where $A(\tilde{p},\tilde{x}) := \sum p_i x_i$, $G(\tilde{p},\tilde{x}) := \prod x_i^{p_i}$, are the generalized arithmetic and geometric means, respectively.

Applying theorems 1 and 2 with $f(x) = -\log x$, one obtains the following converses of A–G inequality:

$$1 \leqslant \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leqslant \exp\left(\frac{(b-a)^2}{4ab}\right) \qquad \text{(cf. [2]);} \tag{4}$$

$$1 \leqslant \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leqslant \frac{(a+b)^2}{4ab}. \tag{5}$$

Since $1 + x \leqslant e^x$, $x \in R$, putting $x = \frac{(b-a)^2}{4ab}$, one can see that the inequality (5) is stronger than (4) for each $a, b \in R^+$.

An even stronger converse of A–G inequality can be obtained applying Theorem C.

Theorem 5. *Let $\tilde{p}, \tilde{x}, A(\tilde{p},\tilde{x}), G(\tilde{p},\tilde{x})$ be defined as above and $x_i \in [a,b]$, $0 < a < b$. We have,*

$$1 \leqslant \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leqslant \frac{L(a,b)I(a,b)}{G^2(a,b)} := T(a,b), \tag{6}$$

where $G(a,b) := \sqrt{ab}$, $L(a,b) := \dfrac{b-a}{\log b - \log a}$, $I(a,b) := \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$, are the geometric, logarithmic and identric means, respectively.

Since, for $a \neq b$, $G(a,b) < L(a,b) < I(a,b) < A(a,b)$, one can see that the inequality (6) is stronger than (5).

In addition to the above we quote a converse for G–H inequality, where $H = H(\tilde{p},\tilde{x}) := (\sum p_i/x_i)^{-1}$, denotes the generalized harmonic mean.

Theorem 6. *With the notation of Theorem E, we have*

$$1 \leqslant \frac{G(\tilde{p},\tilde{x})}{H(\tilde{p},\tilde{x})} \leqslant T(a,b).$$

3. Proofs

Proof of Theorem 3. Since $x_i \in [a,b]$, there is a sequence $\{\lambda_i\}$, $\lambda_i \in [0,1]$, such that $x_i = \lambda_i a + (1-\lambda_i)b$, $i = 1, 2, \ldots$. Hence,

$$\begin{aligned}\sum p_i f(x_i) - f\Big(\sum p_i x_i\Big) &= \sum p_i f\big(\lambda_i a + (1-\lambda_i)b\big) - f\Big(\sum p_i\big(\lambda_i a + (1-\lambda_i)b\big)\Big)\\ &\leqslant \sum p_i\big(\lambda_i f(a) + (1-\lambda_i)f(b)\big) - f\Big(a\sum p_i\lambda_i + b\sum p_i(1-\lambda_i)\Big)\\ &= f(a)\Big(\sum p_i\lambda_i\Big) + f(b)\Big(1 - \sum p_i\lambda_i\Big) - f\Big(a\Big(\sum p_i\lambda_i\Big) + b\Big(1 - \sum p_i\lambda_i\Big)\Big).\end{aligned}$$

Denoting $\sum p_i\lambda_i := p$, $1 - \sum p_i\lambda_i := q$, we have that $0 \leqslant p, q \leqslant 1$, $p + q = 1$. Consequently,

$$\begin{aligned}\sum p_i f(x_i) - f\Big(\sum p_i x_i\Big) &\leqslant pf(a) + qf(b) - f(pa+qb)\\ &\leqslant \max_p \big[pf(a) + qf(b) - f(pa+qb)\big] := T_f(a,b),\end{aligned}$$

and the proof of Theorem 3 is done. □

Proof of Theorem 4. Applying Theorems 1 and 2 with $\#\tilde{x} = 2$, for each $p \in [0,1]$ and $a, b \in \tilde{I}$ we get

$$\text{(i)} \quad pf(a) + qf(b) - f(pa+qb) \leqslant D_f(a,b),$$
$$\text{(ii)} \quad pf(a) + qf(b) - f(pa+qb) \leqslant S_f(a,b).$$

Since the right-hand sides of (i) and (ii) do not depend on p, we get at once

$$T_f(a,b) = \max_p \big[pf(a) + qf(b) - f(pa+qb)\big] \leqslant D_f(a,b);$$
$$T_f(a,b) = \max_p \big[pf(a) + qf(b) - f(pa+qb)\big] \leqslant S_f(a,b). \qquad \square$$

Proof of Theorem 5. By Theorem 3, applied with $f(x) = -\log x$, we obtain

$$0 \leqslant \log \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leqslant T_{-\log x}(a,b) = \max_p \big[\log(pa+qb) - p\log a - q\log b\big].$$

By standard argument it is easy to find that the unique maximum is attained at

$$p = \frac{b}{b-a} - \frac{1}{\log b - \log a}.$$

Since $0 < a < b$, we get $0 < p < 1$ and, after some calculation, it follows that

$$0 \leqslant \log \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leqslant \log\Big(\frac{b-a}{\log b - \log a}\Big) - \log(ab) + \frac{b\log b - a\log a}{b-a} - 1,$$

Exponentiating, we obtain the assertion from Theorem 5. $\square$

Proof of Theorem 6. By change of variable $x_i \to \frac{1}{x_i}$, the proof easily follows from (6) since then

$$A(\tilde{p},\tilde{x}) \to \frac{1}{H(\tilde{p},\tilde{x})}; \quad G(\tilde{p},\tilde{x}) \to \frac{1}{G(\tilde{p},\tilde{x})},$$

and the right-hand side of (6) stays unaltered under the transformation. $\square$

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