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ADEQUACY OF LINK FAMILIES

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Abstract. We analyze adequacy of knots and links, utilizing Conway notation, Montesinos tangles and *Linknot* and *KhoHo* computer calculations. We introduce a numerical invariant called adequacy number, and compute adequacy polynomial which is the invariant of alternating link families. According to computational results, adequacy polynomial distinguishes (up to mutation) all families of alternating knots and links generated by links with at most 12 crossings.

1. Introduction

We consider adequacy of nonalternating knots and links (shortly KLs) and their families (classes) given in the Conway notation. First, we explain the Conway notation, introduced in Conway's seminal paper [**1**] published in 1967, and effectively used since (e.g., [**2**]). Conway symbols of knots with up to 10 crossings and links with at most 9 crossings are given in the Appendix of the book [**3**].

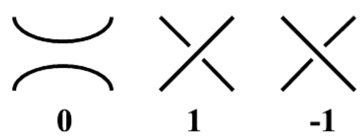

Figure 1. The elementary tangles.

The main building blocks in the Conway notation are elementary tangles. We distinguish three elementary tangles, shown in Fig. 1 and denoted by 0, 1 and -1. All other tangles can be obtained by combining elementary tangles, while 0 and 1 are sufficient for generating alternating knots and links (abbr. KLs). Elementary

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tangles can be combined by the following operations: *sum*, *product*, and *ramification* (Figs. 2-3). Given tangles a and b, the image of a under reflection with mirror line NW–SE is denoted by $-a$, and the sum is denoted by $a+b$. The product $a\,b$ is defined as $a\,b = -a+b$, and *ramification* by $(a,b) = -a-b$.

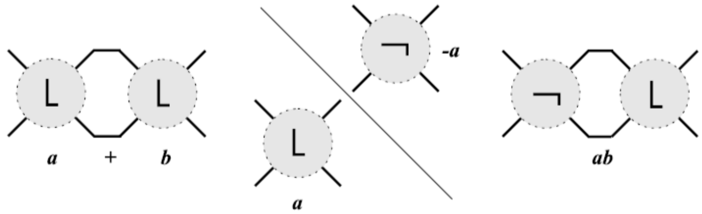

FIGURE 2. A sum and product of tangles.

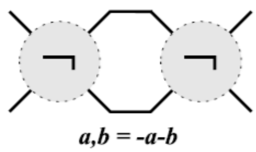

FIGURE 3. Ramification of tangles.

A tangle can be closed in two ways (without introducing additional crossings): by joining in pairs NE and NW, and SE and SW ends of a tangle to obtain a *numerator closure*; or by joining in pairs NE and SE, and NW and SW ends we obtain a *denominator closure* (Fig. 1a,b).

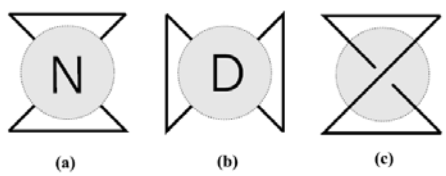

FIGURE 4. (a) Numerator closure; (b) denominator closure; (c) basic polyhedron 1^*.

DEFINITION 1.1. A *rational tangle* is any finite product of elementary tangles. A *rational KL* is a numerator closure of a rational tangle.

DEFINITION 1.2. A tangle is *algebraic* if it can be obtained from elementary tangles using the operations of sum and product. KL is *algebraic* if it is a numerator closure of an algebraic tangle.

A *Montesinos tangle* and the corresponding *Montesinos link*, consisting of n alternating rational tangles t_i, with at least three nonelementary tangles t_k for $k \in \{1, 2, \ldots, n\}$, is denoted by $t_1, t_2, \ldots, t_n$, $n \geqslant 3$, $i = 1, \ldots, n$ (Fig. 5). The number of tangles n is called the *length of the Montesinos tangle*. In particular, if all tangles t_i, $n \geqslant 3$, $i = 1, \ldots, n$ are integer tangles, we obtain pretzel KLs. Every nonalternating Montesinos link can be expressed in the form $t_1, t_2, \ldots, t_m, -t_{m+1}, \ldots, -t_n$, $m \geqslant 3$, $i = 1, \ldots, m$ with all $t_i \neq \pm 1$ and $m \leqslant n$. Such a representation of a nonalternating Montesinos link is called *minimal representation*, with respect to the number of crossings. Throughout the paper, the term "Montesinos link" refers to the minimal representation of a Montesinos link.

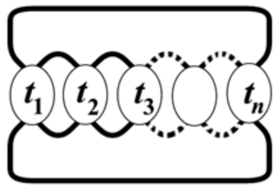

FIGURE 5. Montesinos link $t_1, t_2, \ldots, t_n$.

DEFINITION 1.3. *Basic polyhedron* is a 4-regular, 4-edge-connected, at least 2-vertex connected plane graph.

Basic polyhedron [**1, 2**] of a given KL can be identified by recursively collapsing all bigons in a KL diagram, until none of them remains.

The basic polyhedron 1^* is illustrated in Fig. 1c, and the other basic polyhedra with at most 11 crossings in Figs. 7–9.

DEFINITION 1.4. A link L is *algebraic* or 1^*-*link* if there exists at least one diagram of L which can be reduced to the basic polyhedron 1^* by a finite sequence of bigon collapses. Otherwise, it is a *nonalgebraic* or *polyhedral link*.

Conway notation for polyhedral KLs contains additionally a symbol of a basic polyhedron we are working with. The symbol $n^{*m} = n^{*m}1.1.\ldots.1$, where $*m$ is a sequence of m stars, denotes the m-th basic polyhedron in the list of basic polyhedra with n vertices. A KL obtained from a basic polyhedron n^{*m} by substituting tangles $t_1, \ldots, t_k$, $k \leqslant n$ instead of vertices, is denoted by $n^{*m}t_1 \ldots t_k$, where the number of dots between two successive tangles shows the number of omitted substituents of value 1. For example, $6^*2 : 2 : 2\,0$ means $6^*2.1.2.1.2\,0.1$, and $6^*2\,1.2.3\,2 : -2\,2\,0$ means $6^*2\,1.2.3\,2.1. -2\,2\,0.1$ (Fig. 6).

The complete list of the basic polyhedra with $n \leqslant 11$ crossings used in this paper is illustrated in Figs. 7–9[1].

In comparison with other KL notations (Dowker-Thistlethwaite notation, PD-notation, P-data, *etc.*), Conway notation is the most suitable for utilizing the notion

[1]Two different symbols of the basic polyhedron with 6 vertices are used simultaneously in [**1**].

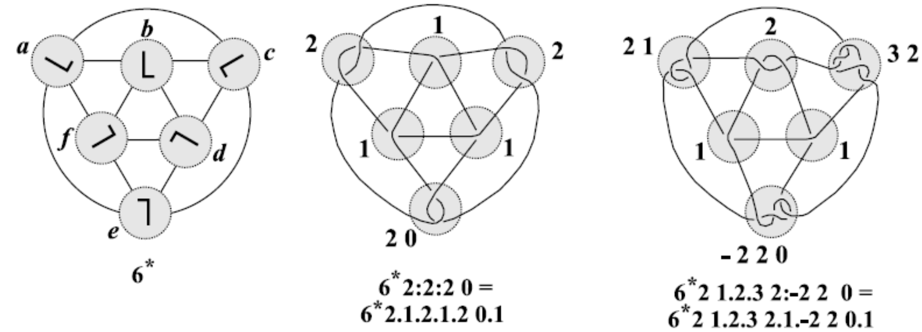

FIGURE 6. Basic polyhedron 6^* and the knots $6^*2.1.2.1.2\,0.1$ and $6^*2\,1.2.3\,2:-2\,2\,0$.

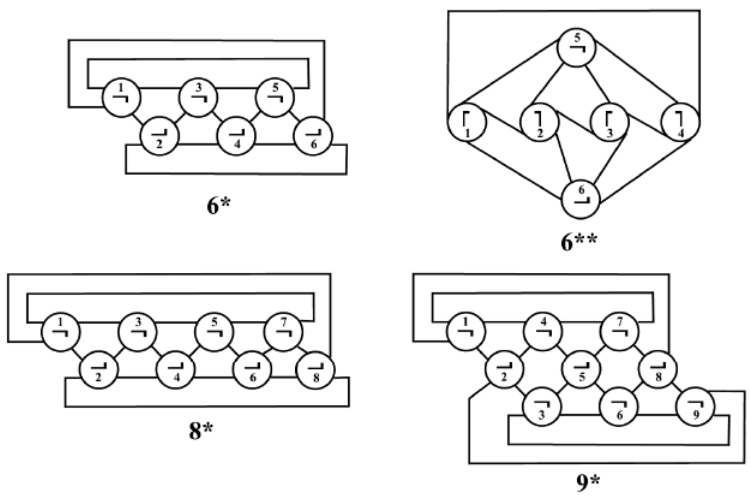

FIGURE 7. Basic polyhedra 6^*, 8^*, and 9^*.

of families of knots and links and analyzing how knot and link properties change inside families.

DEFINITION 1.5. For a link or knot L given in an unreduced[2] Conway notation $C(L)$ denote by S a set of numbers in the Conway symbol excluding numbers denoting basic polyhedron and zeros (determining the position of tangles in the vertices of polyhedron) and let $\tilde{S} = \{a_1, a_2, \ldots, a_k\}$ be a nonempty subset of S. Family $F_{\tilde{S}}(L)$ of knots or links derived from L consists of all knots or links L'

[2]The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

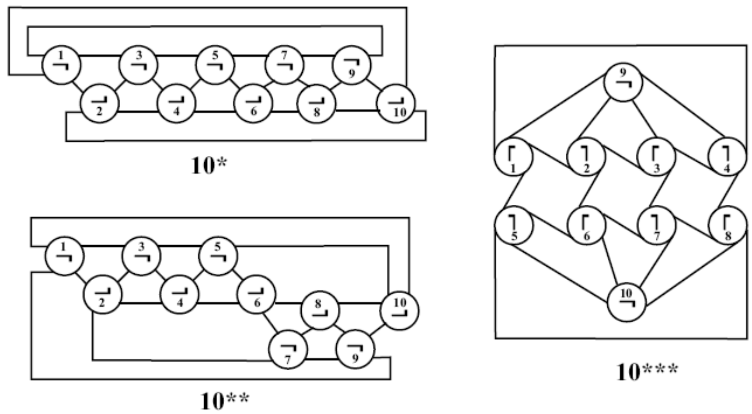

FIGURE 8. Basic polyhedra with $n = 10$ crossings.

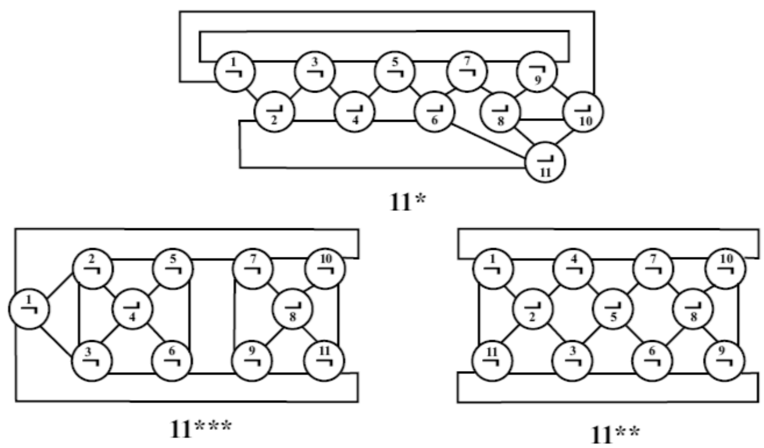

FIGURE 9. Basic polyhedra with $n = 11$ crossings.

whose Conway symbol is obtained by substituting all $a_i \neq \pm 1$, by $\mathrm{sgn}(a_i)|a_i + k_{a_i}|$, $|a_i + k_{a_i}| > 1$, $k_{a_i} \in \mathbb{Z}$.

An infinite subset of a family is called *subfamily*. If all k_{a_i} are even integers, the number of components is preserved within the corresponding subfamilies, i.e., adding full-twists preserves the number of components inside the subfamilies.

DEFINITION 1.6. A link given by Conway symbol containing only tangles ± 1 and ± 2 is called a *source link*. A link given by Conway symbol containing only tangles ± 1, ± 2, or ± 3 is called a *generating link*.

For example, Hopf link 2 (link 2_1^2 in Rolfsen's notation) is the source link of the simplest link family p ($p = 2, 3, \ldots$) (Fig. 10), and Hopf link and trefoil 3 (knot 3_1 in the classical notation) are generating links of this family. A family of KLs is usually derived from its source link by substituting $a_i \in \tilde{S}$, $a_i = \pm 2$, by $\mathrm{sgn}(a_i)(2+k)$, $k = 1, 2, 3, \ldots$ (see Def. 1.6).

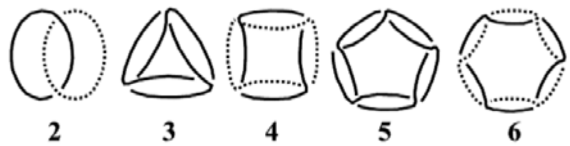

FIGURE 10. Hopf link and its family p ($p = 2, 3, \ldots$).

2. Adequacy of knots and links

In this section we briefly review the adequacy of KLs and discuss relations to the Khovanov homology. Throughout the paper we consider only reduced prime links.

Let D be a diagram of an unoriented, framed link $L \in R^3$. A Kauffman state of a diagram D is a function from the set of crossings of D to the set of signs $\{+1, -1\}$. Graphical interpretation is given by smoothing each crossing of D by introducing markers according to the convention illustrated in Fig. 11. A *state diagram* D_s of a diagram D and Kauffman state s, is a system of circles obtained by smoothing all crossings of D [**4**]. The set of circles in D_s, which are called *state circles*, is denoted by $C(D)$. Points of the state circles corresponding to a smoothed crossing are called *touch-points*. The number of touch-points belonging to a state circle $c \in C(D)$ is called the *length* of c.

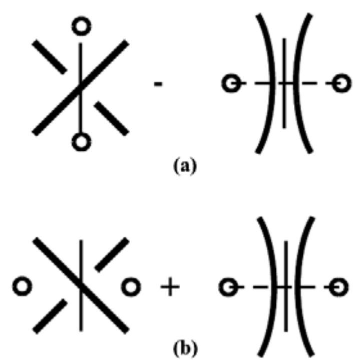

FIGURE 11. (a) $-$marker; (b) $+$marker. The broken lines represent the edges of the associated graph G_s connecting state circles (represented by dots).

Kauffman states s_+ and s_- with all + or all − signs are called *special states*, and their corresponding state diagrams D_{s_+} and D_{s_-} are called *special diagrams*. All other Kauffman states with both + or − signs are called *mixed states*, and to them correspond *mixed state diagrams*.

DEFINITION 2.1. A diagram D is s-adequate if two arcs at every touch-point of D_s belong to different state circles. In particular, a diagram D is +adequate or −adequate if it is s_+ or s_- adequate, respectively. If a diagram is neither +adequate nor −adequate, it is called *inadequate*. If a diagram is both +adequate and −adequate, it is called *adequate*, and if it is only +adequate or −adequate, it is called *semi-adequate* [**5, 6**].

To every state diagram D_s we associate the graph G_s, whose vertices are state circles of D_s and edges are lines connecting state circles via smoothed crossings in D. Now we can restate Definition 2.1 in terms of G_s: D is s-adequate if G_s is loopless. A state graph G_s is called adequate if D_s is s-adequate.

DEFINITION 2.2. A link is *adequate* if it has an adequate (+adequate and −adequate) diagram. A link is *semi-adequate* if it has a + or −adequate diagram. A link is *inadequate* if it is neither + or −adequate [**5, 6**].

The mirror image of a diagram transforms the +adequacy into −adequacy and *vice versa*.

DEFINITION 2.3. A link that has one +adequate diagram and another diagram that is −adequate is called *weakly adequate*.

For example, knot $11n_{146}$ $9^*.-2:.-2$ has −adequate 11-crossing diagram and +adequate 12-crossing diagram $6^*-2.2.-2.2.2\,0.-2\,0$. Another such example is Perko's knot 10_{161} $3:-2\,0:-2\,0=6^*-2\,1.-1.2\,0.-1.2\,0.-1$ (Fig. 14) [**7**].

The torsion in the Khovanov homology carries additional information about knots and their cobordisms, not contained in the Jones polynomial and seems to be the best framework for determining adequacy. Classical theorems on adequacy (Theorems 2.1–2.5) follow almost instantly by using the basic properties of the Khovanov homology [**8, 9**].

A crossing in a link diagram for which there exists a circle in the projection plane intersecting the diagram transversely at that crossing, but not intersecting the diagram at any other point is called *nugatory* crossing. A link diagram is called *reduced* if it has no nugatory crossings. In this paper we work only with reduced KL diagrams. The following theorems hold for reduced alternating link diagrams:

THEOREM 2.1. *A reduced alternating diagram is adequate* [**5, 6, 10**].

Hence, all alternating links are adequate.

THEOREM 2.2. *An adequate diagram has minimal crossing number* [**5, 6, 10**].

Khovanov gave a direct simple proof of Theorem 2.2 in Section 7.7 [**8**].

THEOREM 2.3. *Every nonminimal reduced unlink diagram is inadequate. Semi-adequate reduced link diagrams are nontrivial* [**11**].

A nonminimal diagram of an adequate link can be semi-adequate or inadequate. For example, nonminimal diagram $3\,2\,4\;-2\,2$ of the alternating knot $3\,3\,2\,3$ is semi-adequate, and nonminimal diagram $3\,3\,4\;-1\,2$ of the alternating knot $3\,3\,2$ is inadequate.

A nonminimal diagram of a semi-adequate link also can be semi-adequate or inadequate. For example, nonminimal diagram $3,3,2,2\,-3$ and minimal diagram of the same knot $3,3,2,-2\,-2$ are both semi-adequate; minimal diagram of the knot $2\,1,3,-2$ is semi-adequate, and its nonminimal diagram $2\,1,3,2-$ is inadequate.

PROPOSITION 2.1. *Two adequate diagrams of a link have the same crossing number and the same writhe* [**10**]*, and the same number of circles in the Kauffman state s_+ (and the same for s_-)* [**8**].

DEFINITION 2.4. An alternating diagram of a marked 2-tangle t is called *strongly alternating* if both numerator closure $N(t)$ and denominator closure $D(t)$, are irreducible [**5, 6, 10**].

THEOREM 2.4. *Non-alternating sum of two strongly alternating tangles is adequate* [**5, 6, 10**].

This theorem is very effective in determining adequacy of certain types of link diagrams. For example, all semi-alternating diagrams [**5, 6**] are adequate.

According to Theorem 2.2, minimal diagrams can be used to determine if a link is adequate, but do not provide necessary and sufficient conditions to distinguish semi-adequate links from inadequate ones.

THEOREM 2.5. *A link is inadequate if both coefficients of the terms of highest and lowest degree of its Jones polynomial are different from* ± 1 [**5**].

By adding the q-degree in observation [**7**], we obtain a small improvement of this theorem in terms of the Khovanov homology:

PROPOSITION 2.2. *The link is not $+$adequate* (*resp. $-$adequate*) *if the rank of the homology in the highest* (*resp. lowest*) *nonvanishing q-degree is strictly larger than* 1.

It is worth mentioning the recent results describing torsion in the Khovanov homology of adequate and semi-adequate knots and links. The existence of Z_2-torsion in the Khovanov homology of a large class of adequate links was proved in [**4**]. Existence results were extended, using the modified chromatic graph cohomology, to semi-adequate knots [**12, 13**] and torsion was explicitly computed in [**13, 14**].

3. Adequate links with at most 12 crossings

Using *Knotscape* tables of knots given in the Dowker–Thistlethwaite notation, A. Stoimenow detected all nonalternating adequate knots up to 16 crossings. We consider adequacy of nonalternating links and their families (classes) given in Conway notation.

Adequate nonalternating links with up to 10 crossings are given in the following table:

$n=8$		
$2,2,-2,-2$	$(2,2)-(2,2)$	
2 Links		
$n=9$		
$3,2,-2,-2$	$(3,2)\ -(2,2)$	$(2\,1,2)\ -(2,2)$
$.-(2,2)$		
4 Links		
$n=10$		
$(3,2)\ -(3,2)$	$(3,2)\ -(2\,1,2)$	$(2\,1,2)\ -(2\,1,2)$
3 Knots		
$3,2\,1,-2,-2$	$3,3,-2,-2$	$3,-2,2\,1,-2$
$3,-2,3,-2$	$4,2,-2,-2$	$2,2,2,-2,-2$
$2\,2,2,-2,-2$	$(4,2)\ -(2,2)$	$(3,2\,1)\ -(2,2)$
$(3\,1,2)\ -(2,2)$	$(2\,1,2\,1)\ -(2,2)$	$(3,3)\ -(2,2)$
$(2\,1\,1,2)\ -(2,2)$	$(2,-2,-2)\,(2,2)$	$(2\,2,2)\ -(2,2)$
$(2,2,2)\ -(2,2)$	$(2,2),2,-(2,2)$	$.-(2,2).2$
$.-(2,2).2\,0$	$.-(2,2):2\,0$	$.-(2,2):2$
$103^*-1.-1.-1.-1::.-1$		
22 Links		

All of the KLs in the table, except the polyhedral ones, satisfy Theorem 2.4 or can be obtained by permuting rational tangles in Montesinos links which satisfy this theorem.

Theorem 2.5 and Proposition 2.2 give sufficient, but not necessary conditions for recognizing inadequate links. For example, the first and last coefficient of Jones polynomial of the knot $11n_{95} = 2\,0.-2\,1.-2\,0.2$ are different from ± 1, so it is inadequate [**10**]. However, since these theorems do not give necessary conditions for a link to be inadequate, the main problem remains detection of inadequate links.

For knots with at most 12 crossings every minimal diagram of a semi-adequate knot is semi-adequate. Unfortunately, this is not true for knots with 13 or more crossings: the first example of a semi-adequate knot with a minimal inadequate diagram (Fig. 12) is the knot $13n_{4084}$ $10^{**}.-1.-1.-1:.-2.2.-2$ with the minimal Dowker–Thistlethwaite code

$$\{\{13\},\{6,-10,12,24,20,-18,-26,-22,-4,2,-16,8,-14\}\}.$$

Except this inadequate diagram of writhe 9, it has another semi-adequate minimal diagram $11^{**}.-2::-2\,0:-1.-1.-1$ of writhe 7, with the Dowker–Thistlethwaite code

$$\{\{13\},\{6,12,-16,23,2,17,21,26,11,-4,-25,7,13\}\}$$

[**15, 16**]. The smallest examples of semi-adequate knots without a minimal semi-adequate diagram have 15 crossings. For example, knot $15n_{164563}$ has a unique minimal diagram $10^{**}-1.-2\,0.2\,0::.2\,0.2\,0.-2\,0$ which is inadequate (Fig. 13). However, it has a 16-crossing diagram $11^*2\,0.-1.-2.-1.3\,0.-1.2\,0::-1$ which is semi-adequate [**17**]. This example can be generalized to the family of knot diagrams $10^{**} - 1. - 2\,0.(2k)\,0 :: .2\,0.2\,0. - 2\,0$ and $11^{*}(2k)\,0. - 1. - 2. - 1.3\,0. - 1.2\,0 :: -1$, $k \geqslant 1$ with the same properties, respectively.

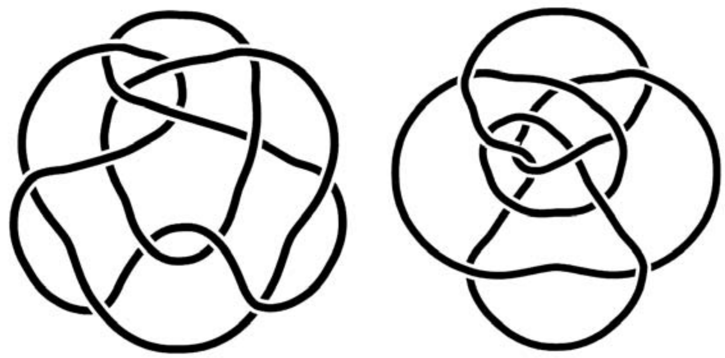

FIGURE 12. Semi-adequate knot $13n_{4084}$ with a minimal inadequate diagram $10^{**}. - 1. - 1. - 1 : . - 2.2. - 2$ and minimal semi-adequate diagram $11^{**}. - 2 :: -2\,0 : -1. - 1. - 1$ [**15, 16**].

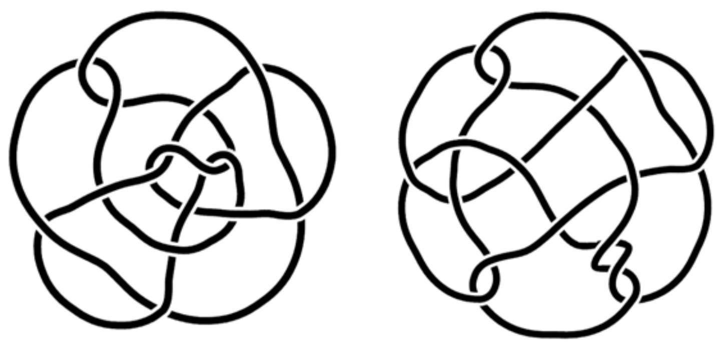

FIGURE 13. Semi-adequate knot $15n_{164563}$ which has only minimal diagram $10^{**} - 1. - 2\,0.2\,0 :: .2\,0.2\,0. - 2\,0$ which is inadequate and nonminimal 16-crossing diagram $11^{*}2\,0. - 1. - 2. - 1.3\,0. - 1.2\,0 :: -1$ which is semi-adequate [**17**].

For knots with at most 12 crossings we checked adequacy of all of their minimal diagrams, while we used only one for KLs with more than 12 crossings.

Note that different minimal diagrams of a given knot or a link can be +adequate or −adequate, hence we get a weakly adequate link.

The smallest[3] nonalternating knot 10_{161} whose minimal diagrams have different writhes is Perko pair: $3:-2\,0:-2\,0$ and $2\,1:-2\,0:-2\,0$ (see Fig. 14), and it is weakly adequate. Knots with this property are scarce in knot theory literature and hard to find. According to our computations, there are only 29 knots with this property among all knots with at most 12 crossings:

- Perko knot 10_{161} with 10 crossings,
- 3 knots: $11n_{116}$, $11n_{135}$, and $11n_{143}$ with 11 crossings, and
- 25 knots: $12n_{349}$, $12n_{382}$, $12n_{394}$, $12n_{398}$, $12n_{417}$, $12n_{430}$, $12n_{436}$, $12n_{519}$, $12n_{535}$, $12n_{552}$, $12n_{579}$, $12n_{594}$, $12n_{617}$, $12n_{624}$, $12n_{629}$, $12n_{638}$, $12n_{640}$, $12n_{644}$, $12n_{647}$, $12n_{650}$, $12n_{655}$, $12n_{739}$, $12n_{764}$, $12n_{850}$, and $12n_{851}$ with 12 crossings.

Notice that all of these knots are nonalgebraic.

According to our computations [**18**], a family of knots generated by the Perko knot, starting with knots $12n_{850}$, $14n_{26229}$, and $16n_{965076}$, given in *Knotscape* notation, has the same property: writhes of diagrams determined by Conway symbols $(2k+1):-2\,0:-2\,0$ and $(2k)\,1:-2\,0:-2\,0$ differ by 2. Diagrams $2\,(2k):-2\,0:-2\,0$ and $2\,(2k-1)\,1:-2\,0:-2\,0$, $k=1,2,3$, corresponding to knots $11n_{135}$, $13n_{3546}$, and $15n_{114094}$ share the same property.

We expect that this property is preserved within infinite families of KLs. For example, consider two families KL diagrams determining the same knot or link, $t\,(k+1):-2\,0:-2\,0$ and $t\,k\,1:-2\,0:-2\,0$, where t is a positive rational tangle[4] whose Conway symbol does not contain basic tangle 1 in the first place, and $k\in\mathbb{N}$. According to [**18**], $t\,(k+1):-2\,0:-2\,0$ is a knot if the tangle $t\,(k+1)$ is of the type [1] or [∞], and 2-component link if it is of the type [0]. Hence, the suitable choice of k and tangle t, yields a knot with two families of minimal diagrams $t\,(k+1):-2\,0:-2\,0$ and $t\,k\,1:-2\,0:-2\,0$ whose writhes differ by 2. As a corollary we get that for every $n\geqslant 10$ there exists at least one nonalternating knot with this property. Two minimal diagrams of the knot obtained for $t=2\,2$ and $k=3$ are illustrated in Fig. 15.

According our computations, all the mentioned knots with two minimal diagrams with different writhes are weakly adequate, so we conjecture

CONJECTURE 3.1. *Every knot which has minimal diagrams with different writhes is weakly adequate.*

At least for small number of crossings, most of nonalternating links are semi-adequate, so we attempt to tabulate adequate nonalternating links and candidates for inadequate links and find some general criteria for adequacy. Adequacy of all minimal diagrams of nonalternating KLs with at most 12 crossings was checked using *LinKnot* [**19**].

Among 202 nonalternating links with at most ten crossings there are only 28 adequate links and three adequate knots. Links with inadequate minimal diagrams are even more rare. There are only three 10 crossing links with inadequate minimal

[3]The term “smallest" refers to the minimal crossing number diagram of a given KL.

[4]A rational tangle is called positive if all numbers in its Conway symbol are positive.

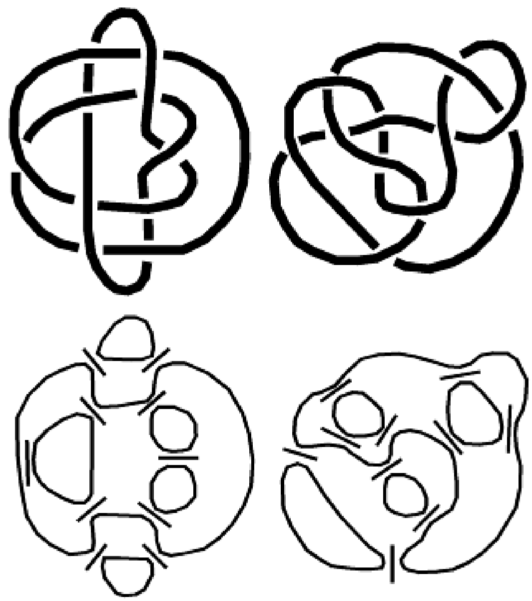

FIGURE 14. Perko pair: weakly-adequate knot with two minimal diagrams $6^*3 : -2\,0 : -2\,0$ and $6^* - 2\,1. - 1.2\,0. - 1.2\,0. - 1$ with the adequacy of different signs, where the first is +adequate, and the other −adequate.

diagrams: $2. - 2\,0. - 2.2\,0$ and $103^* - 1. - 1 :: -1. - 1$ are inadequate according to Theorem 2.5 and Proposition 2.2, and for the link $(2, 2, -2)\,(2, -2)$ we can not decide it is semi-adequate or inadequate.

Although the Khovanov homology is a stronger invariant than the Jones polynomial, so potentially it can give better criteria for adequacy, at the moment we do not have an example of a KL whose inadequacy is detected by the Khovanov homology, but not by the Jones polynomial. For example, Proposition 2.2 gives no new information for KLs with up to 12 crossings[5] based on the computations[6].

According to the computations, all inadequate knots and links with at most 12 crossings are thin in the Khovanov and odd Khovanov homologies.

Particular links, families, or classes of links which have all minimal inadequate diagrams will be referred by us as *candidates for inadequate links* and in some cases Theorem 2.5 or Proposition 2.2 confirm that they indeed are inadequate.

Candidates for inadequate knots occur for the first time among 11-crossing knots: knot $2\,0.{-2}\,1.{-2}\,0.2$ is inadequate according to Theorem 2.5 and Proposition 2.2, but for the knot $2\,0. - 3. - 2\,0.2$ which all minimal diagrams are inadequate, it

[5]The same holds for odd Khovanov homology.

[6]Khovanov homology can be efficiently computed for relatively large links by A. Shumakovitch's *KhoHo* [**20**], or Bar-Natan's *Knot Theory* [**21**]).

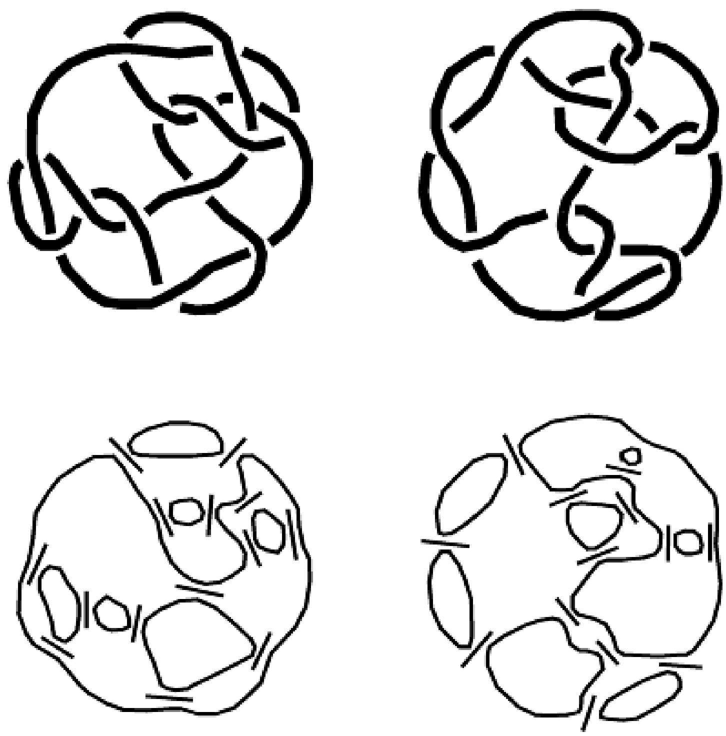

FIGURE 15. Perko-type pair of knot diagrams: weakly-adequate knot with two minimal diagrams $6^*2\,2\,3 : -2\,0 : -2\,0$ and $6^* - 2\,2\,2\,1. - 1.2\,0. - 1.2\,0. - 1$ with the adequacy of different signs, where the first is +adequate, and the other −adequate.

is not possible to make any conclusion, since both leading coefficients of its Jones polynomial are equal to 1.

Among 19 12-crossing knots with an inadequate minimal diagram, 11 knots given in the following table are inadequate according to Theorem 2.5 and Proposition 2.2

$2. - 2\,0. - 2.2\,1\,1\,0$	$2 : (-2, 2\,1)\,0 : -2\,0$	$2 : (2, -2\,1)\,0 : -2\,0$
$2.2. - 2.2\,0. - 2\,1$	$3. - 2\,0. - 2.2\,1\,0$	$3. - 2\,1\,0. - 2.2\,0$
$8^*2\,0. - 2\,0. - 2\,0.2\,0$	$8^* - 2\,1.2\,0. - 2$	$9^*. - 2 : -2\,0. - 2$
$101^* - 2\,0 :: . - 2\,0$	$102^* - 2\,0 :: -2$	

while inadequacy of the remaining 8 knots from the following table remains unknown

$2.-3\,0.-2\,1.2\,0$	$8^*-2\,1\,1::-2\,0$	$8^*2:.-2\,0:.-2\,1\,0$
$8^*-2\,1::-3\,0$	$8^*2:.-2\,1\,0:.-2\,0$	$8^*-2\,0.2:-2\,1\,0$
$8^*-2.2.-2\,0:2\,0$	$8^*-2\,0:-2\,0:-2\,0:2\,0$	

The following 11-crossing links $(2\,1,2)\ -1\ -1\,(2,2)$, $(2,2),-2,-1,(2,-2)$, $6^*3.-2\,0.-2.2\,0$, $6^*(2,-2).2.-2$ are inadequate according to Theorem 2.5 and Proposition 2.2, while 8 links in the following table are candidates for inadequate links:

$(-2\,1,2)\,1\,1\,(2,-2)$	$(-2\,1,2,2)\,(2,-2)$	$(2\,1,2,-2)\,(2,-2)$	$(2,2)\ -1\ -1\ -1\,(2,-2)$
$(2,2,-2)\,(-2\ -1,2)$	$(2,2,-2)\,(2\,1,-2)$	$6^*2\,1.-2\,0.-2.2\,0$	$6^*(2,-2),-2$

Using the same method, we confirm inadequacy of 63 12-crossing links, while the remaining 232 links are candidates for inadequate links.

It is interesting to notice that all inadequate links with at most 12 crossings are Khovanov homology thin. The first three inadequate Khovanov homology thick knots, $14n_{11449}$, $14n_{12713}$, and $14n_{22178}$, occur among 526 inadequate knots with $n=14$ crossings, and all of them are polyhedral. First algebraic inadequate Khovanov homology thick knot is $15n_{5429}$ $-(2,3)\,1,(2\,1,-2)\,1,2+$ (Fig. 16).

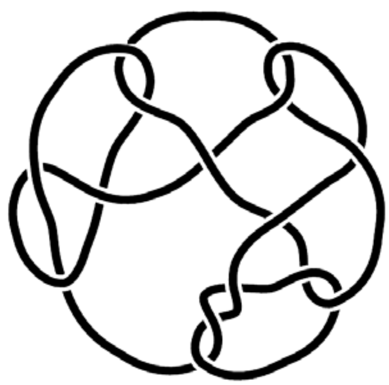

FIGURE 16. Algebraic inadequate Khovanov homology thick knot $15n_{5429}$ $-(2,3)\,1,(2\,1,-2)\,1,2+$.

Tables of adequate nonalternating links with at most 12 crossings in Conway notation can be downloaded in the form of *Mathematica* notebook from the address: `http://www.mi.sanu.ac.rs/vismath/adequate.pdf`

4. Families and classes of links and their adequacy

In this section we analyze adequacy of KL families given in Conway notation.

THEOREM 4.1. *The plus or minus adequacy property is preserved within families of link diagrams.*

PROOF. Let us start from a source link L. If we substitute $a_i \in \tilde{S}$, $a_i \neq \pm 1$, by $sgn(a_i)(|a_i|+1)$ (Definition 1.5), a new state circle of the length 2 appears in one of the states D_{s_+} or D_{s_-}, so the adequacy stays the same. In the other state, the number of state circles remains unchanged and all state circles associated with the new crossing obtain one new touching point. If the crossings of the original tangle a after smoothing correspond to different state circles, the same holds for the tangle $sgn(a_i)(|a_i|+1)$, which does not change the adequacy type. By induction, we conclude that this property holds for every $k_{a_i} \in \mathbb{N}$ (see Def. 1.5). Hence, all link diagrams which belong to the same family of diagrams have the same adequacy. $\square$

PROPOSITION 4.1. *The adequacy type of a link diagram D remains unchanged if we replace every positive rational tangle by* 2*, and every negative rational tangle by* -2.

The proof of this proposition is straightforward.

THEOREM 4.2. *A nonalternating Montesinos link $t_1, t_2, \ldots, t_n$ is semi-adequate if its all but one rational tangle have the same sign. Otherwise, it is adequate.*

DEFINITION 4.1. A tangle $P_n = t_1, t_2, \ldots, t_n$ $(n \geqslant 2)$ is called adequate or semi-adequate if its corresponding link, obtained as its numerator closure, is adequate or semi-adequate, respectively.

DEFINITION 4.2. An alternating tangle $P_n = t_1, t_2, \ldots, t_n$ $(n \geqslant 2)$ is called +alternating if all its rational tangles t_i are positive, and −alternating if they are all negative.

For example, tangle $t_1, -t_2$ is inadequate, if t_1, t_2 are positive rational tangles.

Let us denote source link of the form $2, \ldots, -2, \ldots$, where 2 occurs k times, and -2 occurs l times with $(2)^k, (-2)^l$. As a corollary of Theorem 2.1 and Theorem 4.2, we obtain six classes of source links, with the same adequacy type shown in the table below:

$k \geqslant 3,\ l = 0$	+alternating
$k = 0,\ l \geqslant 3$	−alternating
$k \geqslant 2,\ l \geqslant 2$	adequate
$k = 1,\ l \geqslant 2$	+adequate
$k \geqslant 2,\ l = 1$	−adequate
$k = l = 1$	inadequate

Minimal representatives of these 6 classes are source links $(2,2,2)$, $(-2,-2,-2)$, $(2,2,-2,-2)$, $(-2,-2,2)$, $(2,2,-2)$, and $(2,-2)$, respectively. Based on Proposition 4.1, we view these six source links as the representatives of Montesinos links containing rational tangles with given signs. For example, source link $2,2,-2,-2$ can be used as the representative of all nonalternating adequate Montesinos links of the form $t_1, \ldots, t_k, -t'_1, \ldots, -t'_l$, $(k \geqslant 2,\ l \geqslant 2)$, where t_i, $i = 1,2,\ldots,k$, t'_j, $j = 1,2,\ldots,l$ are positive rational tangles different from 1.

5. Examples: classes of algebraic links and their adequacy

Results in Section 5 and Section 6 follow from Proposition 4.1 and computer computations.

PROPOSITION 5.1. *A link* $P_m\,Q_n = (p_1, p_2, \dots, p_m)\,(q_1, q_2, \dots, q_n)$ *(*$m, n \geqslant 2$*,* $p_i \neq \pm 1$*,* $q_j \neq \pm 1$*,* $i = 1, 2, \dots, m$*,* $j = 1, 2, \dots, n$*) obtained as the product of tangles* P_m *and* Q_n *is adequate if*

- *both* P_m *and* Q_n *are adequate; or*
- *one of them is +alternating, and the other +adequate; or*
- *one of them is −alternating, and the other −adequate.*

It is semi-adequate if

- *one of them is adequate, and the other semi-adequate; or*
- *one of them is +adequate, and the other −adequate; or*
- *if one of them inadequate, and the other an alternating tangle.*

It is candidate for inadequate if

- *both* P_m *and* Q_n *are +adequate or −adequate;*
- *if one of them is inadequate, and the other is not an alternating tangle.*

From the preceding proposition we obtain the following multiplication table, where $*$ denotes the product of tangles P_1 and P_2[7]:

$*$	**+alt**	**−alt**	**adq**	**+adq**	**−adq**	**inadq**
+alt	+alt	adq	adq	adq	+adq	+adq
−alt	adq	−alt	adq	−adq	adq	−adq
adq	adq	adq	adq	−adq	+adq	inadeq
+adq	adq	+adq	+adq	inadeq	+adq	inadeq
−adq	−adq	adq	−adq	−adq	inadeq	inadeq
inadq	−adq	+adq	inadeq	inadeq	inadeq	

For links of the form $P_m\,Q_n = (p_1, p_2, \dots, p_n)\,(q_1, q_2, \dots, q_n)$ we obtain general rules for adequacy, expressed as the multiplication table. Unfortunately, for links of the form $P_1\,P_2\,\dots\,P_k$, with $k \geqslant 2$ we could not determine such a rule.

The following tangles are the minimal representatives of tangles $P_1, \dots, P_k$ ($k \geqslant 2$) with the properties **+alt**, **−alt**, **adq**, **+adq**, **−adq**, and **inadeq**:

1	**+alt**	$2, 2, 2$
2	**−alt**	$-2, -2, -2$
3	**adq**	$2, 2, -2, -2$
4	**+adq**	$-2, -2, 2$
5	**−adq**	$2, 2, -2$
6	**inadeq**	$2, -2$

[7]The product $P_1\,P_2$ of inadequate tangles P_1 and P_2 is omitted, since it represents a non-minimal diagram of an alternating link.

If we denote the properties **+alt**, **−alt**, **adq**, **+adq**, **−adq**, and **inadeq** by numbers 1–6, for $k = 3$, then we have the following statement:

PROPOSITION 5.2. *The links $P_1\,P_2\,P_3$ are adequate for the following properties of tangles P_1, P_2, P_3:*

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 1	1, 2, 2	1, 2, 3	1, 3, 1
1, 3, 2	1, 3, 3	1, 4, 1	1, 4, 2	1, 4, 3	1, 5, 2	1, 6, 2	2, 1, 2
2, 1, 3	2, 2, 2	2, 2, 3	2, 2, 5	2, 3, 2	2, 3, 3	2, 5, 2	2, 5, 3
3, 1, 3	3, 1, 4	3, 2, 3	3, 2, 5	3, 3, 3	4, 1, 4	5, 2, 5	

semi-adequate for:

1, 1, 5	1, 1, 6	1, 2, 4	1, 2, 5	1, 3, 4	1, 3, 5	1, 4, 4	1, 4, 5
1, 5, 1	1, 5, 3	1, 5, 4	1, 6, 1	1, 6, 3	1, 6, 4	2, 1, 4	2, 1, 5
2, 2, 4	2, 2, 6	2, 3, 4	2, 3, 5	2, 4, 2	2, 4, 3	2, 4, 5	2, 5, 4
2, 5, 5	2, 6, 2	2, 6, 3	2, 6, 5	3, 1, 5	3, 1, 6	3, 2, 4	3, 2, 6
3, 3, 4	3, 3, 5	3, 4, 3	3, 4, 5	3, 5, 3	3, 5, 4	4, 1, 5	4, 1, 6
4, 2, 4	4, 2, 5	4, 2, 6	4, 3, 4	4, 5, 4	5, 1, 5	5, 1, 6	5, 2, 6
5, 3, 5	5, 4, 5	6, 1, 6	6, 2, 6				

and candidates for inadequate for:

1, 2, 6	1, 3, 6	1, 4, 6	1, 5, 5	1, 5, 6	1, 6, 5	1, 6, 6	2, 1, 6
2, 3, 6	2, 4, 4	2, 4, 6	2, 5, 6	2, 6, 4	2, 6, 6	3, 3, 6	3, 4, 4
3, 4, 6	3, 5, 5	3, 5, 6	3, 6, 3	3, 6, 4	3, 6, 5	3, 6, 6	4, 3, 5
4, 3, 6	4, 4, 4	4, 4, 5	4, 4, 6	4, 5, 5	4, 5, 6	4, 6, 4	4, 6, 5
4, 6, 6	5, 3, 6	5, 4, 6	5, 5, 5	5, 5, 6	5, 6, 5	5, 6, 6	6, 3, 6
6, 4, 6	6, 5, 6	6, 6, 6					

The results hold for all sequences a, b, c ($a, b, c \in \{1, 2, \dots, 6\}$) and their reverses. Analogous tables are obtained by computer calculations for all $k \leqslant 6$.

For a given nonalternating Montesinos tangle P, the tangle P' obtained by replacing every rational positive or negative tangle t_i with the tangle $sign(t_i) \times 2$ will be called the source Montesinos tangle.

PROPOSITION 5.3. *The links $P_1\,P_2\,\dots\,P_k$ and $P'_1\,P'_2\,\dots\,P'_k$ have the same adequacy.*

Next, we consider links of the form $P_1, P_2, \dots, P_k$ ($k \geqslant 3$), where P_i ($i = 1, \dots, k$) are Montesinos tangles of the length greater then 1. Since permutation of Montesinos tangles preserves the sign of adequacy, the result holds for every sequence a, b, c ($a, b, c \in \{1, 2, \dots, 6\}$) and all of its permutations.

PROPOSITION 5.4. *The links P_1, P_2, P_3 are adequate for the following properties of Montesinos tangles P_1, P_2, P_3:*

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 2	1, 2, 3	1, 3, 3	1, 3, 4
1, 4, 4	2, 2, 2	2, 2, 3	2, 2, 5	2, 3, 3	2, 3, 5	2, 5, 5	3, 3, 3
3, 3, 4	3, 3, 5	3, 3, 6	3, 4, 4	3, 4, 5	3, 4, 6	3, 5, 5	3, 5, 6
3, 6, 6	4, 4, 4	4, 4, 5	4, 4, 6	4, 5, 5	4, 5, 6	4, 6, 6	5, 5, 5
5, 5, 6	5, 6, 6	6, 6, 6					

semi-adequate for:

$1,2,4$	$2,2,4$	$2,2,6$	$2,3,4$	$2,3,6$	$2,4,4$	$2,4,5$	$2,4,6$
$2,5,6$	$2,6,6$	$1,1,5$	$1,1,6$	$1,2,5$	$1,3,5$	$1,3,6$	$1,4,5$
$1,4,6$	$1,5,5$	$1,5,6$	$1,6,6$				

and candidates for inadequate for $1,2,6$.

Analogous results are obtained by computer calculations for all $k \leqslant 6$.

Next we consider links of the form $(P_1, P_2, \ldots, P_m)\,(Q_1, Q_2, \ldots, Q_n)$ $(m, n \geqslant 2)$, where P_i and Q_j $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ are Montesinos tangles.

For $m = n = 2$, and the sequence $(a, b)\,(c, d)$, $a, b, c, d \in \{1, 2, \ldots, 6\}$, where a and b, and c and d can commute and all sequences $(a, b)\,(c, d)$ can be reversed, we obtain the following result:

Proposition 5.5. *Links given in Conway notation by $(P_1, P_2)\,(Q_1, Q_2)$ are adequate for the following properties of Montesinos tangles P_1, P_2, Q_1, Q_2:*

$(1,1)\,(1,1)$	$(1,1)\,(1,2)$	$(1,1)\,(1,3)$	$(1,1)\,(1,4)$	$(1,1)\,(2,2)$	$(1,1)\,(2,3)$	$(1,1)\,(3,3)$
$(1,1)\,(3,4)$	$(1,1)\,(4,4)$	$(1,2)\,(1,2)$	$(1,2)\,(1,3)$	$(1,2)\,(1,4)$	$(1,2)\,(2,2)$	$(1,2)\,(2,3)$
$(1,2)\,(2,5)$	$(1,2)\,(3,3)$	$(1,2)\,(3,4)$	$(1,2)\,(3,5)$	$(1,2)\,(3,6)$	$(1,2)\,(4,4)$	$(1,2)\,(4,5)$
$(1,2)\,(4,6)$	$(1,2)\,(5,5)$	$(1,2)\,(5,6)$	$(1,2)\,(6,6)$	$(1,3)\,(1,3)$	$(1,3)\,(1,4)$	$(1,3)\,(2,2)$
$(1,3)\,(2,3)$	$(1,3)\,(3,3)$	$(1,3)\,(3,4)$	$(1,3)\,(4,4)$	$(1,4)\,(1,4)$	$(2,2)\,(2,2)$	$(2,2)\,(2,3)$
$(2,2)\,(2,5)$	$(2,2)\,(3,3)$	$(2,2)\,(3,5)$	$(2,2)\,(5,5)$	$(2,3)\,(2,3)$	$(2,3)\,(2,5)$	$(2,3)\,(3,3)$
$(2,3)\,(3,5)$	$(2,3)\,(5,5)$	$(2,5)\,(2,5)$	$(3,3)\,(3,3)$			

semi-adequate for:

$(1,1)\,(1,5)$	$(1,1)\,(1,6)$	$(1,1)\,(2,5)$	$(1,1)\,(3,5)$	$(1,1)\,(3,6)$	$(1,1)\,(4,5)$	$(1,1)\,(4,6)$
$(1,1)\,(5,5)$	$(1,1)\,(5,6)$	$(1,1)\,(6,6)$	$(1,2)\,(1,5)$	$(1,2)\,(1,6)$	$(1,3)\,(1,5)$	$(1,3)\,(1,6)$
$(1,3)\,(2,5)$	$(1,3)\,(3,5)$	$(1,3)\,(3,6)$	$(1,3)\,(4,5)$	$(1,3)\,(4,6)$	$(1,3)\,(5,5)$	$(1,3)\,(5,6)$
$(1,3)\,(6,6)$	$(1,4)\,(1,5)$	$(1,4)\,(1,6)$	$(1,4)\,(2,2)$	$(1,4)\,(2,3)$	$(1,4)\,(2,5)$	$(1,4)\,(3,3)$
$(1,4)\,(3,4)$	$(1,4)\,(3,5)$	$(1,4)\,(3,6)$	$(1,4)\,(4,4)$	$(1,4)\,(4,5)$	$(1,4)\,(4,6)$	$(1,4)\,(5,5)$
$(1,4)\,(5,6)$	$(1,4)\,(6,6)$	$(2,4)\,(2,5)$	$(2,4)\,(3,3)$	$(2,4)\,(3,5)$	$(2,4)\,(5,5)$	$(3,3)\,(3,5)$
$(3,3)\,(5,5)$	$(3,4)\,(3,5)$	$(3,4)\,(5,5)$	$(4,4)\,(5,5)$	$(1,1)\,(2,4)$	$(1,2)\,(2,4)$	$(1,2)\,(2,6)$
$(1,3)\,(2,4)$	$(1,5)\,(2,2)$	$(1,5)\,(2,3)$	$(1,5)\,(2,4)$	$(1,5)\,(3,3)$	$(1,5)\,(3,4)$	$(1,5)\,(4,4)$
$(2,2)\,(2,4)$	$(2,2)\,(2,6)$	$(2,2)\,(3,4)$	$(2,2)\,(3,6)$	$(2,2)\,(4,4)$	$(2,2)\,(4,5)$	$(2,2)\,(4,6)$
$(2,2)\,(5,6)$	$(2,2)\,(6,6)$	$(2,3)\,(2,4)$	$(2,3)\,(2,6)$	$(2,3)\,(3,4)$	$(2,3)\,(3,6)$	$(2,3)\,(4,4)$
$(2,3)\,(4,5)$	$(2,3)\,(4,6)$	$(2,3)\,(5,6)$	$(2,3)\,(6,6)$	$(2,5)\,(2,6)$	$(2,5)\,(3,3)$	$(2,5)\,(3,4)$
$(2,5)\,(3,5)$	$(2,5)\,(3,6)$	$(2,5)\,(4,4)$	$(2,5)\,(4,5)$	$(2,5)\,(4,6)$	$(2,5)\,(5,5)$	$(2,5)\,(5,6)$
$(2,5)\,(6,6)$	$(3,3)\,(3,4)$	$(3,3)\,(4,4)$	$(3,5)\,(4,4)$			

and candidates for inadequate for:

$(1,1)\,(2,6)$	$(1,3)\,(2,6)$	$(1,4)\,(2,4)$	$(1,4)\,(2,6)$	$(1,5)\,(1,5)$	$(1,5)\,(1,6)$	$(1,5)\,(2,5)$
$(1,5)\,(2,6)$	$(1,5)\,(3,5)$	$(1,5)\,(3,6)$	$(1,5)\,(4,5)$	$(1,5)\,(4,6)$	$(1,5)\,(5,5)$	$(1,5)\,(5,6)$
$(1,5)\,(6,6)$	$(1,6)\,(1,6)$	$(1,6)\,(2,2)$	$(1,6)\,(2,3)$	$(1,6)\,(2,4)$	$(1,6)\,(2,5)$	$(1,6)\,(2,6)$
$(1,6)\,(3,3)$	$(1,6)\,(3,4)$	$(1,6)\,(3,5)$	$(1,6)\,(3,6)$	$(1,6)\,(4,4)$	$(1,6)\,(4,5)$	$(1,6)\,(4,6)$
$(1,6)\,(5,5)$	$(1,6)\,(5,6)$	$(1,6)\,(6,6)$	$(2,4)\,(2,4)$	$(2,4)\,(2,6)$	$(2,4)\,(3,4)$	$(2,4)\,(3,6)$
$(2,4)\,(4,4)$	$(2,4)\,(4,5)$	$(2,4)\,(4,6)$	$(2,4)\,(5,6)$	$(2,4)\,(6,6)$	$(2,6)\,(2,6)$	$(2,6)\,(3,3)$
$(2,6)\,(3,4)$	$(2,6)\,(3,5)$	$(2,6)\,(3,6)$	$(2,6)\,(4,4)$	$(2,6)\,(4,5)$	$(2,6)\,(4,6)$	$(2,6)\,(5,5)$
$(2,6)\,(5,6)$	$(2,6)\,(6,6)$	$(3,3)\,(3,6)$	$(3,3)\,(4,5)$	$(3,3)\,(4,6)$	$(3,3)\,(5,6)$	$(3,3)\,(6,6)$

$(3,4)\,(3,4)$	$(3,4)\,(3,6)$	$(3,4)\,(4,4)$	$(3,4)\,(4,5)$	$(3,4)\,(4,6)$	$(3,4)\,(5,6)$	$(3,4)\,(6,6)$
$(3,5)\,(3,5)$	$(3,5)\,(3,6)$	$(3,5)\,(4,5)$	$(3,5)\,(4,6)$	$(3,5)\,(5,5)$	$(3,5)\,(5,6)$	$(3,5)\,(6,6)$
$(3,6)\,(3,6)$	$(3,6)\,(4,4)$	$(3,6)\,(4,5)$	$(3,6)\,(4,6)$	$(3,6)\,(5,5)$	$(3,6)\,(5,6)$	$(3,6)\,(6,6)$
$(4,4)\,(4,4)$	$(4,4)\,(4,5)$	$(4,4)\,(4,6)$	$(4,4)\,(5,6)$	$(4,4)\,(6,6)$	$(4,5)\,(4,5)$	$(4,5)\,(4,6)$
$(4,5)\,(5,5)$	$(4,5)\,(5,6)$	$(4,5)\,(6,6)$	$(4,6)\,(4,6)$	$(4,6)\,(5,5)$	$(4,6)\,(5,6)$	$(4,6)\,(6,6)$
$(5,5)\,(5,5)$	$(5,5)\,(5,6)$	$(5,5)\,(6,6)$	$(5,6)\,(5,6)$	$(5,6)\,(6,6)$	$(6,6)\,(6,6)$	

Analogous results are obtained by computer calculations for $m, n \leqslant 4$.

Furthermore, we consider links of the form $P_1, t_1, t_2, \ldots, t_n$, where P_1 is a Montesinos tangle, and t_i, $i = 1, 2, \ldots, n$, $n \geqslant 2$ are rational tangles. If $P = t_1, t_2, \ldots, t_n$, we have the following statement:

- links of the given form are adequate if $\{P_1, P\} \in \{\{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,2\}, \{2,3\}, \{2,5\}, \{3,3\}\}$;
- semi-adequate if $\{P_1, P\} \in \{\{1,5\}, \{1,6\}, \{2,4\}, \{2,6\}, \{3,4\}, \{3,5\}, \{4,5\}\}$;
- and candidates for inadequate if $\{P_1, P\} \in \{\{3,6\}, \{4,4\}, \{4,6\}, \{5,5\}, \{5,6\}, \{6,6\}\}$.

We can generalize the previous family to links with many Montesinos tangles $P_1,\ldots,P_m$ and many rational tangles t_i, $i = 1, 2, \ldots, n$, $n \geqslant 2$ denoted by $P_1, \ldots, P_m, t_1, t_2, \ldots, t_n$. If $P = t_1, t_2, \ldots, t_n$, for $m = 2$ we have the following statement:

Proposition 5.6. *Links of the given form are adequate if* $(\{P_1, P_2\}, P)$ *is:*

$(\{1,1\},1)$	$(\{1,1\},2)$	$(\{1,1\},3)$	$(\{1,1\},4)$	$(\{1,2\},1)$	$(\{1,2\},2)$
$(\{1,2\},3)$	$(\{1,2\},4)$	$(\{1,2\},5)$	$(\{1,2\},6)$	$(\{1,3\},1)$	$(\{1,3\},2)$
$(\{1,3\},3)$	$(\{1,3\},4)$	$(\{1,4\},1)$	$(\{2,2\},1)$	$(\{2,2\},2)$	$(\{2,2\},3)$
$(\{2,2\},5)$	$(\{2,3\},1)$	$(\{2,3\},2)$	$(\{2,3\},3)$	$(\{2,3\},5)$	$(\{2,5\},2)$
$(\{3,3\},1)$	$(\{3,3\},2)$	$(\{3,3\},3)$	$(\{3,4\},1)$	$(\{3,5\},2)$	$(\{4,4\},1)$
$(\{5,5\},2)$					

semi-adequate if $(\{P_1, P_2\}, P)$ *is:*

$(\{1,1\},5)$	$(\{1,1\},6)$	$(\{1,3\},5)$	$(\{1,3\},6)$	$(\{1,4\},2)$	$(\{1,4\},3)$
$(\{1,4\},4)$	$(\{1,4\},5)$	$(\{1,4\},6)$	$(\{1,5\},1)$	$(\{1,5\},2)$	$(\{1,5\},3)$
$(\{1,5\},4)$	$(\{1,6\},1)$	$(\{2,2\},4)$	$(\{2,2\},6)$	$(\{2,3\},4)$	$(\{2,3\},6)$
$(\{2,4\},1)$	$(\{2,4\},2)$	$(\{2,4\},3)$	$(\{2,4\},5)$	$(\{2,5\},1)$	$(\{2,5\},3)$
$(\{2,5\},4)$	$(\{2,5\},5)$	$(\{2,5\},6)$	$(\{2,6\},2)$	$(\{3,3\},4)$	$(\{3,3\},5)$
$(\{3,4\},2)$	$(\{3,4\},3)$	$(\{3,4\},5)$	$(\{3,5\},1)$	$(\{3,5\},3)$	$(\{3,5\},4)$
$(\{3,6\},1)$	$(\{3,6\},2)$	$(\{4,4\},2)$	$(\{4,4\},3)$	$(\{4,4\},5)$	$(\{4,5\},1)$
$(\{4,5\},2)$	$(\{4,6\},1)$	$(\{4,6\},2)$	$(\{5,5\},1)$	$(\{5,5\},3)$	$(\{5,5\},4)$
$(\{5,6\},1)$	$(\{5,6\},2)$	$(\{6,6\},1)$	$(\{6,6\},2)$		

and candidates for inadequate if $(\{P_1, P_2\}, P)$ *is:*

$(\{1,5\},5)$	$(\{1,5\},6)$	$(\{1,6\},2)$	$(\{1,6\},3)$	$(\{1,6\},4)$	$(\{1,6\},5)$
$(\{1,6\},6)$	$(\{2,4\},4)$	$(\{2,4\},6)$	$(\{2,6\},1)$	$(\{2,6\},3)$	$(\{2,6\},4)$
$(\{2,6\},5)$	$(\{2,6\},6)$	$(\{3,3\},6)$	$(\{3,4\},4)$	$(\{3,4\},6)$	$(\{3,5\},5)$
$(\{3,5\},6)$	$(\{3,6\},3)$	$(\{3,6\},4)$	$(\{3,6\},5)$	$(\{3,6\},6)$	$(\{4,4\},4)$

$(\{4,4\},6)$	$(\{4,5\},3)$	$(\{4,5\},4)$	$(\{4,5\},5)$	$(\{4,5\},6)$	$(\{4,6\},3)$
$(\{4,6\},4)$	$(\{4,6\},5)$	$(\{4,6\},6)$	$(\{5,5\},5)$	$(\{5,5\},6)$	$(\{5,6\},3)$
$(\{5,6\},4)$	$(\{5,6\},5)$	$(\{5,6\},6)$	$(\{6,6\},3)$	$(\{6,6\},4)$	$(\{6,6\},5)$

Other examples, obtained by direct computer calculations include links of the form $P_1\,p\,P_2$, where P_1, P_2 are Montesinos tangles, and p is a positive p-twist. These links are adequate if $\{P_1,P_2\} \in \{\{1,1\},\{1,3\},\{1,4\},\{3,3\},\{3,4\},\{4,4\}\}$, candidates for inadequate if $\{P_1,P_2\} \in \{\{2,5\},\{2,6\}\}$, and semi-adequate otherwise.

6. Adequacy of polyhedral links

This section contains computational results about adequacy of polyhedral links derived from the basic polyhedron 6^*, and how it changes with substituting Montesinos tangles in vertices.

PROPOSITION 6.1. *Every polyhedral link derived from the basic polyhedron 6^*, with one Montesinos tangle P_1 and positive rational tangles in other vertices is adequate if $P_1 \in \{1,2,3,4\}$, and semi-adequate if $P_1 \in \{5,6\}$.*

PROPOSITION 6.2. *For every adequate polyhedral link derived from the basic polyhedron 6^*, with two Montesinos tangles P_1, P_2 and positive rational tangles in other vertices, $P_1 \notin \{5,6\}$ and $P_2 \notin \{5,6\}$.*

Since we do not have sufficient conditions for adequacy, we consider different conditions on polyhedral source links. For example, the following results hold for nonalternating links derived from the basic polyhedron 6^* with two Montesinos tangles P_1 and P_2 and positive rational tangles in the remaining vertices:

- a link of the form $6^*P_1.P_2.t_1.t_2.t_3.t_4$ is adequate if $\{P_1,P_2\} \in \{\{1,1\},\{1,2\},\{1,3\},\{1,4\},\{3,3\},\{3,4\},\{4,4\}\}$, a candidate for inadequate if $\{P_1,P_2\} \in \{\{2,2\},\{5,5\},\{5,6\},\{6,6\}\}$, and semi-adequate otherwise;
- a link of the form $6^*P_1.P_2\,0.t_1.t_2.t_3.t_4$ is adequate if $\{P_1,P_2\} \in \{\{1,1\},\{1,2\},\{1,3\},\{1,4\}\}$, a candidate for inadequate if $\{P_1,P_2\} \in \{\{2,2\}\}$, and semi-adequate otherwise;
- a link of the form $6^*P_1.t_1.t_2.P_2\,0.t_3.t_4$ is adequate if $P_1 \notin \{5,6\}$ and $P_2 \notin \{5,6\}$, and semi-adequate otherwise;
- a link of the form $6^*P_1.t_1.t_2.P_2.t_3.t_4$ is adequate if $P_1 \notin \{5,6\}$ and $P_2 \notin \{5,6\}$, and a candidate for inadequate if $\{P_1,P_2\} = \{5,6\}$.

PROPOSITION 6.3. *A link of the form $6^*P_1.P_2.P_3.t_1.t_2.t_3$ is adequate for the following triples (P_1,P_2,P_3):*

1, 1, 1	1, 1, 2	1, 1, 3	1, 1, 4	1, 2, 1	1, 2, 3	1, 2, 4	1, 3, 1
1, 3, 3	1, 3, 4	1, 4, 1	1, 4, 3	1, 4, 4	2, 1, 1	2, 1, 3	2, 1, 4
3, 1, 1	3, 1, 2	3, 1, 3	3, 1, 4	3, 2, 1	3, 2, 3	3, 2, 4	3, 2, 5
3, 2, 6	3, 3, 1	3, 3, 3	3, 3, 4	3, 4, 1	4, 1, 1	4, 1, 2	4, 1, 3
4, 1, 4	4, 2, 1	4, 2, 3	4, 2, 4	4, 2, 5	4, 2, 6	4, 3, 1	4, 3, 3
4, 3, 4	4, 4, 1	5, 2, 3	5, 2, 4	5, 2, 5	5, 2, 6	6, 2, 3	6, 2, 4
6, 2, 5	6, 2, 6						

a candidate for inadequate for:

$1,5,5$	$1,5,6$	$1,6,5$	$1,6,6$	$2,3,5$	$2,3,6$	$2,4,2$	$2,4,3$
$2,4,4$	$2,4,5$	$2,4,6$	$2,5,5$	$2,5,6$	$2,6,2$	$2,6,3$	$2,6,4$
$2,6,5$	$2,6,6$	$3,4,2$	$3,5,5$	$3,5,6$	$3,6,2$	$3,6,3$	$3,6,4$
$3,6,5$	$3,6,6$	$4,4,2$	$4,5,5$	$4,5,6$	$4,6,2$	$4,6,3$	$4,6,4$
$4,6,5$	$4,6,6$	$5,3,2$	$5,4,2$	$5,5,1$	$5,5,2$	$5,5,3$	$5,5,4$
$5,5,5$	$5,5,6$	$5,6,1$	$5,6,2$	$5,6,3$	$5,6,4$	$5,6,5$	$5,6,6$
$6,3,2$	$6,4,2$	$6,5,1$	$6,5,2$	$6,5,3$	$6,5,4$	$6,5,5$	$6,5,6$
$6,6,1$	$6,6,2$	$6,6,3$	$6,6,4$	$6,6,5$	$6,6,6$		

and semi-adequate otherwise.

In addition to polyhedral links with Montesinos tangles, derived from the basic polyhedron 6^*, we consider polyhedral links containing only rational tangles.

PROPOSITION 6.4. *Non-alternating link derived from the basic polyhedron* 6^* *is a candidate for inadequate if it is obtained from one of the following source links by replacing* 2*-tangles by positive rational tangles* t_i*,* $i \in \{1, \dots, 6\}$*,* $t_i \neq 1$

$6^*2.-2\,0.-2.2\,0$	$6^*2.2.-2.2.-2\,0$	$6^*-2.2.-2\,0.2.2$
$6^*2.2.-2.2\,0.-2$	$6^*-2.2\,0.-2.2\,0.2$	$6^*2.-2.2.2.2\,0.-2\,0$
$6^*2.-2.2\,0.-2.-2.-2\,0$	$6^*2.-2.-2.-2.2.-2\,0$	$6^*2.-2.2.2.2.-2\,0$
$6^*2.-2.-2.-2\,0.2.-2\,0$	$6^*2.-2\,0.-2.-2\,0.-2.2\,0$	$6^*2.-2\,0.-2.2\,0.-2.2\,0$

and semi-adequate otherwise[8].

7. Adequacy of mixed states and adequacy number

The definition of adequacy can be extended to an arbitrary mixed state of a link diagram D containing both positive and negative markers.

According to Definition 2.1, a state s of the diagram D is an *adequate state* if two segments of D_s obtained by smoothing the same crossing belong to different state circles.

THEOREM 7.1. *Every link diagram has at least two adequate states.*

PROOF. Every alternating link diagram is adequate, so its states s_+ and s_- are adequate. Note that every nonalternating link diagram can be transformed into an alternating diagram and its mirror image by crossing changes which correspond to changes between positive and negative markers. Hence, two adequate states of a nonalternating diagram can be obtained by appropriate choice of markers corresponding to crossing changes transforming the nonalternating diagram to the alternating one. □

[8] Knot $6^*2.-2.2\,0.-2.-2.-2\,0$ is recognized as potential inadequate, i.e., as a knot without minimal $+$ or $-$adequate diagram by Thistlethwaite in 1988 [**11**], but to this knot Theorem 2.5 and Proposition 2.2 cannot be applied. From 12 links from this table, the five of them: $6^*2.-2\,0.-2.2\,0$, $6^*2.2.-2.2.-2\,0$, $6^*2.2.-2.2\,0.-2$, $6^*2.-2.2.2.2\,0.-2\,0$, and $6^*2.-2.-2.-2.2.-2\,0$ are inadequate, according to Theorem 2.5 and Proposition 2.2

The first link that has an adequate state other than s_+ and s_- is the knot 4_1 (2 2) and it is illustrated in Fig. 17.

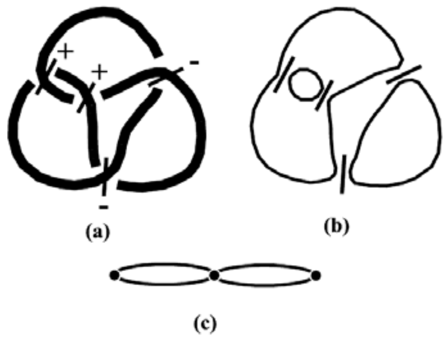

FIGURE 17. (a) Minimal diagram of the figure-eight knot with two +markers, and two −markers; (b) state circles; (c) the associated loopless graph G_s.

The minimal diagram of inadequate knot $2\,0.-3.-2\,0.2$ has eleven adequate states. First two are obtained from the alternating diagram $2\,0.3.2\,0.2$ and its mirror image. The remaining nine adequate states can be obtained from other adequate diagrams, one corresponding to the minimal diagram $2\,0.-3.-2\,0.-2$ and the other to the nonminimal diagram $-2\,0.3.2\,0.-2$ which is reducible to 10-crossing nonalternating knot 10_{124} $(5,3,-2)$ (Fig. 18).

THEOREM 7.2. *Vertex connectivity of every loopless graph G_{s_+} or G_{s_-} corresponding to an alternating diagram D is greater than* 1*. Vertex connectivity of every loopless graph G_{s_+} or G_{s_-} corresponding to a nonalternating minimal diagram D is* 1*.*

The same statement is not true for loopless graphs G_s obtained from other states. For example, the loopless graph G_s corresponding to the minimal nonalternating diagram of the knot $10_{155} = -3:2:2$ (Fig. 19) has the vertex connectivity 4.

DEFINITION 7.1. The minimal number of adequate states taken over all diagrams of a link L is called the *adequacy number* of link L and denoted by $a(L)$.

LEMMA 7.1. *All minimal diagrams of an alternating link have the same number of adequate states.*

Since changing marker in one crossing is equivalent to the crossing change, we conclude that the number of adequate states is invariant of a link diagram independent from the signs of crossings. This means that the number of adequate states is the same for every alternating diagram and all nonalternating diagrams obtained from it by crossing changes. Moreover, this can be generalized to families

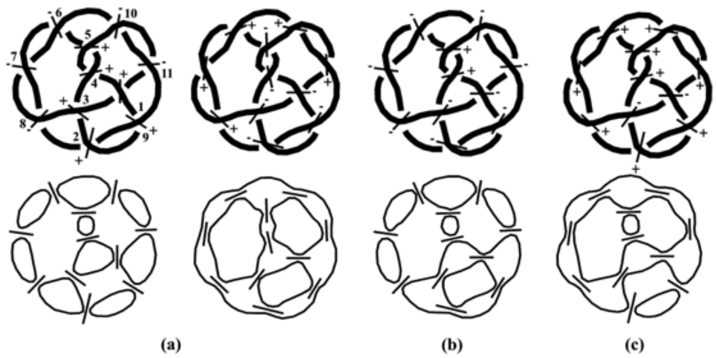

FIGURE 18. (a) Two adequate states of the inadequate knot diagram $2\,0.-3.-2\,0.2$ obtained from the alternating knot $2\,0.3.2\,0.2$; (b) adequate state of the same diagram corresponding to the minimal diagram $2\,0.-3.-2\,0.-2$; (c) its adequate state corresponding to the nonminimal diagram $-2\,0.3.2\,0.-2$, which is reducible to 10-crossing nonalternating knot 10_{124} $(5,3,-2)$.

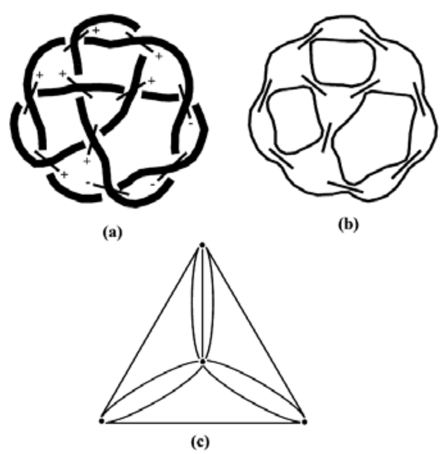

FIGURE 19. (a) Minimal diagram of the knot 10_{155} with markers; (b) state circles; (c) the associated graph G_s with the vertex connectivity 4.

of link diagrams, since adding a bigon to the chain of bigons does not change the adequacy of a diagram.

LEMMA 7.2. *The number of adequate states $a(L)$ is the invariant of a family of alternating links L and it is realized on every minimal diagram belonging to the link family.*

THEOREM 7.3. *The only links L whose adequacy number is $a(L) = 2$ are links of the family p $(p = 2, 3, 4, 5, \dots)$*[9]*, i.e., the links 2^2_1, 3_1, 4^2_1, 5_1,...*

Adequacy number of two minimal diagrams corresponding to a nonalternating link can be different. The minimal diagram $3, 2\,1, -2$ of the knot 8_{20} has 6 adequate states, while its other minimal diagram $.2. - 2\,0. - 1 : . - 1$ has 8 adequate states.

$n = 2$	2 2					
$n = 4$	2 2 3					
$n = 5$	2 1 2 4					
$n = 6$	2 2 2 5	2 1 1 2 5	2, 2, 2 5			
$n = 7$	2 1 2 2 6	2 1 1 1 2 7	2, 2, 2+ 8	2 1, 2, 2 6	.2 7	
$n = 8$	2 2 2 2 8	2 1 2 1 2 8	2 2 1 1 2 8	2 1 1 1 1 2 9	2, 2, 2, 2 12	2 2, 2, 2 7
	2 1 1, 2, 2 9	2 1, 2 1, 2 8	2, 2, 2 + + 9	2 1, 2, 2+ 9	(2, 2) (2, 2) 8	.21 10
	.2 : 2 9	.2.2 8	.2 : 2 0 8	.2.2 0 8		
$n = 9$	2 2 1 2 2 9	2 2 2 1 2 10	2 1 2 1 1 2 10	2 2 1 1 1 2 11	2 1 1 1 1 1 2 12	2 1, 2 1, 2 1 12
	2 1 2, 2, 2 10	2 2 1, 2, 2 11	2 1 1 1, 2, 2 13	2 1, 2, 2, 2 9	2 2, 2 1, 2 11	2 1 1, 2 1, 2 10
	2 1, 2, 2 + + 10	2, 2, 2, 2+ 16	2 2, 2, 2 + 1 12	2 1 1, 2, 2+ 13	2 1, 2 1, 2+ 11	(2 1, 2) (2, 2) 11
	(2, 2) (2, 2) 10	(2, 2) 1 (2, 2) 12	.2 2 11	.2 1 1 13	.2 1 : 2 12	.2 1 : 2 0 12
	.2 1.2 0 11	.2.2 0.2 10	2 : 2 0 : 2 0 10	2 0 : 2 0 : 2 0 9	.2.2.2 10	2 : 2 : 2 9
	.2.2.2 0 9	2 : 2 : 2 0 9	.(2, 2) 14	8*2 12	8*2 0 13	

The next theorem gives the integer sequences corresponding to adequacy numbers of certain classes KLs. For every sequence is given its number from the *The On-Line Encyclopedia of Integer Sequences* [**22**], where four sequences are not present in the *Encyclopedia*.

THEOREM 7.4. (1) *A rational KL of the form $2 \dots 2$, where 2 occurs n times $(n = 1, 2, \dots)$ has the adequacy number f_{n+1}, where f_{n+1} is the*

[9] See Fig. 10.

Fibonacci number f_{n+1}, given by sequence $f_{n+1} = f_n + f_{n-1}$, $f_0 = 1$, $f_1 = 1$ (sequence A000045);

(2) *A rational KL of the form $2\,1\,\ldots1\,2$, where 1 occurs n times $(n = 1, 2, \ldots)$ has the adequacy number a_{n+4}, where $a_n = a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 2$ (Padovan sequence A000931);*

(3) *A rational KL of the form $2\,\ldots\,2\,1\,2\,\ldots\,2$, where each block of numbers 2 contains n numbers $(n = 1, 2, \ldots)$ has the adequacy number f_{n+2}^2, where f_n is the n^{th} member of the Fibonacci sequence (sequence A007598);*

(4) *A rational link of the form $2\,1\,2\,1\ldots2$, where the block $2\,1$ occurs n times $(n \geqslant 1)$ has the adequacy number $a_n = 2^n$ (sequence A000079);*

(5) *A pretzel link of the form $2, 2\,\ldots, 2$ of the length n $(n \geqslant 3)$ has the adequacy number $a_n = 2^n - n$ (sequence A000325);*

(6) *A Montesinos link of the form $2, 2\,\ldots, 2+$, where 2 occurs n times $(n \geqslant 3)$ has the adequacy number 2^n (sequence A000079);*

(7) *A Montesinos link of the form $2, 2, \ldots, 2 + +$, where 2 occurs n times $(n \geqslant 3)$ has adequacy number $a_n = 2^n + 1$ (sequence A000051);*

(8) *A Montesinos link of the form $2\,2, 2, 2, \ldots, 2$ of the length $n+1$ $(n \geqslant 2)$ has the adequacy number $a_n = 3 \times 2^n - 2n - 1$;*

(9) *A Montesinos link of the form $2\,1, 2, 2, \ldots, 2$ of the length $n+1$ $(n \geqslant 2)$ has the adequacy number $a_n = 2^n - n + 1$ (sequence A132045);*

(10) *A link of the form $(2, \ldots, 2)\,(2, 2)$ where the length of the pretzel tangle $(2, \ldots, 2)$ is n $(n \geqslant 2)$ is $a_n = 2^{n+1} - n + 2$;*

(11) *A link of the form $(2, \ldots, 2+)\,(2, 2)$ where in the pretzel tangle $(2, \ldots, 2+)$ number 2 occurs n times $(n \geqslant 2)$ is $a_n = 2^{n+1} + 2$ (sequence A052548);*

(12) *A link of the form $(2, \ldots, 2{+}{+})\,(2, 2)$ where in the pretzel tangle $(2, \ldots, 2{+}{+})$ number 2 occurs n times $(n \geqslant 2)$ is $a_n = 2^{n+1}+3$ (sequence A062709);*

(13) *A link of the form $(2, \ldots, 2)\,(2, 2, 2)$ where the length of the pretzel tangle $(2, \ldots, 2)$ is n $(n \geqslant 3)$ is $a_n = 5 \times 2^n - 4n + 3$;*

(14) *A link of the form $6^*(2, \ldots, 2)$, where the length of the tangle $(2, \ldots, 2)$ is n $(n \geqslant 2)$ is $a_n = 2^{n+2} - n$ (sequence A132753);*

(15) *A link of the form $6^*(2\,2, 2, \ldots, 2)$, where the length of the tangle $(2\,2, 2, \ldots, 2)$ is n $(n \geqslant 2)$ is $a_n = 3 \times 2^{n+1} - 2n + 1$.*

Proof. Since the proofs follow similar ideas, we prove only parts (1) and (5) of Theorem 7.4:

(1) It is easy to show that Hopf link 2 (2_1^2) has 2, figure-eight knot $2\,2$ (4_1) has 3, and the link $2\,2\,2$ (6_3^2) has 5 adequate states (Fig. 20a). Let us consider the rational tangles T_n, T_{n+1}, T_{n+2} and the links L_n, L_{n+1}, and L_{n+2} of the form $2\ldots2$ obtained as their numerator closures, where 2 occurs n, $n+1$ and $n+2$ times, respectively ($n \geqslant 1$). In order to obtain an adequate state of the link L_{n+2}, the last bigon of the tangle T_{n+2} must have the markers of the same sign: both positive or both negative. If it has both positive markers, the number of the adequate states obtained by its addition equals the number of adequate states of the link L_{n+1} (Fig. 20b). If it has both negative markers, in order to preserve adequacy by the addition of the last bigon, its preceding bigon must have both negative markers, and

the number of adequate states obtained in this way equals the adequacy number of the link L_n (Fig. 20c). Hence, the adequacy numbers of the links L_n, L_{n+1}, and L_{n+2} satisfy the recursion generating Fibonacci sequence.

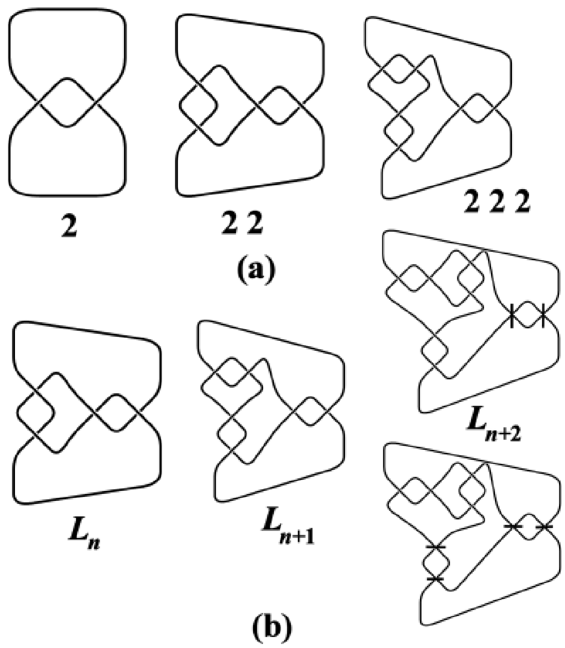

FIGURE 20. (a) Hopf link 2 (2_1^2), figure-eight knot 2 2, (4_1), and link 2 2 2 (6_3^2); (b) addition of the last bigon preserving adequacy.

(5) In order to show that a pretzel link of the form $2, 2 \ldots, 2$ of the length n ($n \geqslant 3$) has the adequacy number $a_n = 2^n - n$, first notice that all tangles 2 with markers of different signs result in nonadequate diagrams, since there appears a self-touch point of the corresponding state circle. Hence, in an adequate state both markers in all bigons must be of the same sign. For such states we have 2^n possibilities for the choice of markers. Among them, nonadequate diagrams will be obtained if exactly one bigon contains markers of one sign, and all the other markers are of opposite sign. For such choice of markers we have n possibilities. Hence, among the mentioned 2^n states, n will be nonadequate, so the adequacy number of the pretzel link of the form $2, 2 \ldots, 2$ is $a_n = 2^n - n$. □

8. Adequacy polynomial as an invariant of alternating link families

Graphs corresponding to adequate link diagrams, called adequate state graphs, can be used for defining a polynomial invariant of alternating link families.

DEFINITION 8.1. A *cut-vertex* (or articulation vertex) of a connected graph is a vertex whose removal disconnects the graph [**23**]. In general, a cut-vertex is a vertex of a graph whose removal increases the number of components [**24**]. A graph with no cut-vertices is called a *biconnected graph* [**25**]. A *block* is a maximal biconnected subgraph of a given graph.

The following transformations can be applied to the adequate state graphs, until the graph cannot be reduced to a graph with smaller number of vertices:

- (multiple edge reduction) replace every edge of the multiplicity greater than 2 by a single edge;[10]
- (edge chain collapse) replace maximal part of every chain consisting from edges with vertices of degree 2 by a new edge connecting the beginning vertex of the first and ending vertex of the last edge;
- (block move) every block can be moved along the edges of the remaining part of the graph.

From every adequate state graph G we obtain the reduced adequate state graph $\overline{G}$. Recently, several graph invariants have been categorified following Khovanov's link homology construction, such as the chromatic and the Tutte polynomial. Chromatic cohomology for graphs defined by Helme-Guizon and Rong [**26**] provides a link between link homology and well developed theory of the Hochschild homology [**27**]. Based on the results about torsion in chromatic graph cohomology [**14**] we get the following theorem:

THEOREM 8.1. *Block move preserves torsion in $H^{1,v-2}_{A_2}(G)$ in the first chromatic graph cohomology of a graph G with v vertices, computed over algebra of truncated polynomials $A_2 = \mathbb{Z}[x]/(x^2)$*

Fig. 21 illustrates reduction of the graph with 16 vertices (Fig. 21a) to the graph with 13 vertices (Fig. 21b), or to its equivalent graph (Fig. 21c) obtained from it by block moves, which has the same torsion and chromatic polynomial as the graph (Fig. 21b).

Consider an arbitrary minimal diagram of an alternating link L. Let G_i denote the corresponding state graphs for all adequate states of a diagram D_L and $\overline{G}_i$ reduced state graphs ($i = 1, 2, \dots, a(L)$), where $a(L)$ is the adequacy number of L (Def. 7.1).

DEFINITION 8.2. The *adequacy polynomial* of any alternating diagram D_L is a polynomial in two variables determined by $A(x, y) = \sum_{i=1}^{a(L)} x^{\overline{t}_i}\overline{P}_i(y)$, where $\overline{P}_i(y) = P(\overline{G}_i)$ denotes the chromatic polynomial of a reduced state graph G_i and $\overline{t}_i$ is the power of $\mathbb{Z}_2$ torsion of the first chromatic graph cohomology $H^{1,h}_{A_m}(G_i)$ over algebra of truncated polynomials A_m $=\mathbb{Z}[x]/x^2 = 0$ in the grading $h = (m-1)(v-2)+1$, where v denotes the number of vertices of the graph G_i.

[10] Since chromatic polynomial of a graph and graph homology does not recognize multiple edges, this step is not necessary for further computations [**14**].

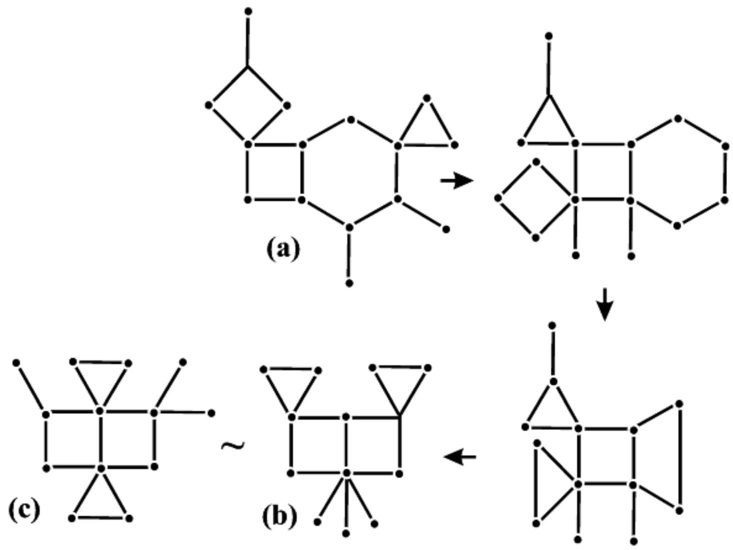

FIGURE 21. Reduction of the graph (a) to the graph (b), and graph (c) equivalent to (b).

THEOREM 8.2. *Adequacy polynomial is the same for all minimal diagrams of all alternating links belonging to the same family, which satisfy the condition*[11] $|a| + k_a \geqslant 3$.

The computation of adequacy polynomial is illustrated on the example of link 3 1 5 4 (Figs. 22-23). This link has 3 different minimal diagrams: 3 1 5 4[12], $((1,(1,3),1,1,1,1),1,1,1,1)$, and $((1,1,(3,1),1,1,1),1,1,1,1)$ (Fig. 22). For the reduced adequate state graphs $\overline{G}_i$ $(i = 1, 2, \ldots, 6)$ corresponding to the first minimal diagram, the sequence $(1, 2, 2, 1, 1, 0)$ represents powers of $\mathbb{Z}_2$-torsion $\overline{t}_i$ for $m = 3$, and the following is the list of chromatic polynomials:

1) $6y - 15y^2 + 14y^3 - 6y^4 + y^5$,
2) $4y - 12y^2 + 13y^3 - 6y^4 + y^5$,
3) $-4y + 16y^2 - 25y^3 + 19y^4 - 7y^5 + y^6$,
4) $-18y + 81y^2 - 156y^3 + 168y^4 - 110y^5 + 44y^6 - 10y^7 + y^8$,
5) $-2y + 5y^2 - 4y^3 + y^4$,
6) $-9y + 27y^2 - 33y^3 + 21y^4 - 7y^5 + y^6$,

[11]Please compare this additional condition with the definition of a family of link diagrams (Def. 1.5): according to the additional condition, all chains of bigons must be of the length greater then 2.

[12]This diagram can be also written as $(((3,1),1,1,1,1,1),1,1,1,1)$.

so the adequacy polynomial is

$$\begin{aligned}A(3\,1\,5\,4) = &-9y - 14xy + 27y^2 + 71xy^2 + 4x^2y^2 - 33y^3 - 146xy^3\\ &- 12x^2y^3 + 21y^4 + 163xy^4 + 13x^2y^4 - 7y^5 - 109xy^5\\ &- 6x^2y^5 + y^6 + 44xy^6 + x^2y^6 - 10xy^7 + xy^8.\end{aligned}$$

This polynomial is invariant of link family $p\,1\,q\,r$ $(p,q,r \geqslant 3)$.

If we compute the adequacy polynomial from the second or third diagram, we obtain the same sequence $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_6 = (1,2,2,1,1,0)$ and the same list of chromatic polynomials, so the final result remains the same.

CONJECTURE 8.1. *Adequacy polynomial distinguishes all alternating link families (up to mutation).*

This conjecture is verified for all alternating links with at most $n = 12$ crossings. If the conjecture does not hold in general, one may consider various adequacy polynomials obtained by taking into consideration other gradings in the first homology or the whole groups (possibly higher in homology), or changing algebra. Moreover, depending on the algebra, one may consider torsions other then $\mathbb{Z}_2$, if they exist.

The adequacy polynomial of any family of alternating links can be computed from any minimal diagram of the link L representing this family, with all chains of bigons of length 3. For subfamilies we use links with some parameters equal to 2, and the remaining ones equal to 3. For the general Conway symbol $p\,1\,q\,r$ $(p,q,r \geqslant 2)$, we distinguish the following cases:

(1) $2\,1\,2\,2$ with $A(x,y) = y + 2xy - 2y^2 - 5xy^2 - 4x^2y^2 + y^3 + 6xy^3 + 8x^2y^3 - 4xy^4 - 5x^2y^4 + xy^5 + x^2y^5$;

(2) $p\,1\,2\,2$ with $A(x,y) = -8y - 8xy + 25y^2 + 26xy^2 - 4x^2y^2 - 32y^3 - 33xy^3 + 8x^2y^3 + 21y^4 + 21xy^4 - 5x^2y^4 - 7y^5 - 7xy^5 + x^2y^5 + y^6 + xy^6$, $p \geqslant 3$;

(3) $2\,1\,q\,2$ with $A(x,y) = -2xy - 4x^2y + 9xy^2 + 8x^2y^2 - 14xy^3 - 5x^2y^3 + 11xy^4 + x^2y^4 - 5xy^5 + xy^6$, $q \geqslant 3$;

(4) $2\,1\,2\,r$ with $A(x,y) = 10xy - 8x^3y - 27xy^2 + 28x^3y^2 + 29xy^3 - 38x^3y^3 - 15xy^4 + 25x^3y^4 + 3xy^5 - 8x^3y^5 + x^3y^6$, $r \geqslant 3$;

(5) $p\,1\,q\,2$ with $A(x,y) = -9y^2 - 2xy^2 - 4x^2y^2 + 27y^3 + 5xy^3 + 8x^2y^3 - 33y^4 - 4xy^4 - 5x^2y^4 + 21y^5 + xy^5 + x^2y^5 - 7y^6 + y^7$, $p,q \geqslant 3$;

(6) $p\,1\,2\,r$ with $A(x,y) = -9y + 6xy + 16x^2y + 27y^2 - 17xy^2 - 60x^2y^2 - 33y^3 + 19xy^3 + 92x^2y^3 + 21y^4 - 10xy^4 - 75x^2y^4 - 7y^5 + 2xy^5 + 35x^2y^5 + y^6 - 9x^2y^6 + x^2y^7$, $p,r \geqslant 3$;

(7) $2\,1\,q\,r$ with $A(x,y) = 8xy + 8x^2y - 20xy^2 - 32x^2y^2 + 20xy^3 + 54x^2y^3 - 10xy^4 - 50x^2y^4 + 2xy^5 + 27x^2y^5 - 8x^2y^6 + x^2y^7$, $q,r \geqslant 3$;

(8) $p\,1\,q\,r$ with $A(x,y) = -9y - 14xy + 27y^2 + 71xy^2 + 4x^2y^2 - 33y^3 - 146xy^3 - 12x^2y^3 + 21y^4 + 163xy^4 + 13x^2y^4 - 7y^5 - 109xy^5 - 6x^2y^5 + y^6 + 44xy^6 + x^2y^6 - 10xy^7 + xy^8$, $p,q,r \geqslant 3$.

The adequacy polynomial can be defined in the same way for families of virtual links. Equivalents of Theorem 8.2 and Conjecture 8.1 hold for alternating virtual links.

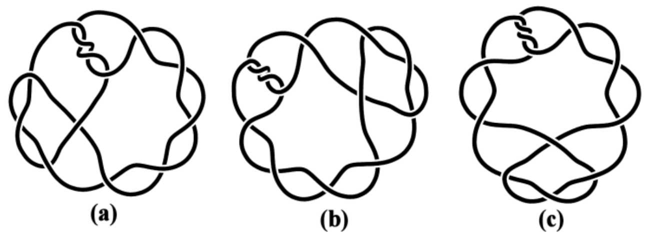

FIGURE 22. Minimal diagrams (a) 3 1 5 4; (b) ((1, (1, 3), 1, 1, 1, 1), 1, 1, 1, 1); (c) ((1, 1, (3, 1), 1, 1, 1), 1, 1, 1, 1).

The equivalent of Conjecture 8.1 is verified by computer calculations for all families of virtual knots derived from real knots with at most 8 crossings.

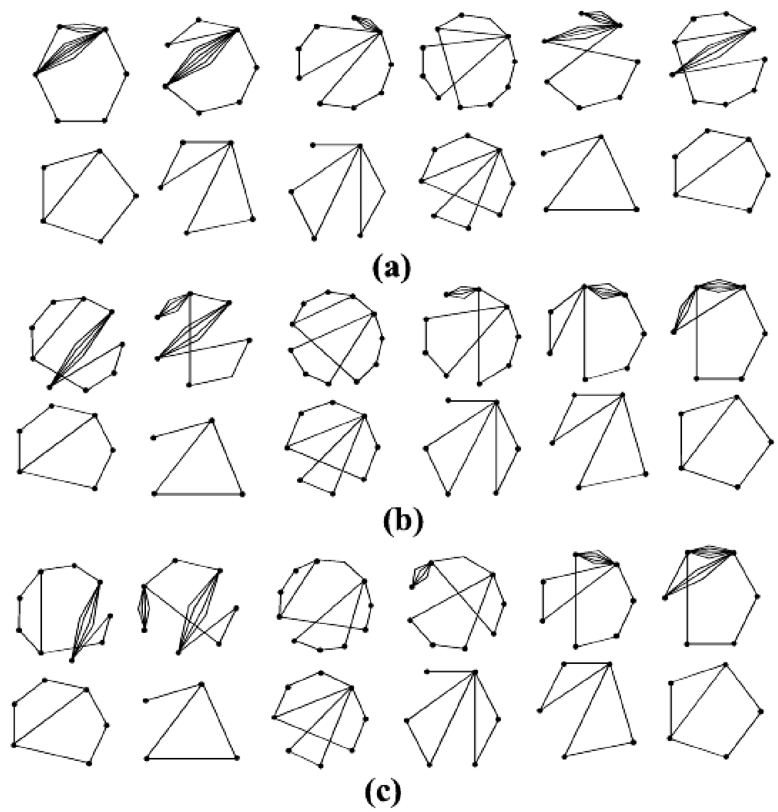

FIGURE 23. Adequate state graphs of the diagrams (a) 3 1 5 4, (b) ((1, (1, 3), 1, 1, 1, 1), 1, 1, 1, 1), (c) ((1, 1, (3, 1), 1, 1, 1), 1, 1, 1, 1) and their corresponding reduced graphs.

The definition of the adequacy polynomial (Definition 8.2) contains the first chromatic graph homology in the specific grading coming from the interpretation of Hochschild homology as the chromatic graph homology of a polygon, i.e., $H^{1,(m-1)(v-2)+1}_{A_m}(G)$, where G is a graph and v denotes the number of its vertices and $A_m = \mathbb{Z}[x]/x^m$ for $m \geqslant 3$. The reason why we have excluded algebra A_2 is that it cannot distinguish some generating links (e.g., $2\,2\,1\,1\,1\,2$ from $2\,1\,1, 2\,1, 2$). According to the computations for all generating links with $n \leqslant 12$ crossings, for $3 \leqslant m \leqslant 5$ the adequacy polynomial distinguishes all families of alternating links with at most $n = 12$ crossings (up to mutation).

Notice that the adequacy polynomials of the family $3\,1\,3\,3$ computed for $m = 2, 3, \ldots, 8$ are the same, but this is not true in general: according to the computer calculations for $2 \leqslant m \leqslant 8$, the family $.p$ ($p > 2$) will have two different polynomials:

$$\begin{aligned} &2y - 10xy - 10x^2y - 4y^2 + 21xy^2 + 27x^2y^2 \\ &\qquad + 2y^3 - 14xy^3 - 31x^2y^3 + 3xy^4 + 20x^2y^4 - 7x^2y^5 + x^2y^6 \text{ for odd } m, \end{aligned}$$

$$\begin{aligned} &2y - 4xy - 10x^2y - 6x^4y - 4y^2 + 10xy^2 + 27x^2y^2 + 11x^4y^2 + 2y^3 \\ &- 8xy^3 - 31x^2y^3 - 6x^4y^3 + 2xy^4 + 20x^2y^4 + x^4y^4 - 7x^2y^5 + x^2y^6 \text{ for even } m. \end{aligned}$$

The adequacy polynomial can be redefined to include the second or all chromatic graph cohomology groups $H^{2,(m-1)(v-2)}_{A_m}(G)$, which for $m = 2, 4, 6$ distinguish all alternating link families corresponding to links with at most 12 crossings.

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