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EHRESMANN CONNECTION IN THE GEOMETRY OF NONHOLONOMIC SYSTEMS

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Abstract. This article deals with a dynamic system whose motion is constrained by nonholonomic, reonomic, affine constraints. The article analyses the geometrical properties of the "reactions" of nonholonomic constraints in Voronets's equations of motion. The analysis shows their link with the torsion of the Ehresmann connection, which is defined by the nonholonomic constraints.

1. Differential Equations of Motion

We consider nonholonomic mechanical system on a configuration manifold Q^n, where the local coordinates are $q = (q^1, \dots, q^n)$. The motion of the system is constrained by the affine constraints

$$\dot{q}^\nu = a^\nu_\alpha \dot{q}^\alpha + a^\nu, \quad \alpha = 1, \dots, m;\ \nu = m+1, \dots, n \tag{1}$$

where $a^\nu_\alpha : (q,t) \mapsto a^\nu_\alpha(q,t)$ and $a^\nu : (q,t) \mapsto a^\nu(q,t)$ are smooth functions. At each point $(q,t) \in Q^n \times R$, the constraints (1) define a $(n-m)$-dimensional time-dependent tangent subspace $D_{q,t} \subset T_q Q^n \times R$ (assuming that the subspace dimensions are the same at each point), i.e., $(n-m)$-dimensional distribution.

A curve $\gamma : (a,b) \ni t \mapsto \gamma(t) \in Q^n$, is said to satisfy the constraints if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for all t. The set $\mathfrak{M}$ comprising all the points at which the tangent vector belongs to the distribution is called the integral set of the distribution.

The nonholonomic constraints (1) define the 1-forms ω on an extended configuration manifold $Q^n \times R$ in the following manner: $\omega^\nu = dq^\nu - a^\nu_\alpha dq^\alpha - a^\nu dt$.

For the sake of brevity, we will express them as follows ($t = q^0$):

$$\omega^\nu = dq^\nu - a^\nu_{\hat{\alpha}} dq^{\hat{\alpha}},\ a^\nu = a^\nu_0,\ \hat{\alpha} = 0, 1, \dots, m.$$

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Dedicated to the memory of Professor Antun Bilimovich, whose books and works in the area of Rational Mechanics are a part of the foundation of my professional education and serve as a model for my work. His textbook [1] is the source of my initial knowledge of the dynamics of nonholonomic systems.

The linear space N^* of the 1-forms on the manifold, which represents a module over the ring of smooth functions, will be called the differential system. If dim N^* is equal at all points, then it is taken as the dimension of the differential system. The differential system may also be given with $(m+1)$ linearly independent vector fields

$$\xi_{\hat{\alpha}} = a^{\nu}_{\hat{\alpha}} \frac{\partial}{\partial q^{\nu}} + \frac{\partial}{\partial q^{\hat{\alpha}}}$$

which annul the 1-forms ω^{ν}. The linearly independent vectors $\xi_{\hat{\alpha}}$ in each tangent space $T_{(q,t)}(Q^n \times R)$ define the $(m+1)$-dimensional vector subspace $D_{(q,t)}$. The differentiable field of these subspaces $D : (q,t) \mapsto D_{q,t}$ is called the distribution.

For the linearly independent vector fields $\xi_{\hat{\alpha}}$ we say that they span the distribution D, and that the corresponding fields of 1-forms ω^{ν} span the codistribution or annulator $D^{\perp}$.

The differential equations of motion of the mechanical system determined by the Lagrangian $L : TQ^n \times R \ni (\dot{q}, q, t) \mapsto L(\dot{q}, q, t) \in R$, whose motion is constrained by nonholonomic constraints (1), are derived from the basic dynamic equation

$$\Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}, \delta q \Big) = 0$$

where δq is a random vector from $T_q Q^n$ (virtual displacement). Bearing in mind the constraints imposed on the coordinates of vector δq by nonholonomic connections

$$\delta q^{\nu} = a^{\nu}_{\alpha} \delta q^{\alpha}$$

the basic dynamic equation has the following form

$$\Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{\alpha}} - \frac{\partial L}{\partial q^{\alpha}} \Big) \delta q^{\alpha} + a^{\nu}_{\alpha} \Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{\nu}} - \frac{\partial L}{\partial q^{\nu}} \Big) \delta q^{\alpha} = 0.$$

Let us introduce a "connected" Lagrangian $\tilde{L}$ by using the equality (1) and excluding the coordinates $\dot{q}^{\nu}$:

$$L(\dot{q}, q, t) \to \tilde{L}(\dot{q}^1, \ldots, \dot{q}^m, q^1, \ldots, q^n, t).$$

The equations of motion may now be written in the following form (in scientific literature, these equations are known as Voronets's equations [1] [3]:

$$\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}^{\alpha}} - \frac{\partial \tilde{L}}{\partial q^{\alpha}} - \frac{\partial \tilde{L}}{\partial q^{\nu}} a^{\nu}_{\alpha} - \Bigg(\widetilde{\frac{\partial L}{\partial \dot{q}^{\nu}}} \Bigg) (B^{\nu}_{\alpha\beta} \dot{q}^{\beta} + B^{\nu}_{\alpha}) = 0, \tag{2}$$

where

$$B^{\nu}_{\alpha\beta} = \frac{\partial a^{\nu}_{\alpha}}{\partial q^{\beta}} - \frac{\partial a^{\nu}_{\beta}}{\partial q^{\alpha}} + \frac{\partial a^{\nu}_{\alpha}}{\partial q^{\mu}} a^{\mu}_{\beta} - \frac{\partial a^{\nu}_{\beta}}{\partial q^{\mu}} a^{\mu}_{\alpha} \tag{3}$$

$$B^{\nu}_{\alpha} = \frac{\partial a^{\nu}_{\alpha}}{\partial t} - \frac{\partial a^{\nu}}{\partial q^{\alpha}} + \frac{\partial a^{\nu}_{\alpha}}{\partial q^{\mu}} a^{\mu} - \frac{\partial a^{\nu}}{\partial q^{\mu}} a^{\mu}_{\alpha} \tag{4}$$

[1] P. V. Voronets, 1871–1923

Along with the initially set conditions, equations (1) and (2) completely define the motion of the system. Obviously, the system of coefficient (3) is skew-symmetric. In further we shall denote

$$B^{\nu}_{\alpha\beta}\dot{q}^{\beta} + B^{\nu}_{\alpha} = B^{\nu}_{\alpha\hat{\beta}}\dot{q}^{\hat{\beta}}, \quad \alpha = 1,\dots,m,\ \hat{\beta} = 0,\dots,m,\ \dot{q}^{0} = 1.$$

where $t = q^0$, $a^{\mu} = a^{\mu}_{0}$, $B^{\nu}_{\alpha} = B^{\nu}_{\alpha 0}$.

2. Ehresmann Connection

In order to examine the geometrical properties of the nonholonomic connections, we need to analyse the fibre bundle $(Q^n, R^m, S^{(n-m)}, \pi)$, where Q^n is the total space, R^m is the base space, $S^{(n-m)}$ is the fiber, $\pi : Q^n \to R^{(n-m)}$ projection (surjective submersion).

The submersion of π induces the submersion of $\pi_* : TQ \to TR$. The kernel of the submersion is the vertical subspace $\mathrm{Ker}(\pi_*) = V \in TQ$. The complement of the vertical subspace is the horizontal subspace H. At any point $q \in Q$ is $V_q \oplus H_q = T_qQ$, $V_q \cap H_q = \{0\}$.

The separation of the horizontal subspace is linked with the structure of the linear connection at the fiber bundle, i.e., with the parallel transport. The connection can be defined in several ways and we will introduce it via the vector-valued 1-form.

Definition. The Ehresmann[2] connection is a vector-valued differential 1-form Ω on Q^n such that

$$(1)\ \Omega : T_qQ^n \to V_q \ \text{(vertical valued)},\ \ (2)\ \Omega(v) = v,\ v \in V \ \text{(projection)}.$$

We will introduce the Ehresmann connection on $Q^n \times R$ via the differential form

$$\Omega := \omega^{\nu}\frac{\partial}{\partial q^{\nu}}, \quad \omega^{\nu} = dq^{\nu} - a^{\nu}_{\alpha}dq^{\alpha} - a^{\nu}dt.$$

The vertical vector component ξ (we will mark it as ξ_v) is

$$\xi_v = \Omega(\xi) = (\xi^{\nu} - a^{\nu}_{\hat{\alpha}}\xi^{\hat{\alpha}})\frac{\partial}{\partial q^{\nu}},\ \hat{\alpha} = 0, 1, \dots, m.$$

The map Ω is obviously vectorial and it can be easily seen that the projection is vectorial as well.

The horizontal vectors have the following form

$$\xi_h = \xi - \xi_v \Rightarrow \xi_h = \xi^{\hat{\alpha}}\Big(a^{\nu}_{\hat{\alpha}}\frac{\partial}{\partial q^{\nu}} + \frac{\partial}{\partial q^{\hat{\alpha}}}\Big).$$

It should be noted that these vectors extend the distribution. We can see that the horizontal lift is

$$\mathrm{hlift}\Big(\frac{\partial}{\partial q^{\hat{\alpha}}}\Big) = \frac{\partial}{\partial q^{\hat{\alpha}}} + a^{\nu}_{\hat{\alpha}}\frac{\partial}{\partial q^{\nu}}.$$

The collection of all the horizontal vectors at point $q \in Q^n$ forms a horizontal subspace. The separation of the horizontal subspace basically represents the definition of the Ehresmann connection.

[2]Charles Ehresmann, 1905–1979

One of the important characteristics of the connection is its curvature. The curvature of the connection Ω is the two-form on Q^n with values in TQ^n, [4], [7]

$$T(X,Y) = \Omega[X_h, Y_h].$$

By using a known identity valid for a 1-form differential θ

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) + \theta([X,Y])$$

we get

$$T(X,Y) = d\omega^\nu(X_h, Y_h)\frac{\partial}{\partial q^\nu}.$$

In our case it is

$$d\omega^\nu = -\frac{\partial a^\nu_{\hat\alpha}}{\partial q^{\hat\beta}} dq^{\hat\beta} \wedge dq^{\hat\alpha} - \frac{\partial a^\nu_{\hat\alpha}}{\partial q^\mu} dq^\mu \wedge dq^{\hat\alpha}.$$

The value of the differential form $d\omega$ on vectors $\xi_h \in TQ$, $\eta_h \in T(Q^n \times R)$ is

$$d\omega(\xi_h, \eta_h) = B^\nu_{\alpha\beta}\xi^\alpha\eta^\beta + B^\nu_\alpha \xi^\alpha \eta^0.$$

This proves the following

PROPOSITION. *Coefficients $B^\nu_{\alpha\beta} + B^\nu_\alpha$ in Voronets's equations represent the curvature of the Ehresmann connection on the vectors $\xi_h \in H \subset TQ^n$, $\eta_h \in H \subset T(Q^n \times R)$.*

EXAMPLE. Let us consider the rolling of a heavy homogenous ball on a horizontal plane, revolving around its axis [4]. Let us assume that the radius of the ball $r(\cdot)$ and the angular speed of the plane $\Omega(\cdot)$ are given smooth functions of time. We will set the inertial (immovable) coordinate system $Oxyz$ in such a manner that the z-axis is vertical, pointed upwards. We will determine the position of the ball by the coordinates (x, y) of the point at which the ball A touches the plane, and by the Euler angles (ψ, θ, φ). The configuration manifold of the system is $R^2 \times SO(3)$, which can be considered as the main stratification with the base R^2 and the fiber $SO(3)$, at which the structural group acts on itself from the right. Based on the rolling conditions (without sliding), we get the nonholonomic constraints in the following form

$$\omega_x(t) = -\frac{1}{a(t)}\dot y + \frac{\Omega(t)}{a(t)}x, \quad \omega_y(t) = \frac{1}{a(t)}\dot x + \frac{\Omega(t)}{a(t)}y, \tag{5}$$

$t \in (\alpha, \beta) \subset R$, where

$$\begin{aligned} \omega_x &= \dot\theta\cos\psi + \dot\varphi\sin\psi\sin\theta \\ \omega_y &= \dot\theta\sin\psi - \dot\varphi\cos\psi\sin\theta \\ \omega_z &= \dot\varphi\cos\theta + \dot\psi. \end{aligned} \tag{6}$$

Substituting (6) into (5) gives

$$\begin{aligned} \dot\varphi &= \frac{\cos\psi}{a\sin\theta}\dot x - \frac{\sin\psi}{a\sin\theta}\dot y + \frac{\Omega(x\sin\psi - y\cos\psi)}{a\sin\theta} \\ \dot\theta &= \frac{\sin\psi}{a}\dot x - \frac{\cos\psi}{a}\dot y + \frac{\Omega(x\cos\psi + y\sin\psi)}{a} \end{aligned}$$

The Lagrangian is

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{mk^2}{2}\left(\omega_x^2+\omega_y^2+\omega_z^2\right) \\ &= \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{mk^2}{2}\left[\dot{\psi}^2+\dot{\varphi}^2+2\dot{\varphi}\dot{\psi}\cos\theta+\dot{\theta}^2\right] \end{aligned}$$

(mk^2 is inertia about any axis).

From Lagrange's equations with multipliers [1]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}=\sum \lambda_\nu \frac{\partial \Phi_\nu}{\partial q^i}$$

we get

$$\frac{\partial L}{\partial \dot{\psi}}=mk^2\left(\dot{\psi}+\dot{\varphi}\cos\theta\right)=mk^2\omega_z=\text{const}\,.$$

The reduced Lagrangian $\tilde{L}$, without an additive constant, is

$$\tilde{L}(x,y,\dot{x},\dot{y})=\frac{m}{2}\left[\left(1+\frac{k^2}{a^2}\right)\left(\dot{x}^2+\dot{y}^2\right)+\frac{2k^2\Omega}{a^2}(y\dot{x}-x\dot{y})+\frac{k^2\Omega^2}{a^2}\left(x^2+y^2\right)\right].$$

The functions that determine the curvature are ($q^1=x$, $q^2=y$, $q^3=\psi$, $q^4=\varphi$, $q^5=\theta$)

$$B^4_{12}=-\frac{\cos\theta}{a^2\sin^2\theta},\quad B^4_{13}=\frac{\sin\psi}{a\sin\theta},\quad B^4_{23}=-\frac{cos\psi}{a\sin\theta}$$

$$B^4_1=\frac{\dot{a}\cos\psi}{a^2\sin\theta}-\frac{\Omega\sin\psi}{a\cos\theta}+\frac{\Omega\cos\theta}{a^2\sin^2\theta}x,\quad B^4_2=\frac{\dot{a}\sin\psi}{a^2\sin\theta}+\frac{\Omega\cos\psi}{a\sin\theta}+\frac{\Omega\cos\theta}{a^2\sin^2\theta}y,$$

$$B^4_3=-\frac{\Omega(x\cos\psi+y\sin\psi)}{a\sin\theta},\quad B^5_{12}=0,\quad B^5_{13}=\frac{\cos\psi}{a},\quad B^5_{23}=\frac{\sin\psi}{a},$$

$$B^5_1=-\frac{1}{a^2}(\dot{a}\sin\psi+a\cos\psi),\quad B^5_2=\frac{1}{a^2}(\dot{a}\cos\psi-a\sin\psi)\,;\; B^5_3=\frac{\Omega}{a}(x\sin\psi-y\cos\psi)$$

The distribution D is determined by (horizontal vectors)

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial x}-\frac{\cos\psi}{a\sin\theta}\frac{\partial}{\partial\varphi}+\frac{\sin\psi}{a}\frac{\partial}{\partial\theta}, \\ \xi_2 &= \frac{\partial}{\partial y}-\frac{\sin\psi}{a\sin\theta}\frac{\partial}{\partial\varphi}-\frac{\cos\psi}{a}\frac{\partial}{\partial\theta}, \\ \xi_0 &= \frac{\partial}{\partial t}+\frac{\Omega(x\sin\psi+y\cos\psi)}{a\sin\theta}\frac{\partial}{\partial\varphi}-\frac{\Omega(x\cos\psi+y\sin\psi)}{a}\frac{\partial}{\partial\theta}. \end{aligned}$$

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