PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 92(106) (2012), 53–78 DOI: 10.2298/PIM1206053K

GRAM'S LAW AND THE ARGUMENT OF THE RIEMANN ZETA FUNCTION

Maxim Korolev

Communicated by Žarko Mijajlović

Abstract. Some new statements concerning the behavior of the argument of the Riemann zeta function at the Gram points are proved. We apply these statements to prove Selberg's formulas connected with Gram's Law.

1. Introduction

The notion 'Gram's Law' has different senses in different papers. Thus, we begin this paper with a short survey. This survey contains the results concerning the peculiar phenomenon observed by Jörgen Pedersen Gram [**1**] in 1903.

In what follows, we need several definitions. Suppose that $t > 0$ and let $\vartheta(t)$ be an increment of any fixed continuous branch of the argument of the function $\pi^{-s/2}\Gamma\bigl(\frac{s}{2}\bigr)$ as t varies along the segment connecting the points $s = \frac{1}{2}$ and $s = \frac{1}{2} + it$. Then Hardy's function $Z(\tau) = e^{i\vartheta(\tau)}\zeta\bigl(\frac{1}{2} + i\tau\bigr)$ is real for real τ and its real zeros coincide with the ordinates of zeros of $\zeta(s)$ lying on the critical line $\operatorname{Re} s = \frac{1}{2}$. Further, if t is not an ordinate of a zero of $\zeta(s)$, then the function $S(t) = \pi^{-1}\arg\zeta\bigl(\frac{1}{2} + it\bigr)$ is defined as an increment of any continuous branch of $\pi^{-1}\arg\zeta(s)$ along the polygonal arc connecting the points $2, 2 + it$ and $\frac{1}{2} + it$. Otherwise, $S(t)$ is defined by the relation

$$S(t) = \tfrac{1}{2}\lim_{h\to 0}\bigl(S(t+h) + S(t-h)\bigr).$$

Let $N(t)$ be the number of zeros of $\zeta(s)$ in the rectangle $0 < \operatorname{Im} s \leqslant t$, $0 \leqslant \operatorname{Re} s \leqslant 1$ counted with multiplicities. At the points of discontinuity $N(t)$ is defined as follows:

$$N(t) = \tfrac{1}{2}\lim_{h\to 0}\bigl(N(t+h) + N(t-h)\bigr).$$

2010 *Mathematics Subject Classification:* Primary 11M06.

Supported by the Russian Fund of Fundamental Research (grant no. 11-01-00759-a).

The equation

$$N(t) = \frac{1}{\pi}\,\vartheta(t) + 1 + S(t) \tag{1.1}$$

is known as the Riemann–von Mangoldt formula and is true for any $t > 0$.

By $\varrho_n = \beta_n + i\gamma_n$ we denote the complex zeros of $\zeta(s)$ lying in the upper half-plane and ordered as follows: $0 < \gamma_1 < \gamma_2 < \cdots \leqslant \gamma_n \leqslant \gamma_{n+1} \leqslant \cdots$. Finally, let c_n be the real and positive zeros of $Z(t)$ indexed in ascending order and counted with their multiplicities.

Though the first three positive ordinates of zeros of $\zeta(s)$ had been already counted by Riemann, this fact was revealed only in 1932 by Siegel [**2**]. It seems that the first mathematical publication devoted to the calculation of zeta zeros belongs to Gram [**3**] (1895). He established that $\gamma_1 = 14.135$, $\gamma_2 = 20.82$, $\gamma_3 = 25.1$, but his method was too laborious and unfit for detecting of higher zeros. In 1902, Gram invented a more acceptable method for detecting the zeros of $\zeta(s)$.

The key idea of this method is the following. Let $A(t)$ and $B(t)$ be the real and imaginary parts of $\zeta\bigl(\frac{1}{2} + it\bigr)$ correspondingly. Then

$$\zeta\bigl(\tfrac{1}{2} + it\bigr) = e^{-i\vartheta(t)}Z(t) = Z(t)(\cos\vartheta(t) - i\sin\vartheta(t)),$$

and hence $A(t) = Z(t)\cos\vartheta(t)$, $B(t) = -Z(t)\sin\vartheta(t)$. We consider the real zeros of $B(t)$. These zeros are of two types. The zeros of the first type are the ordinates γ_n of zeros of $\zeta(s)$ lying on the critical line, and the zeros of the second type are the roots of the equation $\sin\vartheta(t) = 0$. Using Stirling's formula in the form

$$\vartheta(t) = \frac{t}{2}\ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O\Bigl(\frac{1}{t}\Bigr),$$

and considering the values $t > 7$, it is possible to show that the roots of the above equation generate the unbounded monotonic sequence: $t_0 = 9.6669\ldots$, $t_1 = 17.8456\ldots$, $t_2 = 23.1703\ldots$, $t_3 = 27.6702\ldots$, $\ldots$. Here t_n denotes the n-th Gram point, i.e., the unique solution of the equation $\vartheta(t_n) = (n-1)\pi$. Therefore, the value $\zeta\bigl(\frac{1}{2} + it_n\bigr)$ is real and

$$\zeta\bigl(\tfrac{1}{2} + it_n\bigr) = A(t_n) = Z(t_n)\cos\pi(n-1) = (-1)^{n-1}Z(t_n).$$

Suppose now that $A(t)$ has the same sign at the points t_{n-1} and t_n for some n. Then the values $Z(t_{n-1})$ and $Z(t_n)$ are of opposite sign. Hence, $Z(t)$ vanishes at the odd number of points between t_{n-1} and t_n.

Using the Euler–MacLaurin summation formula, Gram established that $A(t_n) > 0$ for $n = 1, 2, \ldots, 15$, and proved that all zeros of $\zeta(s)$ in the strip $0 < t < 66$ lie on the critical line. This method allowed him also to find approximately the ordinates $\gamma_1, \gamma_2, \ldots, \gamma_{15}$. Thus Gram established that there is exactly one zero c_n in each interval $G_n = (t_{n-1}, t_n]$, $n = 1, 2, \ldots, 15$ and, moreover, that $t_{n-1} < c_n < t_n$. However, he assumed also that this law is not universal: '*...the values* $A(t_n)$ *are positive for all* t_n *lying between* 10 *and* 65*. It seems that the function* $A(t)$ *is positive for most part of* t *under consideration. Obviously, the reason is that the first term of the sum* $\sum_1^n n^{-1/2}\cos(t\log n)$ *leads to the dominance of positive summands. If it is so, the regularity in the relative location of* c_n *and* t_n *will hold true for some* *time for the roots c lying closely to* c_{15} *until the equilibrium will set in*' (see [**1**]; for simplicity, we use here the notation of the present paper).

The phrase 'Gram's law' appeared for the first time in Hutchinson's paper [**4**]. He used this notion to underline the property that c_n and c_{n+1} are separated by the Gram point t_n. Hutchinson undertook wider calculations of zeros of $\zeta(s)$ in order to check the validity of Gram's assumption. He found the two first values of n that do not satisfy Gram's law: $n = 127$ and $n = 136$. Namely, he established that

$$t_{127} < \gamma_{127} < \gamma_{128} < t_{128}, \qquad t_{134} < \gamma_{135} < \gamma_{136} < t_{135}.$$

Ten years later, Titchmarsh continued in [**5, 6**] Hutchinson's calculations using Brunsviga, National and Hollerith machines. They found a lot of new exceptions from Gram's law, but the proportion of these exceptions did not exceed 4.5%[1]. The paper [**5**] contains also the first theoretical results concerning Gram's law. Thus, Titchmarsh proved that the inequality $A(t_n) = (-1)^{n-1}Z(t_n) > 0$ fails for infinitely many n. Moreover, he proved that the sequence of fractions $\tau_n = \frac{c_n - t_n}{t_{n+1} - t_n}$ is unbounded. The last assertion means that there are infinitely many zeros c_n lying outside the corresponding intervals G_n.

Though the rule with infinitely many exceptions is not a law in a rigorous sense, the notion 'Gram's law' is widely used now, but in different senses. We will also use this notion for any assertion concerning the relative location of ordinates of zeros of $\zeta(s)$ and Gram points. Further we present a kind of 'classification' of 'Gram's laws'.

Definition 1.1. Gram's interval $G_n = (t_{n-1}, t_n]$ satisfies to the Gram's Strict Law (GSL) iff G_n contains a zero c_n of $Z(t)$.

This definition is close to that of Hutchinson and Titchmarsh. But here we allow the coincidence of t_n with a zero of Hardy's function. The reason is that now only little is known about the number of such coincidences (or noncoincidences). It seems that $Z(t_n)$ does not vanish for every n, i.e., $c_m \neq t_n$ for any m and n. But we only know that $Z(t_n) \neq 0$ for at least $(4 - o(1))N(\ln N)^{-1}$ values of n, $1 \leqslant n \leqslant N$ (see [**7**])[2]. The unboundedness of the fractions τ_n implies that GSL fails for infinitely many cases. Unfortunately, it is still unknown whether the number of cases when GSL holds true is finite or infinite.

The Definition 1.1 contains a very rigid restriction. Namely, the index of the interval G_n and the index of a zero c belonging to G_n must be equal. If we omit this restriction, we come to the second version of Gram's law.

---

[1]Titchmarsh mentioned 43 exceptions that he had found during his calculation of the first 1041 zeros of $Z(t)$ lying in the interval $0 < t \leqslant 1468$. However, there are 1042 zeros of Hardy's function and 1041 Gram's points t_n between $t = 0$ and $t = 1468$, and there are 45 values of n such that $(-1)^{n-1}Z(t_n) < 0$.

[2]Selberg's theorem formulated without a proof in [**8**] and cited below implies that $Z(t_n) \neq 0$ for a positive proportion of n. It's interesting to note that the values $Z(t_n)$ are very close to 0 for some n. For example, the minima of $|Z(t_n)|$ for $n \leqslant 10^5$ and $n \leqslant 10^6$ are equal to $1.238 \cdot 10^{-5}$ ($n = 97\,281$) and to $8.908 \cdot 10^{-8}$ ($n = 368\,383$) respectively.

DEFINITION 1.2. Gram's interval G_n satisfies to Gram's law (GL) iff G_n contains exactly one (simple) zero of $Z(t)$.

It is possible to show that GSL and GL are not equivalent. The failure (validity) of one statement for a given n does not imply the failure (validity) of the other statement. For example, $G_1, \dots, G_{126}$ satisfy both GSL and GL; G_{127} does not satisfy neither GSL, nor GL; further, G_{128} satisfies GSL, but does not satisfy GL; finally, G_{3359}, G_{3778}, G_{4542} satisfy GL, but do not satisfy GSL.

The determination of zeros of $Z(t)$ in a given interval (a, b) is usually reduced to an evaluation of number of sign-changes of $Z(t)$ in (a, b). Therefore, this method allows one to determine only the parity of the number of zeros. For example, the inequality $Z(a)Z(b) < 0$ guarantees the existence of an odd number of zeros in (a, b) counted with multiplicity. Therefore, it seems natural to consider one more type of Gram's law.

DEFINITION 1.3. Gram's interval G_n satisfies Gram's Weak Law (GWL) iff G_n contains an odd number of zeros of $Z(t)$.

Obviously, GL implies GWL, but the reverse statement is not correct. For example, if $n = 2147$, then G_n contains exactly three zeros of Hardy's function, namely c_{n-1}, c_n and c_{n+1}. The inequality $Z(t_{n-1})Z(t_n) < 0$ is sufficient (but not necessary) condition for GWL. Therefore, Titchmarsh's formula (see [**9**])

$$\sum_{n \leqslant N} Z(t_{n-1})Z(t_n) \sim -2(\gamma + 1)N$$

implies that GWL holds true for infinitely many cases (here γ denotes Euler's constant).

The statement 'GSL holds true for all $n \geqslant n_0$' implies the boundedness of the fractions τ_n as $n \to +\infty$. The last fact contradicts to some properties of $S(t)$ established by Bohr and Landau [**10**] in 1913.

In the middle of the 40's, Selberg invented a new powerful method of researching of the function $S(t)$ (see [**11**,**12**]) and obtained a lot of very deep results concerning the distribution of zeros of $\zeta(s)$. In particular, in [**8**] he formulated (without a proof) the following theorem: there exist absolute constants K and N_0 such that for $N > N_0$, $1 \leqslant n \leqslant N$, the numbers $Z(t_{n-1})$ and $Z(t_n)$ are of different sign in more than KN cases, and of the same sign in more than KN cases. This theorem implies that both GWL and GL fail for a positive proportion of cases, and that GWL holds true for a positive proportion of cases.

Denote by $\nu_k = \nu_k(N)$ the number of Gram's intervals G_n, $1 \leqslant n \leqslant N$, that contain exactly k ordinates of zeros of $\zeta(s)$ (here we consider all the zeros in the critical strip, but not only those lying on the critical line). It is not difficult to prove that Selberg's theorem implies the following relations:

$$\nu_0 + \nu_2 + \nu_4 + \cdots \geqslant KN, \tag{1.2}$$

$$\nu_1 + \nu_3 + \nu_5 + \cdots \geqslant KN, \tag{1.3}$$

These inequalities are weaker than Selberg's original assertion. The reason is that Selberg's theorem deals with the ordinates in open intervals (t_{n-1}, t_n) (instead of the interval G_n) and hence with nonvanishing of $Z(t)$ at the endpoints of such intervals for a positive proportion of n.

Further, (1.2) implies a weaker estimate

$$\nu_0 + \nu_2 + \nu_3 + \nu_4 + \cdots \geqslant KN. \tag{1.4}$$

This inequality shows that the positive proportion of Gram's intervals contain an 'abnormal' number (i.e., $\neq 1$) of ordinates. Both (1.2) and (1.4) imply that GWL and GL fail for positive proportion of n.

As far as the author knows, the proofs of either Selberg's theorem or formulas (1.2), (1.3) have never been published. Estimate (1.4) was proved by Trudgian [**14**] in 2009[3]. He also pointed out in [**14**] that (1.4) implies the inequality

$$\nu_0 > K_1 N \tag{1.5}$$

for any fixed K_1, $0 < K_1 < \frac{1}{2}K$, and for $N \geqslant N_0(K_1)$. Indeed, the following identities hold true:

$$0 \cdot \nu_0 + 1 \cdot \nu_1 + 2 \cdot \nu_2 + 3 \cdot \nu_3 + \ldots = N + S(t_N + 0), \tag{1.6}$$

$$\nu_0 + \nu_1 + \nu_2 + \nu_3 + \ldots = N. \tag{1.7}$$

It is easy to see that (1.6) expresses the fact that the number of zeros whose ordinates are positive and do not exceed t_N, is equal to $N(t_N + 0) = N + S(t_N + 0)$, and (1.7) expresses the fact that the number of G_n contained in $(0, t_N]$ is equal to N. Subtracting (1.7) from (1.6) we find: $\nu_0 = \nu_2 + 2\nu_3 + 3\nu_4 + \cdots - S(t_N + 0)$. Adding ν_0 to both parts and using the classical estimate $S(t) = O(\ln t)$ (see [**13**]) we get:

$$\begin{aligned} 2\nu_0 &= \nu_0 + \nu_2 + 2\nu_3 + 3\nu_4 + \ldots \\ &\geqslant \nu_0 + \nu_2 + \nu_3 + \nu_4 + \cdots + O(\ln N) \geqslant KN + O(\ln N). \end{aligned}$$

This proves (1.5). Similarly to (1.2) and (1.4), the inequality (1.5) implies that both GWL and GL fail for a positive proportion of cases.

It is interesting to note the following. It is expected that $\nu_1(N) \geqslant cN$ or even $\nu_1(N) \sim c_1 N$ as N grows. However, a weaker relation $\nu_1(N) \to +\infty$ as $N \to +\infty$ is still unproved. Thus, we don't know whether the number of cases when GL holds true is finite or not.

There are some reasons to think that Selberg interpreted Gram's Law in a way different from Titchmarsh's one and different from GSL, GL and GWL. In dealing with Gram's Law, Titchmarsh considered only the real zeros of Hardy's function. We think that Selberg considered all the zeros of $\zeta(s)$ in the critical strip. The weighltly arguments that sustain this point of view will be introduced later. Now we give here some preliminary remarks.

Let γ_n be an ordinate of a zero of $\zeta(s)$ in the critical strip. Then we determine a unique integer $m = m(n)$ such that $t_{m-1} < \gamma_n \leqslant t_m$, and set $\Delta_n = m - n$.

---

[3]We note that the inequalities $\nu_0 \gg N$, $\nu_2 + \nu_3 + \nu_4 + \cdots \gg N$ were formulated without a proof by Fujii in [**15**].

Definition 1.4. We say that Gram–Selberg's Phenomenon (GSP) is observed for the ordinate γ_n iff $\Delta_n = 0$, i.e., iff $t_{n-1} < \gamma_n \leqslant t_n$.

It seems likely that the property of γ_n to satisfy the condition $\Delta_n = 0$ was called by Selberg 'Gram's Law'.

The above result of Selberg implies that there is a positive proportion of cases when GSP is not observed. However, it is possible to say much more about GSP. Selberg established the formulas

$$\sum_{n\leqslant N} \Delta_n^{2k} = \frac{(2k)!}{k!}\frac{N}{(2\pi)^{2k}}\,(\ln\ln N)^k + O\big(N(\ln\ln N)^{k-1/2}\big), \tag{1.8}$$

$$\sum_{n\leqslant N} \Delta_n^{2k-1} = O\big(N(\ln\ln N)^{k-1}\big), \tag{1.9}$$

where $k \geqslant 1$ is a fixed integer. In view of (1.8),(1.9) he assumed that the inequalities

$$\frac{1}{\Phi(n)}\sqrt{\ln\ln n} < |\Delta_n| \leqslant \Phi(n)\sqrt{\ln\ln n}$$

hold true for 'almost all' n. Here $\Phi(x)$ denotes any fixed positive unbounded function. In particular, this assumption implies that GSP is not observed in 'almost all' cases.

It follows from the remark in [**16**, p. 355] that Selberg had found a proof of his own assumption long before 1989, but he did not publish it. For a version of a proof of Selberg's assumption, see author's papers [**17**, **18**].

Thus, we can't expect the occurrence of GSP in positive proportion of cases. The reason is that Definition 1.4 impose a rigid restriction on γ_n (the ordinate should belong to Gram's interval with the same number). Thus, GSP is a very rare phenomenon. It is natural to ask whether GSP occurs in infinitely many cases or not. Since no such results had been published, some quantitative statements about the frequency of occurrence of GSP seem to have some interest (see the recent paper of the author [**19**]).

Now we place the results concerning Gram's law in the table below.

The present paper contains some new statements concerning the behavior of the function $S(t)$ at the Gram points. We use them to prove Selberg's formulas (1.8), (1.9) and apply them to other problems connected with Gram's law. The paper is organized as follows.

First, Section 2 contains auxiliary assertions. In Section 3, the sum

$$\sum_{N<n\leqslant N+M} \big(S(t_{n+m}+0) - S(t_n+0)\big)^{2k} \tag{1.10}$$

is evaluated. Here $k \geqslant 0$, $m > 0$ are sufficiently large integers, that may grow slowly with N (Theorem 3.1). The correct bound (in the sense of order of growth) for the sum (1.10) with $m = 1$ is also given here (Theorem 3.2). We note that the bounds of such type are contained in [**15**]. But they hold true only for a 'long' interval of summation: $1 \leqslant n \leqslant N$ or $N < n \leqslant N+M$, $M \asymp N$. The statements of the present paper are valid for a 'short' interval, namely for the case $M \asymp N^{\alpha+\varepsilon}$, $\alpha = \frac{27}{82} = \frac{1}{3} - \frac{1}{246}$.

<table>
<tr><td></td><td>holds true</td><td>fails</td></tr>
<tr><td>GSL</td><td rowspan="2">it's unknown, whether
the number of such cases
is finite or not</td><td>for infinitely many cases:
– Titchmarsh [5], 1935;</td></tr>
<tr><td>GL</td><td>for infinitely many cases:
– Titchmarsh [5], 1935;
for positive proportion of cases:
– Selberg [8], 1946;
– Fujii [15], 1987;
– Trudgian [14], 2009.</td></tr>
<tr><td>GWL</td><td>for infinitely many cases:
– Titchmarsh [9], 1934;
for positive proportion of cases:
– Selberg [8], 1946;</td><td>for positive proportion of cases:
– Selberg [8], 1946;
– Fujii [15], 1987;
– Trudgian [14], 2009.</td></tr>
<tr><td>GSP</td><td>for infinitely many cases</td><td>for infinitely many cases:
– Titchmarsh [5], 1935;
for 'almost all' cases:
– Selberg [16], 1989.</td></tr>
</table>

Further, Theorem 3.3 in Section 3 gives a true order of magnitude of the sum

$$\sum_{N<n\leqslant N+M}\bigl|S(t_n+0)-S(t_{n-1}+0)\bigr|.$$

This statement is based on Theorems 3.1 and 3.2 and plays the key role in the proof of the inequalities $\nu_0(N+M)-\nu_0(N)\gg M$, $\sum_{k\geqslant 2}\bigl(\nu_k(N+M)-\nu_k(N)\bigr)\gg M$ (Theorem 3.4). Analogues of these bounds in the case of 'long' intervals of summation were formulated (without a proof) for the first time by Selberg in [**8**] as corollaries of his theorem cited above.

In Section 4, a nontrivial bound for the alternating sum

$$T_k=\sum_{N<n\leqslant N+M}S^k(t_n+0)\bigl(S(t_n+0)-S(t_{n-1}+0)\bigr)$$

is given (Theorem 4.1). This estimate leads to a new proof of Selberg's formulas (1.8), (1.9) (see Theorem 4.2) and of the assumption that $\Delta_n\neq 0$ for 'almost all' n. The proofs of these facts differ from those given in the author's previous paper [**18**]. They don't use the information about the number of solutions of the inequalities $a<\Delta_n\leqslant b$ with the condition $N<n\leqslant N+M$. It is likely that the proof of Theorem 4.2 presented here is close to Selberg's original proof.

Finally, in Section 5 we try to motivate our assumption that Selberg considered all the complex zeros of $\zeta(s)$ in dealing with Gram's law in [**8**]. This argumentation leads us to a new equivalent of 'almost Riemann hypothesis' (see Theorem 5.1; 'almost Riemann hypothesis' claims that almost all complex zeros of $\zeta(s)$ lie on the critical line).

Throughout the paper, ε denotes an arbitrary small positive number, $0 < \varepsilon < 10^{-3}$; $N_0(T)$ denotes the number of zeros of $\zeta\bigl(\frac{1}{2} + it\bigr)$ in the strip $0 < t \leqslant T$; $N \geqslant N_1(\varepsilon) > 0$ is a sufficiently large integer; $L = \ln\ln N$, M is an arbitrary integer satisfying the conditions $N^{\alpha+\varepsilon_1} \leqslant M \leqslant N^{\alpha+\varepsilon}$, $\alpha = \frac{27}{82}$, $\varepsilon_1 = 0.9\varepsilon$; $\theta, \theta_1, \theta_2, \dots$ are complex numbers whose absolute values do not exceed 1 and which are, generally speaking, different in different relations. In some cases we use for brevity the notation $\Delta(n)$ for the value $S(t_n + 0)$.

2. Auxiliary lemmas

LEMMA 2.1. *The following relations hold true for any $x \geqslant 2$:*

$$\sum_{p\leqslant x} \frac{\ln p}{p} < \ln x, \qquad \sum_{p\leqslant x} \frac{1}{p} = \ln\ln x + c + \frac{\theta}{\ln^2 x};$$

here $c = 0.26\dots$ and $-\frac{1}{2} < \theta < 1$.

For a proof, see [**20**].

LEMMA 2.2. *Suppose that $0 < \kappa < \frac{1}{2}$, $0 < c < \frac{1}{2} - \kappa$, μ, ν are integers such that $\mu, \nu \geqslant 0$, $\mu + \nu = 2k$, $k \geqslant 1$, $N \geqslant \exp(9\kappa^{-1})$, $M \geqslant \exp(3kc^{-1})$, $y = M^{c/k}$. Furthermore, let $p_1, \dots, p_\nu, q_1, \dots, q_\mu$ take values of prime numbers from the interval $(1, y]$ and satisfy the condition $p_1 \dots p_\nu \neq q_1 \dots q_\mu$. Finally, suppose that $|a(p)| \leqslant \delta$ for $p \leqslant y$. Then the sum S,*

$$S = \sum_{N<n\leqslant N+M} \sum_{\substack{p_1,\dots,p_\nu\\ q_1,\dots,q_\mu}} \frac{a(p_1)\dots a(p_\nu)\overline{a}(q_1)\dots\overline{a}(q_\mu)}{\sqrt{p_1\dots q_\mu}} \left(\frac{q_1\dots q_\mu}{p_1\dots p_\nu}\right)^{it_n},$$

satisfies the bound $|S| < (\delta y^{3/2})^{2k} \ln N$.

For a proof, see [**17**].

LEMMA 2.3. *Let $k \geqslant 1$ be an integer, $y > e^3$, and let $p_1, \dots, p_k, q_1, \dots, q_k$ take values of prime numbers from the interval $(1, y]$. Then the following relation holds:*

$$\sum_{p_1\dots p_k = q_1\dots q_k} \frac{a(p_1)\dots a(p_k)\overline{a}(q_1)\dots\overline{a}(q_k)}{\sqrt{p_1\dots q_k}} = k!\bigl(\sigma_1^k + \theta_k k^2 \sigma_1^{k-2}\sigma_2\bigr),$$

where $\sigma_j = \sum_{p\leqslant y} \bigl(\frac{1}{p}|a(p)|^2\bigr)^j$, $j = 1, 2$, $-1 \leqslant \theta_k \leqslant 0$, and $\theta_1 = 0$.

For a proof, see [**21, 22**].

Suppose that $x = t_N^{0.1\varepsilon}$. For positive t and y we define

$$V(t) = V_y(t) = \frac{1}{\pi} \sum_{p<y} \frac{\sin(t\ln p)}{\sqrt{p}}, \qquad R(t) = S(t) + V(t).$$

LEMMA 2.4. *Suppose that k is an integer, $1 \leqslant k \leqslant \frac{1}{192}\ln x$, $y = x^{1/(4k)}$, $V(t) = V_y(t)$. Then the following inequality holds*

$$\sum_{N<n\leqslant N+M} R^{2k}(t_n + 0) \leqslant (Ae^{-4}k)^{2k} M, \quad \text{where } A = e^{21}\varepsilon^{-1.5}.$$

LEMMA 2.5. *Suppose that k is an integer, $1 \leqslant k \leqslant \sqrt{L}$. Then the following relations hold:*

$$\sum_{N<n\leqslant N+M} S^{2k}(t_n+0) = \frac{(2k)!}{k!}\frac{ML^k}{(2\pi)^{2k}}\big(1+\theta A^k L^{-0.5}\big),$$

$$\Big|\sum_{N<n\leqslant N+M} S^{2k-1}(t_n+0)\Big| \leqslant \frac{3.5}{\sqrt{B}}\,(Bk)^k ML^{k-1},$$

$$\sum_{N<n\leqslant N+M} S^{2k}(t_n+0) \leqslant 2\Big(\frac{kA}{\pi^2 e}\Big)^k ML^k,$$

where $B = A^2e^{-8}$, and A is defined in Lemma 2.4.

For the proofs of these two lemmas, see [**17**] (the substitution of t_n to t_n+0 does not affect the truth of the result; the reason is that the above substitution does not affect the functions that approximate $S(t)$ in the proofs of Lemmas 2.4 and 2.5).

LEMMA 2.6. *Suppose that m is an integer, $1 \leqslant m \leqslant M$. Then the equality*

$$t_{n+m} - t_n = \frac{\pi m}{\vartheta'(t_N)} + \frac{3\theta M}{N(\ln N)^2}$$

holds true for $N < n \leqslant N+M$.

PROOF. By Lagrange's mean value theorem, we have

$$\pi m = \vartheta(t_{n+m}) - \vartheta(t_n) = (t_{n+m}-t_n)\vartheta'(\xi), \quad t_{n+m}-t_n = \frac{\pi m}{\vartheta'(\xi)}$$

for some ξ, $t_n < \xi < t_{n+m}$. Since $\vartheta'(t)$, $\vartheta''(t)$ are monotonic for $t > 7$, by the inequality

$$\frac{t_N}{2\pi}\ln\frac{t_N}{2\pi} > N$$

we get:

$$\begin{aligned} 0 &< \frac{1}{\vartheta'(t_N)} - \frac{1}{\vartheta'(\xi)} = \frac{\vartheta'(\xi)-\vartheta'(t_N)}{\vartheta'(t_N)\vartheta'(\xi)} \\ &\leqslant \frac{(t_{N+M}-t_N)\vartheta''(t_N)}{\big(\vartheta'(t_N)\big)^2} = \frac{\pi M}{\vartheta'(\zeta)}\frac{\vartheta''(t_N)}{\big(\vartheta'(t_N)\big)^2} < \frac{\pi M\vartheta''(t_N)}{\big(\vartheta'(t_N)\big)^3} < \frac{3M}{N(\ln N)^2}. \end{aligned}$$

This proves the lemma.

LEMMA 2.7. *Suppose that $0 < h_0 < \frac{1}{2}$ is a sufficiently small constant, $0<h<h_0$, $h\ln x > 2$, and let $V(x;h) = \sum_{p\leqslant x}\frac{1}{p}\sin^2\big(\frac{1}{2}h\ln p\big)$. Then the following relation holds: $V(x;h) = \frac{1}{2}\ln(h\ln x) + 1.05\theta$.*

PROOF. Setting $y = e^{\lambda/h}$ for some $1 < \lambda < 2$, we obtain:

$$V(x;h) = \biggl(\sum_{p\leqslant y} + \sum_{y<p\leqslant x}\biggr)\frac{\sin^2\bigl(\frac{1}{2}h\ln p\bigr)}{p} =$$
$$= \sum_{p\leqslant y}\frac{\sin^2\bigl(\frac{1}{2}h\ln p\bigr)}{p} + \frac{1}{2}\sum_{y<p\leqslant x}\frac{1-\cos(h\ln p)}{p} = V_1 + \frac{1}{2}(V_2 - V_3).$$

The application of Lemma 2.1 yields:

$$V_1 \leqslant \Bigl(\frac{h}{2}\Bigr)^2\sum_{p\leqslant y}\frac{(\ln p)^2}{p} \leqslant \Bigl(\frac{h}{2}\Bigr)^2(\ln y)\sum_{p\leqslant y}\frac{\ln p}{p} < \Bigl(\frac{h}{2}\ln y\Bigr)^2 = \frac{\lambda^2}{4},$$
$$V_2 = \biggl(\sum_{p\leqslant x} - \sum_{p\leqslant y}\biggr)\frac{1}{p} = \ln(h\ln x) - \ln\lambda + 2\theta\Bigl(\frac{h}{\lambda}\Bigr)^2.$$

Now we divide the domain of p in V_3 into the intervals of the form $a < p \leqslant b$, where $b \leqslant 2a$, $a = 2^k y$, $k = 0, 1, 2, \ldots$. Thus we get: $V_3 = \mathrm{Re}\sum_{y<p\leqslant x}\frac{p^{ih}}{p} = \mathrm{Re}\sum_a V_3(a)$, $V_3(a) = \sum_{a<p\leqslant b}\frac{p^{ih}}{p}$. Setting $\mathbb{C}(u) = \sum_{a<p\leqslant u}\frac{1}{p} = \ln\ln u - \ln\ln a + 2\theta\ln^{-2}a$ and applying the Abel's summation formula, we have

$$V_3(a) = \mathbb{C}(b)b^{ih} - \int_a^b \mathbb{C}(u)\,du^{ih} = \int_a^b \frac{u^{ih}du}{u\ln u} + \frac{3\theta_1}{\ln^2 a},$$

and therefore

$$V_3 = \mathrm{Re}(j) + 3\theta_2\sum_{k\geqslant 0}\frac{1}{(k\ln 2 + \ln y)^2}, \quad j = \int_y^x \frac{u^{ih}du}{u\ln u}.$$

Integration by parts yields:

$$j = \frac{1}{ih}\biggl(\frac{x^{ih}}{\ln x} - \frac{y^{ih}}{\ln y} - \int_y^x u^{ih}d\frac{1}{\ln u}\biggr), \quad |j| \leqslant \frac{2}{h\ln y} = \frac{2}{\lambda}.$$

Finally, we obtain

$$|V_3| \leqslant \frac{2}{\lambda} + \frac{9}{\ln y} = \frac{2+9h}{\lambda}, \quad V(x;h) = \frac{1}{2}\ln(h\ln x) + v(x;h),$$

where

$$|v(x;h)| \leqslant \frac{\lambda^2}{4} + \frac{1}{\lambda} + \frac{1}{2}\ln\lambda + \frac{9h}{\lambda} + \frac{h^2}{\lambda^2}.$$

Setting $\lambda = 1.5$, we arrive at the assertion of the lemma. □

LEMMA 2.8. *The inequality* $N_0(t) > (0.4 + 7\cdot 10^{-3})N(t)$ *holds for* $t > t_0 > 1$.

For a proof, see [**23**].

3. On the mean values of the quantities $S(t_{n+m}+0)-S(t_n+0)$

Suppose that $m \geqslant 1$ is an integer. Let us consider the union of m adjacent Gram's intervals $G_{n+1}, G_{n+2}, \ldots, G_{n+m}$, that is the interval $(t_n, t_{n+m}]$. By Riemann–von Mangoldt's formula (1.1), the number of ordinates in $(t_n, t_{n+m}]$ is equal to

$$\begin{aligned} N(t_{n+m}+0)-N(t_n+0) &= \frac{1}{\pi}\big(\vartheta(t_{n+m})-\vartheta(t_n)\big)+S(t_{n+m}+0)-S(t_n+0) \\ &= m+S(t_{n+m}+0)-S(t_n+0). \end{aligned} \tag{3.1}$$

The number of γ_n that do not exceed a given bound equals asymptotically to the number of Gram points in the same domain. Then it is natural to call the number m as an 'expected' number of ordinates of zeros of $\zeta(s)$ in the interval $(t_n, t_{n+m}]$. Hence, the difference

$$S(t_{n+m}+0)-S(t_n+0) \tag{3.2}$$

is a deviation of the 'true' number of ordinates from the 'expected' number.

The below Theorem 3.1 shows that this deviation often takes a very large values (of order $\sqrt{\ln m}$, for example). This fact was observed for the first time by Fujii [**15**] for the case when the interval of summation is long ($1 \leqslant n \leqslant N$) and when m grows with N. He proved that the distribution function for the normalized differences (3.2) tends to the Gaussian distribution as $N \to +\infty$.

THEOREM 3.1. *Let k and m be integers that satisfy the conditions $k \geqslant 1$, $k\varepsilon^{-1}\exp(\lambda k^2) \leqslant m \leqslant c\ln N$, where $\lambda = (2Be\pi^2)^2$, B is defined in Lemma 2.5, and c is a sufficiently small absolute constant. Then the following relation holds:*

$$\sum_{N<n\leqslant N+M}\big(S(t_{n+m}+0)-S(t_n+0)\big)^{2k} = \frac{(2k)!}{k!}M\Big(\frac{1}{2\pi^2}\ln\frac{m\varepsilon}{k}\Big)^k\Big(1+\frac{6\theta\sqrt{B}\,4^k k\sqrt{k}}{\sqrt{\ln(m\varepsilon/k)}}\Big).$$

PROOF. Let $x = t_N^{0.1\varepsilon}$, $y = x^{1/(4k)}$, $V(t) = V_y(t)$. By Lemma 2.6, $t_{n+m}-t_n = h+\varepsilon_n$, where

$$h = \frac{\pi m}{\vartheta'(t_N)}, \quad |\varepsilon_n| \leqslant \frac{3M}{N(\ln N)^2}.$$

By Lagrange's mean value theorem and the inequalities

$$|V'(t)| \leqslant \frac{1}{\pi}\sum_{p<y}\frac{\ln p}{\sqrt{p}} < \sqrt{y} = x^{1/(8k)} < N^{\varepsilon/80}$$

we get

$$V(t_{n+m}) = V(t_n+h+\varepsilon_n) = V(t_n+h)+\varepsilon_n V'(t_n+h+\theta\varepsilon_n) = V(t_n+h)+\theta_1 N^{-2/3}.$$

Hence

$$V(t_{n+m})-V(t_n) = \frac{2}{\pi}W(t_n)+\theta_2 N^{-2/3},$$

$$W(t) = \frac{1}{2}\sum_{p<y}\frac{\sin((t+h)\ln p)-\sin(t\ln p)}{\sqrt{p}} = \sum_{p<y}\frac{\sin\big(\frac{1}{2}h\ln p\big)}{\sqrt{p}}\cos\big((t+\tfrac{1}{2}h)\ln p\big).$$

Using the trivial bound $|W(t)| < \sqrt{y}$ and Lagrange's mean value theorem, we obtain

$$\begin{aligned}
&\bigl(V(t_{n+m}) - V(t_n)\bigr)^{2k} \\
&\qquad = \bigl(\tfrac{2}{\pi}\bigr)^{2k} W^{2k}(t_n) + \theta k 2^{2k-1}\bigl(|W(t_n)|^{2k-1}N^{-2/3} + N^{-4k/3}\bigr) \\
&\qquad = \bigl(\tfrac{2}{\pi}\bigr)^{2k} W^{2k}(t_n) + \theta_1 x N^{-2/3}.
\end{aligned}$$

Summing over n and denoting the corresponding sum by W_1, we have

$$W_1 = \sum_{N<n\leqslant N+M} \bigl(V(t_{n+m}) - V(t_n)\bigr)^{2k} = \bigl(\tfrac{2}{\pi}\bigr)^{2k} W_2 + \theta_2 N^{-1/3},$$

where $W_2 = \sum_{N<n\leqslant N+M} W^{2k}(t_n)$. Next, we put $W(t) = \frac{1}{2}\bigl(U(t) + \overline{U}(t)\bigr)$, where

$$U(t) = \sum_{p<y} \frac{a(p)}{\sqrt{p}}\, p^{it}, \quad a(p) = p^{ih/2}\sin\bigl(\tfrac{1}{2}h\ln p\bigr).$$

Then

$$\begin{aligned}
W_2 &= 2^{-2k}\sum_{\nu=0}^{2k}\binom{2k}{\nu} w_\nu, \\
w_\nu &= \sum_{N<n\leqslant N+M}\ \sum_{\substack{p_1,\ldots,p_\nu<y\\ q_1,\ldots,q_\mu<y}} \frac{a(p_1)\ldots a(p_\nu)\overline{a}(q_1)\ldots\overline{a}(q_\mu)}{\sqrt{p_1\ldots q_\mu}}\left(\frac{p_1\ldots p_\nu}{q_1\ldots q_\mu}\right)^{it_n},
\end{aligned}$$

where $\mu = 2k-\nu$. By setting $\kappa = \frac{1}{4}$, $c = (k\ln y)(\ln M)^{-1}$, $\delta = 1$ in Lemma 2.2, we obviously have $0 < c < \frac{1}{4} = \frac{1}{2} - \kappa$, $N \geqslant e^{36} = \exp(9\kappa^{-1})$, $y = x^{1/(4k)} > e^4$,

$$\exp\bigl(3kc^{-1}\bigr) = \exp\left(\frac{3\ln M}{\ln y}\right) \leqslant \exp\bigl(\tfrac{3}{4}\ln M\bigr) < M.$$

Thus, the conditions of Lemma 2.2 are satisfied. Hence, for $\nu \neq k$ we have

$$|w_\nu| \leqslant (y^{3/2})^{2k}\ln N = x^{3/4}\ln N < x.$$

The contribution of the terms of w_k that obey the condition $p_1\ldots p_k \neq q_1\ldots q_k$ is estimated as above. Therefore,

$$W_2 = 2^{-2k}\binom{2k}{k} Mw + \theta 2^{-2k}\sum_{\nu=0}^{2k}\binom{2k}{\nu} x = 2^{-2k}\binom{2k}{k} Mw + \theta x,$$

where

$$w = \sum_{p_1\ldots p_k = q_1\ldots q_k} \frac{a(p_1)\ldots\overline{a}(q_k)}{\sqrt{p_1\ldots q_k}}.$$

By Lemma 2.3, $w = k!(\sigma_1^k + \theta_k k^2\sigma_1^{k-2}\sigma_2)$, where

$$\sigma_1 = \sum_{p<y}\frac{1}{p}\sin^2\bigl(\tfrac{1}{2}h\ln p\bigr), \quad \sigma_2 \leqslant \sum_p \frac{1}{p^2} < \frac{1}{2}$$

and $-1 \leqslant \theta_k \leqslant 0$. Since

$$h\ln y = \frac{2\pi m\ln y}{\ln t_N}\bigl(1+o(1)\bigr) = \frac{\pi m\varepsilon}{20k}\bigl(1+o(1)\bigr) > 2,$$
$$h \leqslant \frac{2\pi m}{\ln N}\bigl(1+o(1)\bigr) \leqslant 2\pi c\bigl(1+o(1)\bigr),$$

the conditions of Lemma 2.7 are satisfied for $h_0 = 7c$ and for a sufficiently small c. Hence,

$$\sigma_1 = \frac{1}{2}\ln\,(h\ln y) + 1.05\theta = \frac{1}{2}\ln\frac{\pi m\varepsilon}{20k}\bigl(1+o(1)\bigr) + 1.05\theta = \frac{1}{2}\ln\frac{m\varepsilon}{k} + 2\theta_1.$$

Further, the inequality $\ln\,(m\varepsilon/k) \geqslant 100k$ implies the following bounds for σ_1, w, W_2 and W_1:

$$\sigma_1^k \leqslant \Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k\Bigl(1+\frac{4}{\ln\,(m\varepsilon/k)}\Bigr)^k \leqslant \Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k\Bigl(1+\frac{1}{25k}\Bigr)^k < 1.1\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k,$$
$$w \leqslant k!\sigma_1^k < 1.1k!\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k,$$
$$W_2 \leqslant 2^{-2k}\frac{(2k)!}{k!}\,M\cdot 1.1\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + x,$$
$$W_1 \leqslant \pi^{-2k}\frac{(2k)!}{k!}\,M\cdot 1.1\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + x + N^{-1/3} < 2\Bigl(\frac{2k}{\pi^2 e}\Bigr)^k\Bigl(\ln\frac{m\varepsilon}{k}\Bigr)^k M.$$

Moreover,

$$\begin{aligned}
w &= k!\biggl(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k} + 2\theta\Bigr)^k + \theta_k k^2\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\ + 2\Bigr)^{k-2}\biggr) = \\
&= k!\biggl(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + 2\theta k\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\ + 2\Bigr)^{k-1} + \theta_k k^2\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\ + 2\Bigr)^{k-2}\biggr) = \\
&= k!\biggl(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + 4\theta k\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\ + 2\Bigr)^{k-1}\biggr),\\
W_1 &= \Bigl(\frac{2}{\pi}\Bigr)^{2k}\biggl(2^{-2k}\binom{2k}{k}k!M\biggl(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + 4\theta k\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k} + 2\Bigr)^{k-1}\biggr) + \theta x\biggr) + \\
&+ \theta N^{-1/3}\\
&= \frac{(2k)!}{k!}\frac{M}{\pi^{2k}}\biggl(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + 4.1\theta k\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k} + 2\Bigr)^{k-1}\biggr).
\end{aligned}$$

Denoting by W_0 the initial sum of the theorem and noting that

$$S(t_{n+m}+0) - S(t_n+0) = -\bigl(V(t_{n+m}) - V(t_n)\bigr) + \bigl(R(t_{n+m}+0) - R(t_n+0)\bigr),$$

we get

$$\begin{aligned}
\bigl(S(t_{n+m}+0) - S(t_n+0)\bigr)^{2k} = \bigl(V(t_{n+m}) - V(t_n)\bigr)^{2k}&\\
+ \theta k 2^{2k-1}\Bigl(\bigl(R(t_{n+m}+0) - R(t_n+0)\bigr)^{2k}&\\
+\bigl|V(t_{n+m}) - V(t_n)\bigr|^{2k-1}\bigl|R(t_{n+m}+0) - R(t_n+0)\bigr|\Bigr)&,
\end{aligned}$$

$$W_0 = \sum_{N<n\leqslant N+M} \bigl(S(t_{n+m}+0) - S(t_n+0)\bigr)^{2k} = W_1 + \theta_1 k 2^{2k-1}(W_3+W_4),$$

where

$$W_3 = \sum_{N<n\leqslant N+M} \bigl(R(t_{n+m}+0) - R(t_n+0)\bigr)^{2k},$$

$$W_4 = \sum_{N<n\leqslant N+M} \bigl|V(t_{n+m}) - V(t_n)\bigr|^{2k-1}\bigl|R(t_{n+m}+0) - R(t_n+0)\bigr|.$$

By Lemma 2.4,

$$\begin{aligned} W_3 &\leqslant 2^{2k-1} \sum_{N<n\leqslant N+M} \bigl(R^{2k}(t_{n+m}+0) + R^{2k}(t_n+0)\bigr) \\ &\leqslant 2^{2k} \sum_{N<n\leqslant N+2M} R^{2k}(t_n+0) \\ &\leqslant 2^{2k}\cdot 2M(\sqrt{B}k)^{2k} = 2M(2\sqrt{B}k)^{2k}. \end{aligned}$$

Further, combining the above bounds for W_1 and W_3 with Hölder's inequality, we have

$$\begin{aligned} W_4 \leqslant W_1^{1-1/(2k)} W_3^{1/(2k)} &\leqslant 2^{1/(2k)}\, 2\sqrt{B}k\cdot 2^{1-1/(2k)}\Bigl(\frac{2k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2} M \\ &= 4\sqrt{B}k\Bigl(\frac{2k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2} M. \end{aligned}$$

Therefore,

$$\begin{aligned} k2^{2k-1}\bigl(W_3+W_4\bigr) &\leqslant k2^{2k-1}\cdot 4Mk\sqrt{B}\Bigl(\frac{2k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2} M \\ &\times\left(1+\Bigl(\frac{k\sqrt{\lambda}}{\ln(m\varepsilon/k)}\Bigr)^{k-1/2}\right) < 4.1Mk^2\sqrt{B}\Bigl(\frac{8k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2}. \end{aligned}$$

Finally, we get

$$\begin{aligned} W_0 &= \frac{(2k)!}{k!}\frac{M}{\pi^{2k}}\left(\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}\Bigr)^k + 4.1k\theta_1\Bigl(\frac{1}{2}\ln\frac{m\varepsilon}{k}+2\Bigr)^{k-1}\right) \\ &+ 4.1\theta_2\sqrt{B}k^2\Bigl(\frac{8k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2} M = \frac{(2k)!}{k!}M\Bigl(\frac{1}{2\pi^2}\ln\frac{m\varepsilon}{k}\Bigr)^k\bigl(1+\theta(\delta_1+\delta_2)\bigr), \end{aligned}$$

where

$$\delta_1 = 4.1k\Bigl(1+\frac{4}{\ln(m\varepsilon/k)}\Bigr)^{k-1}\frac{2}{\ln(m\varepsilon/k)} < \frac{8.6k}{\ln(m\varepsilon/k)},$$

$$\delta_2 = \frac{\pi^{2k}k!}{(2k)!}\frac{4.1\sqrt{B}k^2}{\bigl(\frac{1}{2}\ln(m\varepsilon/k)\bigr)^k}\Bigl(\frac{8k}{\pi^2 e}\ln\frac{m\varepsilon}{k}\Bigr)^{k-1/2} \leqslant 5.8\sqrt{B}\frac{4^k k\sqrt{k}}{\sqrt{\ln(m\varepsilon/k)}}.$$

It remains to note that

$$\delta_1+\delta_2 < 6\sqrt{B}\frac{4^k k\sqrt{k}}{\sqrt{\ln(m\varepsilon/k)}}.$$

Thus the theorem is proved. $\square$

COROLLARY 3.1. *Suppose that k and m are integers such that*

$$1 \leqslant k \leqslant 0.1 \ln\ln\ln N, \quad k\varepsilon^{-1}\exp(\varkappa) < m \leqslant c\ln N,$$

where c is a sufficiently small absolute constant, and $\varkappa$ is a maximum of the numbers $(2Be\pi^2k)^2$ and $12^2B\,4^{2k}k^3$. Then the following inequality holds:

$$\sum_{N<n\leqslant N+M} \big(S(t_{n+m}+0)-S(t_n+0)\big)^{2k} > \frac{M}{2}\frac{(2k)!}{k!}\Big(\frac{1}{2\pi^2}\ln\frac{m\varepsilon}{k}\Big)^k.$$

In particular, if $\varepsilon^{-1}\exp\big((2Be\pi^2)^2\big) < m \leqslant c\ln N$, then

$$\sum_{N<n\leqslant N+M} \big(S(t_{n+m}+0)-S(t_n+0)\big)^{2} > \frac{M}{2\pi^2}\ln(m\varepsilon),$$

and if $m=[\mu]+1$, $\mu=\varepsilon^{-1}\exp(e^{76}\varepsilon^{-6})$, then

$$\sum_{N<n\leqslant N+M} \big(S(t_{n+m}+0)-S(t_n+0)\big)^{2} > 1.01e^{73}\varepsilon^{-6}M.$$

THEOREM 3.2. *Let k be an integer such that $1 \leqslant k \leqslant \frac{1}{192}\ln x$. Then the following inequality holds:*

$$\sum_{N<n\leqslant N+M} \big(S(t_n+0)-S(t_{n-1}+0)\big)^{2k} \leqslant 2Mk(4k\sqrt{B})^{2k}.$$

PROOF. Using the same arguments as above, in the case $m=1$ we obtain

$$W_1 = \sum_{N<n\leqslant N+M} \big(V(t_n)-V(t_{n-1})\big)^{2k} \leqslant \pi^{-2k}\frac{(2k)!}{k!}M\sigma_1^k + x,$$

where $\sigma_1 = \sum_{p<y}\frac{1}{p}\sin^2\big(\frac{1}{2}h\ln p\big)$, $h=\frac{\pi}{\vartheta'(t_N)}$. Using the relation

$$\frac{h}{2}\ln y = \frac{\pi}{\ln t_N + O(1)}\frac{\ln x}{4k} < \frac{\ln x}{k\ln t_N} < \frac{\varepsilon}{10k},$$

and applying Lemma 2.1, we get

$$\sigma_1 \leqslant \frac{h^2}{4}\sum_{p<y}\frac{\ln^2 p}{p} < \Big(\frac{h}{2}\ln y\Big)^2 < \Big(\frac{\varepsilon}{10k}\Big)^2,$$

$$W_1 < \frac{(2k)!}{k!}M\Big(\frac{\varepsilon}{10\pi k}\Big)^{2k} + x < \frac{3}{2}\Big(\frac{\varepsilon^2}{25\pi^2 ek}\Big)^k M + x < \varepsilon^{2k}M.$$

The application of Hölder's inequality to the initial sum W_0 of the theorem yields:

$$\begin{aligned} W_0 &= \sum_{N<n\leqslant N+M} \big(S(t_n+0)-S(t_{n-1}+0)\big)^{2k} \\ &\leqslant W_1 + k2^{2k-1}\big(W_3 + W_1^{1-1/(2k)}W_3^{1/(2k)}\big), \end{aligned}$$

where the sum $W_3 = \sum_{N<n\leqslant N+M}\big(R(t_n+0)-R(t_{n-1}+0)\big)^{2k}$ was estimated in the proof of Theorem 3.1. Using the above bounds of W_1 and W_3, we obtain:

$$k2^{2k-1}\big(W_3 + W_1^{1-1/(2k)}W_3^{1/(2k)}\big) \leqslant 1.5Mk(4k\sqrt{B})^{2k},$$
$$W_0 \leqslant \varepsilon^{2k}M + 1.5Mk(4k\sqrt{B})^{2k} < 2Mk(4k\sqrt{B})^{2k}.$$

The theorem is proved. □

The above theorems imply the lower bound for the 'first moment' of the differences $S(t_n+0) - S(t_{n-1}+0)$.

THEOREM 3.3. *There exists a positive constant $c_1 = c_1(\varepsilon)$ such that*

$$\sum_{N<n\leqslant N+M} \big|S(t_n+0) - S(t_{n-1}+0)\big| > c_1 M.$$

PROOF. Let us put for brevity $r(n) = S(t_n+0) - S(t_{n-1}+0)$ and denote by V_k the sum of kth powers of $|r(n)|$. Further, let $m = [\mu]+1$, where $\mu = \varepsilon^{-1}\exp\big(e^{76}\varepsilon^{-6}\big)$. Combining the identity

$$S(t_{n+m}+0) - S(t_n+0) = r(n+1) + r(n+2) + \cdots + r(n+m)$$

with Cauchy's inequality, we get $\big(S(t_{n+m}+0) - S(t_n+0)\big)^2 \leqslant m\sum_{\nu=1}^{m} r^2(n+\nu)$. Summing both parts of the above relation over n, we have

$$\sum_{N<n\leqslant N+M} \big(S(t_{n+m}+0) - S(t_n+0)\big)^2 \leqslant m\sum_{\nu=1}^{m}\sum_{N<n\leqslant N+M} r^2(n+\nu)$$
$$\leqslant m\sum_{\nu=1}^{m}\sum_{N<n\leqslant N+M+m} r^2(n) = m^2\sum_{N<n\leqslant N+M+m} r^2(n).$$

Hence, by the Corollary of Theorem 3.1, we get:

$$\sum_{N<n\leqslant N+M+m} r^2(n) \geqslant m^{-2}\sum_{N<n\leqslant N+M}\big(S(t_{n+m}+0) - S(t_n+0)\big)^2 \geqslant 1.01e^{73}\varepsilon^{-6}m^{-2}\,M.$$

Since $|r(n)| \leqslant |S(t_n)| + |S(t_{n-1})| \leqslant 18\ln N$ for $N < n \leqslant N+M+m$ (see [**24**]), we have:

$$V_2 = \sum_{N<n\leqslant N+M} r^2(n) \geqslant 1.01e^{73}\varepsilon^{-6}m^{-2}\,M - m(18\ln N)^2 > c_2 M,$$

where $c_2 = e^{73}\varepsilon^{-6}m^{-2}\,M$. Further, an application of Hölder's inequality to the sum V_2 yields

$$V_2 = \sum_{N<n\leqslant N+M} |r(n)|^{2/3}|r(n)|^{4/3}$$
$$\leqslant \bigg(\sum_{N<n\leqslant N+M} |r(n)|\bigg)^{2/3}\bigg(\sum_{N<n\leqslant N+M} |r(n)|^4\bigg)^{1/3} = V_1^{2/3}V_4^{1/3}.$$

Therefore, $V_1 \geqslant V_2^{3/2} V_4^{-1/2}$. Using both the above bound for V_2 and the inequality of Theorem 3.2 with $k = 2$, we obtain:

$$V_4 \leqslant 2^{14} B^2 M = c_4 M, \quad V_1 = \sum_{N<n\leqslant N+M} |r(n)| \geqslant \frac{(c_2 M)^{3/2}}{(c_4 M)^{1/2}} = c_1 M, \quad c_1 = c_2^{3/2} c_4^{-1/2}.$$

The theorem is proved. □

The following assertion is an analogue (for the short interval of summation) of corollary of Selberg's theorem cited in Section 1.

THEOREM 3.4. *There exist positive constants K_1 and K_2 such that for $N < n \leqslant N+M$, there are more than $K_1 M$ cases when the interval G_n does not contain any ordinate of a zero of $\zeta(s)$, and more than $K_2 M$ cases when the interval G_n contains at least two ordinates, i.e.,*

$$\nu_0(N+M) - \nu_0(N) \geqslant K_1 M, \quad \sum_{k\geqslant 2} \big(\nu_k(N+M) - \nu_k(N)\big) \geqslant K_2 M.$$

PROOF. Since $r(n) = S(t_n + 0) - S(t_{n-1} + 0)$ is an integer and $r(n) \geqslant -1$ for any n, then the equality (3.1) implies that the interval G_n does not contain any ordinate iff $r(n) = -1$ and contains more than one ordinate iff $r(n) \geqslant 1$. In other words, the number M_1 of 'empty' Gram's intervals is equal to the number of n such that $r(n)$ is negative, and the number M_2 of Gram's intervals that contain two or more ordinates is equal to the number of positive $r(n)$.

Using the relation

$$\frac{1}{2}\big(|r(n)| - r(n)\big) = \begin{cases} 1, & \text{if } r(n) < 0, \\ 0, & \text{if } r(n) > 0, \end{cases}$$

together with the estimate of Theorem 3.3 we get

$$\begin{aligned} M_1 = \sum_{N<n\leqslant N+M} \frac{1}{2}\big(|r(n)| - r(n)\big) &= \frac{1}{2} \sum_{N<n\leqslant N+M} \big|S(t_n+0) - S(t_{n-1}+0)\big| \\ &- \frac{1}{2} \sum_{N<n\leqslant N+M} \big(S(t_n+0) - S(t_{n-1}+0)\big) \geqslant \frac{c_1}{2} M - 9\ln N > K_1 M, \end{aligned}$$

where $K_1 = \frac{2}{5} c_1$. Further, M_2 is equal to the number of nonzero terms of the sum

$$W = \sum_{N<n\leqslant N+M} \frac{1}{2}\big(|r(n)| + r(n)\big).$$

The application of Theorem 3.2 and Cauchy's inequality yields:

$$W \leqslant \sqrt{M_2} \sqrt{\sum_{N<n\leqslant N+M} r^2(n)} \leqslant \sqrt{M_2}\sqrt{32BM}.$$

Since $W > K_1 M$, then $M_2 \geqslant K_2 M$, where $K_2 = \frac{K_1^2}{32B}$. The theorem is proved. □

Remark 3.1. The constants K_1, K_2 are too small. It is easy to see that they do not exceed $\exp\bigl(-e^{75}\varepsilon^{-6}\bigr)$. At the same time, the calculations of zeros of $\zeta(s)$ show that likely $K_1 > 0.1$, $K_2 > 0.1$. Thus, it is of some interest to prove the analogue of Theorem 3.4 with $K_1, K_2 \cong 0.001 - 0.01$.

4. The alternating sums connected with the function $S(t)$.

Here we study the sums of the following type:

$$T_k = \sum_{N<n\leqslant N+M} S^k(t_n+0)\bigl(S(t_n+0) - S(t_{n-1}+0)\bigr).$$

Theorem 3.4 implies that the difference $r(n) = S(t_n+0) - S(t_{n-1}+0)$ is negative for a positive proportion of n, $N < n \leqslant N+M$. At the same time, this difference is positive for a positive proportion of n. Hence, the sums T_k are alternating (at least, for even k).

The direct application of Cauchy's inequality does not allow us to take into account the oscillation in the sum T_k, and therefore does not allow us to obtain a nontrivial bound for T_k. Namely, the inequalities

$$|T_{2k}| \leqslant \biggl(\sum_{N<n\leqslant N+M} S^{4k}(t_n+0)\biggr)^{1/2}\biggl(\sum_{N<n\leqslant N+M} r^2(n)\biggr)^{1/2} \ll_k \sqrt{ML^{2k}}\sqrt{M} \ll_k ML^k,$$

give only the trivial bound. Hence, we need to use some additional tools.

Theorem 4.1. *Suppose that k is an integer such that $1 \leqslant k \leqslant \sqrt{L}$. Then the following inequalities hold:*

$$|T_{2k-1}| < 0.02(Ak)^{k+1}ML^{k-1}, \quad |T_{2k}| < 0.02(10A)^{k+1}\frac{(2k)!}{k!}\frac{ML^{k-1/2}}{(2\pi)^{2k}}.$$

Proof. We begin with the sum T_{2k-1}. Setting $a = S(t_n+0)$, $b = r(n) = S(t_n+0) - S(t_{n-1}+0)$ in the easy-to-check identity

$$(a-b)^{2k} = a^{2k} - 2ka^{2k-1}b + \theta k^2 2^{2k-2}\bigl(a^{2k-2}b^2 + b^{2k}\bigr),$$

after some obvious transformations we get:

$$\begin{aligned}2k\,S^{2k-1}(t_n+0)r(n) &= S^{2k}(t_n+0) - S^{2k}(t_{n-1}+0)\\ &\quad + \theta k^2 2^{2k-2}\bigl(r^{2k}(n) + S^{2k-2}(t_n+0)r^2(n)\bigr).\end{aligned}$$

Summing over n, we obtain

$$2k\,T_{2k-1} = S^{2k}(t_{N+M}) - S^{2k}(t_N) + \theta\, k^2 2^{2k-2}\bigl(W_1 + W_2\bigr),$$

where $W_1 = \sum_{N<n\leqslant N+M} r^{2k}(n)$, $W_2 = \sum_{N<n\leqslant N+M} S^{2k-2}(t_n+0)r^2(n)$. By Theorem 3.2 one gets $W_1 \leqslant 2Mk(4k\sqrt{B})^{2k}$. Using Hölder's inequality and the estimates of Lemma 2.5, we have:

$$W_2 \leqslant \biggl(\sum_{N<n\leqslant N+M} S^{2k}(t_n+0)\biggr)^{1-1/k} W_1^{1/k} \leqslant 2\sqrt[3]{3}(4k\sqrt{B})^2\biggl(\frac{kAL}{\pi^2 e}\biggr)^{k-1}M.$$

Hence

$$\begin{aligned}k2^{2k-2}(W_1+W_2) &\leqslant k2^{2k-2}\,2\sqrt[3]{3}\big(4k\sqrt{B}\,\big)^2\Big(\frac{kAL}{\pi^2 e}\Big)^{k-1}M\bigg(1+\frac{k}{\sqrt[3]{3}}\Big(\frac{kA}{4L}\Big)^{k-1}\bigg)\\ &< 2\cdot 4^2\sqrt[3]{3}e^{-8}k^3A^2\Big(\frac{4kAL}{\pi^2 e}\Big)^{k-1} < \frac{1}{30}k^3A^2(0.15kAL)^{k-1}M\\ &\leqslant \frac{1}{30}(Ak)^{k+1}ML^{k-1},\end{aligned}$$

and therefore $|T_{2k-1}| \leqslant \frac{1}{60}(Ak)^{k+1}ML^{k-1} + \frac{1}{k}(9\ln N)^{2k} < \frac{1}{50}(Ak)^{k+1}ML^{k-1}$.

Now we consider the sum T_{2k}. Setting $a = S(t_n+0)$, $b = r(n)$ in the identity

$$(a-b)^{2k+1} = a^{2k+1} - (2k+1)a^{2k}b + \theta k(2k+1)2^{2k-2}\big(|b|^{2k+1} + |a|^{2k-1}b^2\big),$$

after some transformations we get:

$$(2k+1)T_{2k+1} = S^{2k+1}(t_{N+M}+0) - S^{2k+1}(t_N+0) + \theta k(2k+1)2^{2k-2}\big(W_1+W_2\big),$$

where $W_1 = \sum_{N<n\leqslant N+M}|r(n)|^{2k+1}$, $W_2 = \sum_{N<n\leqslant N+M}|S(t_n+0)|^{2k-1}r^2(n)$. We have

$$\begin{aligned}W_1 &= \sum_{N<n\leqslant N+M}|r(n)|^{2k-1}r^2(n)\\ &\leqslant \bigg(\sum_{N<n\leqslant N+M} r^{2k}(n)\bigg)^{1-1/(2k)}\bigg(\sum_{N<n\leqslant N+M} r^{4k}(n)\bigg)^{1/(2k)}\\ &\leqslant \big(2k\big(4k\sqrt{B}\,\big)^{2k}\big)^{1-1/(2k)}\big(4k\big(8k\sqrt{B}\,\big)^{4k}\big)^{1/(2k)}M < 8\sqrt{2}kM\big(4k\sqrt{B}\,\big)^{2k+1}\\ &= \frac{(2k)!}{k!}\frac{M}{(2\pi)^{2k}}(AL)^{k-1/2}\delta_1,\end{aligned}$$

where

$$\delta_1 = \frac{(2\pi)^{2k}k!}{(2k)!}\frac{8\sqrt{2}k(4k\sqrt{B})^{2k+1}}{(AL)^{k-1/2}} \leqslant k^2A\sqrt{AL}\bigg(\frac{(4\pi)^2Ak}{e^7L}\bigg)^k < 1.$$

Next, the application of Lemmas 2.4 and 2.5 yields:

$$\begin{aligned}W_2 &\leqslant \bigg(\sum_{N<n\leqslant N+M} S^{2k}(t_n+0)\bigg)^{1-1/(2k)}\bigg(\sum_{N<n\leqslant N+M} r^{4k}(n)\bigg)^{1/(2k)}\\ &\leqslant \bigg(\frac{1.1}{A}\frac{(2k)!}{(2\pi)^{2k}k!}(AL)^k\bigg)^{1-1/(2k)}(4k)^{1/(2k)}\big(8k\sqrt{B}\big)^2 \leqslant \frac{(2k)!}{k!}\frac{M}{(2\pi)^{2k}}(AL)^{k-1/2}\delta_2,\end{aligned}$$

where

$$\delta_2 = 2\pi\bigg(\frac{k!}{(2k)!}\bigg)^{1/(2k)}\sqrt{\frac{1.1}{A}}\,(4k)^{1/(2k)}\big(8k\sqrt{B}\big)^2 < 0.01(kA)^{3/2}.$$

Let us note that

$$k2^{2k-2}\big(W_1+W_2\big) \leqslant k\,2^{2k-2}\frac{(2k)!}{k!}\frac{M}{(2\pi)^{2k}}\,(AL)^{k-1/2}(\delta_1+\delta_2)$$
$$< \frac{(2k)!}{k!}\frac{ML^{k-1/2}}{(2\pi)^{2k}}\cdot 0.01(10A)^{k+1}.$$

Therefore,

$$(2k+1)|T_{2k}| \leqslant (2k+1)\cdot 0.01(10A)^{k+1}\frac{(2k)!}{k!}\frac{ML^{k-1/2}}{(2\pi)^{2k}} + 2(9\ln N)^{2k+1},$$
$$|T_{2k}| < 0.02(10A)^{k+1}\frac{(2k)!}{k!}\frac{ML^{k-1/2}}{(2\pi)^{2k}}.$$

The theorem is proved. □

LEMMA 4.1. *Let k and n be arbitrary natural numbers and suppose that the interval $G_n=(t_{n-1},t_n]$ contains $(r+1)$ ordinates of zeros of $\zeta(s)$, $r=r(n)\geqslant -1$. Then the following relations hold:*

$$\sum_{t_{n-1}<\gamma_m\leqslant t_n}\Delta_m^{2k} = (r+1)\Delta^{2k}(n)+\theta_1 k2^{2k}\big(|\Delta(n)|^{2k-1}r^2+|r|^{2k+1}\big), \tag{4.1}$$

$$\sum_{t_{n-1}<\gamma_m\leqslant t_n}\Delta_m^{2k-1} = -(r+1)\Delta^{2k-1}(n)+\theta_2 k2^{2k}\big(\Delta^{2k-2}(n)r^2+r^{2k}\big). \tag{4.2}$$

PROOF. First we consider the case $r=-1$. Then G_n does not contain any ordinate, and the sums in the left-hand sides of (4.1),(4.2) are empty. Thus the assertion of lemma is true for $\theta_1=\theta_2=0$.

Now let us consider the case $r\geqslant 0$. Suppose that the inequalities

$$\gamma_{s-1}\leqslant t_{n-1}<\gamma_s\leqslant\gamma_{s+1}\leqslant\ldots\leqslant\gamma_{s+r}\leqslant t_n<\gamma_{s+r+1}$$

hold for some $s\geqslant 1$. Then $\Delta(n)=S(t_n+0)=N(t_n+0)-\pi^{-1}\vartheta(t_n)-1=s+r-n$, and hence

$$\begin{aligned}\Delta_s &= n-s=r-\Delta(n),\\ \Delta_{s+1} &= n-s-1=r-1-\Delta(n),\\ &\ \ \ldots\\ \Delta_{s+r} &= n-s-r=-\Delta(n).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{t_{n-1}<\gamma_m\leqslant t_n}\Delta_m^{2k} &= \sum_{m=s}^{s+r}\Delta_m^{2k}=\sum_{j=0}^{r}\big(j-\Delta(n)\big)^{2k}\\ &= \sum_{j=0}^{r}\big(\Delta^{2k}(n)+\theta\,2kj\,2^{2k-2}(|\Delta(n)|^{2k-1}+j^{2k-1})\big)\\ &= (r+1)\Delta^{2k}(n)+\theta_1 k2^{2k}\big(|\Delta(n)|^{2k-1}r^2+r^{2k+1}\big).\end{aligned}$$

The proof of (4.2) follows the same arguments. The lemma is proved. □

The below theorem reduces the evaluation of the sums (1.8), (1.9) to the calculation of the sums of the quantities $\Delta^k(n) = S^k(t_n + 0)$ (see Lemma 2.5).

THEOREM 4.2. *Let k be an integer such that $1 \leqslant k \leqslant \sqrt{L}$. Then the following relations hold:*

$$\sum_{N<n\leqslant N+M} \Delta_n^{2k} = \frac{(2k)!}{k!}\,\frac{ML^k}{(2\pi)^{2k}}\bigl(1 + \theta(10A)^{k+1}L^{-0.5}\bigr),$$

$$\Biggl|\sum_{N<n\leqslant N+M} \Delta_n^{2k-1}\Biggr| \leqslant e^9(Bk)^k ML^{k-1}.$$

PROOF. We define the numbers μ and ν by the inequalities $\gamma_\mu \leqslant t_N < \gamma_{\mu+1}$, $\gamma_\nu \leqslant t_{N+M} < \gamma_{\nu+1}$. Using the definition of $\Delta(n)$, we obtain:

$$\mu = N(t_N + 0) = \pi^{-1}\vartheta(t_N) + 1 + S(t_N + 0) = N + \Delta(N),$$
$$|\mu - N| = |\Delta(N)| < 9\ln N,$$

and, similarly, $|\nu - (N + M)| = |\Delta(N + M)| < 9\ln N$. These inequalities and the bound $|\Delta_m| < 9\ln N$ (see Lemma 2.7 and a posterior remark in [**18**]) imply that the difference between the sums $\sum_{N<m\leqslant N+M} \Delta_m^{2k}$ and $V = \sum_{\mu<m\leqslant\nu} \Delta_m^{2k}$ does not exceed in modulus $(18\ln N + 1)(9\ln N)^{2k} < 3(9\ln N)^{2k+1}$.

On the other hand, Lemma 4.1 implies that

$$\begin{aligned}
V &= \sum_{N<n\leqslant N+M}\ \sum_{t_{n-1}<\gamma_m\leqslant t_n} \Delta_m^{2k}\\
&= \sum_{N<n\leqslant N+M} \bigl((r(n) + 1)\Delta^{2k}(n) + \theta k\,2^{2k}(|\Delta(n)|^{2k-1}r^2(n) + |r(n)|^{2k+1})\bigr)\\
&= \sum_{N<n\leqslant N+M} \Delta^{2k}(n) + T_{2k} + \theta_1 k\,2^{2k}\bigl(W_1 + W_2\bigr),
\end{aligned}$$

where $r(n) = S(t_n + 0) - S(t_{n-1} + 0)$ and

$$\begin{aligned}
W_1 &= \sum_{N<n\leqslant N+M} |r(n)|^{2k+1} = \sum_{N<n\leqslant N+M} |S(t_n + 0) - S(t_{n-1} + 0)|^{2k+1},\\
W_2 &= \sum_{N<n\leqslant N+M} |\Delta(n)|^{2k-1}r^2(n)\\
&= \sum_{N<n\leqslant N+M} |S(t_n + 0)|^{2k-1}\bigl(S(t_n + 0) - S(t_{n-1} + 0)\bigr)^2.
\end{aligned}$$

Proving Theorem 4.1, we found that

$$|T_{2k}| \leqslant 0.02(10A)^{k+1}\frac{(2k)!}{k!}\,\frac{ML^{k-0.5}}{(2\pi)^{2k}},$$
$$k\,2^{2k}\bigl(W_1 + W_2\bigr) < 0.08(10A)^{k+1}\frac{(2k)!}{k!}\,\frac{ML^{k-0.5}}{(2\pi)^{2k}}.$$

Using both these inequalities and the assertion of Lemma 2.5, we get:

$$\sum_{N<m\leqslant N+M}\Delta_m^{2k} = \sum_{N<n\leqslant N+M}\Delta^{2k}(n) + 0.1\theta_1(10A)^{k+1}\frac{(2k)!}{k!}\frac{ML^{k-0.5}}{(2\pi)^{2k}}$$

$$+ 3\theta_2(9\ln N)^{2k+1} = \frac{(2k)!}{k!}\frac{ML^k}{(2\pi)^{2k}}\bigl(1+\theta_3A^kL^{-0.5}+0.2\theta_4(10A)^{k+1}L^{-0.5}\bigr)$$

$$= \frac{(2k)!}{k!}\frac{ML^k}{(2\pi)^{2k}}\bigl(1+\theta(10A)^{k+1}L^{-0.5}\bigr).$$

Applying the same arguments, we obtain

$$V = \sum_{\mu<m\leqslant\nu}\Delta_m^{2k-1} = -\sum_{N<n\leqslant N+M}\Delta^{2k-1}(n) - T_{2k-1} + \theta k\,2^{2k}\bigl(V_1+V_2\bigr),$$

where $V_1 = \sum_{N<n\leqslant N+M} r^{2k}(n)$, $\sum_{N<n\leqslant N+M}\Delta^{2k-2}(n)r^2(n)$. Proving Theorem 4.1, we also found that

$$k\,2^{2k}\bigl(V_1+V_2\bigr) \leqslant 0.14(Ak)^{k+1}ML^{k-1}, \quad |T_{2k-1}| < 0.02(Ak)^{k+1}ML^{k-1}.$$

By these estimates and by the inequalities of Lemma 2.5 we have:

$$\Biggl|\sum_{N<m\leqslant N+M}\Delta_m^{2k-1}\Biggr| \leqslant \Biggl|\sum_{N<n\leqslant N+M}\Delta^{2k-1}(n)\Biggr| + |T_{2k-1}| + k\,2^{2k}\bigl(V_1+V_2\bigr)$$

$$+ 3(9\ln N)^{2k} \leqslant \frac{3.5}{\sqrt{B}}(Bk)^kML^{k-1} + 0.16(Ak)^{k+1}ML^{k-1}$$

$$= (Bk)^kML^{k-1}\left(\frac{3.5}{\sqrt{B}} + 0.16Ak\Bigl(\frac{e^8}{A}\Bigr)^k\right) < e^9(Bk)^kML^{k-1}.$$

The theorem is proved. □

The approximate expression for the distribution function of a discrete random quantity with the values $\delta_n = \pi\Delta_n\sqrt{2/L}$, $N<n\leqslant N+M$, and the proof of the assertion that $\Delta_n \neq 0$ for 'almost all' n follow now from Theorem 4.2 by standard tools (see, for example, [**17**, Theorem 4]).

5. On some equivalents of 'almost Riemann hypothesis'

The last section is devoted to some new equivalents of the 'almost Riemann hypothesis'. This hypothesis asserts that 'almost all' complex zeros of $\zeta(s)$ lie on the critical line, that is

$$\lim_{T\to+\infty}\frac{N_0(T)}{N(T)} = 1.$$

Moreover, the below arguments imply that Selberg interpreted Gram's law in [**8**] in a way different from those of Titchmarsh. Namely, the below assertions show that Selberg considered all the complex zeros of $\zeta(s)$ (but not only the zeros on the critical line) in handling with the quantities Δ_n. Thus, Selberg's definition of Δ_n is equivalent to our Definition 1.4.

Suppose that $0 < c_1 < c_2 < \ldots \leqslant c_n \leqslant c_{n+1} \leqslant \ldots$ are the ordinates of zeros of $\zeta(s)$, lying on the critical line and counting with theirs multiplicities. For a fixed $n \geqslant 1$, we define the number $m = m(n)$ by the inequalities

$$t_{m-1} < c_n \leqslant t_m \tag{5.1}$$

and set $D_n = m - n$. Of course, if the Riemann hypothesis is true then $c_n = \gamma_n$ and $D_n = \Delta_n$ for any n.

THEOREM 5.1. *The validity of the relation*

$$\sum_{n \leqslant N} |D_n| = o(N^2) \tag{5.2}$$

as $N \to +\infty$, is the necessary and sufficient condition for the truth of the 'almost Riemann hypothesis'.

PROOF. Suppose that the 'almost Riemann hypothesis' is true. Then

$$N(c_n + 0) = (1 + o(1))N_0(c_n + 0) = (1 + o(1))(n + O(\ln n)) = n + o(n)$$

(the term $O(\ln n)$ takes into account the multiplicity of the zero with the ordinate c_n; by Lemma 2.8, this multiplicity is $O(\ln c_n) = O(\ln n)$). On the other hand, by (5.1) we have $m - 1 + S(t_{m-1} + 0) < N(c_n + 0) \leqslant m + S(t_m + 0)$, and hence $D_n = m - n = o(n) + O(\ln m) = o(n)$. Therefore, $\sum_{n \leqslant N} |D_n| = o(N^2)$.

Suppose now that condition (5.2) is satisfied. Noting that $N(c_n+0) \geqslant n+\varepsilon(n)$, where $\varepsilon(n)$ is the number of zeros of $\zeta(s)$ with the condition $0 < \operatorname{Im} s \leqslant c_n$, $\operatorname{Re} s \neq \frac{1}{2}$, we get $n+\varepsilon(n) \leqslant N(t_m+0) = m+S(t_m+0)$. Hence, $0 \leqslant \varepsilon(n) \leqslant |D_n|+|S(t_m+0)|$. Summing this estimate over $n \leqslant 2N$ and applying Cauchy's inequality, we obtain

$$\sum_{n \leqslant 2N} \varepsilon(n) \leqslant \sum_{n \leqslant 2N} |D_n| + \sum_{\substack{n \leqslant 2N \\ m = m(n)}} |S(t_m + 0)| \leqslant \sqrt{2N}\sqrt{W} + o(N^2),$$

where

$$W = \sum_{\substack{n \leqslant 2N \\ m = m(n)}} S^2(t_m + 0).$$

Let μ be the maximum of $m(n)$ for $n \leqslant 2N$. Then $t_{\mu-1} < c_{2N} \leqslant t_\mu$ and hence

$$N(c_{2N} + 0) \geqslant N(t_{\mu-1} + 0) = \mu + O(\ln \mu).$$

By Lemma 2.8, we have for $t = c_{2N}$:

$$N(c_{2N}) \leqslant \left(\tfrac{5}{2} - 10^{-3}\right)N_0(c_{2N}) = \left(\tfrac{5}{2} - 10^{-3}\right)(2N + O(\ln N)),$$

and therefore $\mu < 5N$. Changing the order of summation in W, we obtain:

$$W \leqslant \sum_{l \leqslant 5N} S^2(t_l + 0) \sum_{\substack{n \leqslant 2N \\ m(n) = l}} 1.$$

For a fixed l, the number of n that satisfy the conditions $n \leqslant 2N$, $m(n) = l$, does not exceed the number of all ordinates of zeros of $\zeta(s)$ lying in the interval $(t_{l-1}, t_l]$, that is $N(t_l+0) - N(t_{l-1}+0) = 1 + S(t_l+0) - S(t_{l-1}+0) = 1 + r(l)$. Thus we have

$$W \leqslant \sum_{l\leqslant 5N} S^2(t_l+0)(1+r(l)).$$

Using the first formula of Lemma 2.5 and the estimate of Theorem 4.1, we find that

$$W \leqslant \frac{5N}{2\pi^2}\ln\ln N + O\bigl(N\sqrt{\ln\ln N}\bigr) < \tfrac{1}{3}N\ln\ln N,$$
$$\sum_{n\leqslant 2N}\varepsilon(n) < N\sqrt{\ln\ln N} + o(N^2) = o(N^2).$$

By obvious inequality $\varepsilon(N+1) + \varepsilon(N+2) + \cdots + \varepsilon(2N) \geqslant N\varepsilon(N)$ we get:

$$N\varepsilon(N) = o(N^2), \quad \varepsilon(N) = o(N).$$

Suppose now that t is sufficiently large. Then, defining N from the inequalities $c_{N-1} < t \leqslant c_N$ and using the above relations, we obtain:

$$N(t) \geqslant N-1, \quad N(t) - N_0(t) \leqslant \varepsilon(N) = o(N) = o(N(t)).$$

The theorem is proved.

Corollary 5.1. *The validity of the relation*

$$\sum_{n\leqslant N}|D_n|^k = o\bigl(N^{k+1}\bigr), \quad N\to+\infty$$

for at least one fixed value of $k\geqslant 1$ is the necessary and sufficient condition for the truth of the 'almost Riemann hypothesis'.

The proof is similar to the previous one. The difference is that we should use the inequality

$$N\varepsilon(N) \leqslant \sum_{n\leqslant 2N}|D_n| + N\sqrt{\ln\ln N} \leqslant (2N)^{1-1/k}\biggl(\sum_{n\leqslant 2N}|D_n|^k\biggr)^{1/k} + N\sqrt{\ln\ln N}$$

for the proof of sufficiency.

This assertion shows, in particular, that if Selberg's formulas (1.8), (1.9) hold true after the replacement of the quantities Δ_n by D_n, then the 'almost Riemann hypothesis' is also true.

Corollary 5.2. *The assertion '$D_n = o(n)$ as $n\to+\infty$' is the necessary and sufficient condition for the 'almost Riemann hypothesis'.*

The below theorem shows that the upper bound for D_n causes the main difficulty.

Theorem 5.2. *Suppose that $N_0(t) > \varkappa N(t)$ for any $t > t_0 > 1$ and for some constant $\varkappa$, $0<\varkappa<1$. Then the following inequalities hold for all sufficiently large n:*

$$-9\ln n \leqslant D_n \leqslant \bigl(\tfrac{1}{\varkappa}-1\bigr)n + 9\bigl(\tfrac{1}{\varkappa}+1\bigr)\ln n.$$

PROOF. By (5.1), we get:

$$N(c_n+0) \geqslant N(t_{m-1}+0) = m-1+S(t_{m-1}+0). \tag{5.3}$$

By the assumption of the theorem, we get for $t = c_n + 0$:

$$N(c_n+0) < \tfrac{1}{\varkappa}\, N_0(c_n+0) \leqslant \tfrac{1}{\varkappa}\,(n+\kappa_n), \tag{5.4}$$

where κ_n denotes the multiplicity of the ordinate c_n. Comparing (5.3) and (5.4) and using the inequality $|S(t)| \leqslant 8.9\ln t$, we obtain:

$$m \leqslant \tfrac{1}{\varkappa}\,n + 9(\tfrac{1}{\varkappa}+1)\ln n, \quad D_n = m-n < (\tfrac{1}{\varkappa}-1)n + 9(\tfrac{1}{\varkappa}+1).$$

On the other hand,

$$n \leqslant N_0(c_n+0) \leqslant N(c_n+0) \leqslant N(t_m+0) = m + S(t_m+0),$$

and therefore $D_n \geqslant -S(t_m+0) \geqslant -9\ln n$. The theorem is proved. $\square$

In [**8**], Selberg referred to the formulas

$$\liminf_{n\to+\infty} \Delta_n = -\infty, \quad \limsup_{n\to+\infty} \Delta_n = +\infty, \tag{5.5}$$

as to the result of Titchmarsh from [**5**]. Indeed, the relations (5.5) hold true, and the modern omega-theorems for the function $S(t)$ imply a much deeper result, namely

$$\Delta_n = \Omega_{\pm}\left(\sqrt[3]{\frac{\ln n}{\ln\ln n}}\right),$$

as n grows (see [**18**]). On the other hand, in [**5**], Titchmarsh considered the fractions

$$\tau_n = \frac{c_n - t_n}{t_{n+1} - t_n}$$

instead of the quantities Δ_n (one can easily see that the difference between τ_n and D_n is $O(1)$), and established the unboundedness of τ_n. As far as can be seen, the methods of [**5**] allows one to show only that $\tau_n \neq O(1)$ and $D_n \neq O(1)$, as $n \to +\infty$. A slight modification of these methods and the omega-theorems for $S(t)$ lead to the following assertion

$$D_n = \Omega_{-}\left(\sqrt[3]{\frac{\ln n}{\ln\ln n}}\right), \quad n \to +\infty.$$

So, the problem of unboundedness of D_n from above still remains open.

References

1. J.-P. Gram, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, Acta Math. **27** (1903), 289–304.
2. C. L. Siegel, *Über Riemanns Nachlaß zur analytischen Zahlentheorie*, Quell. Stud. Gesch. Math. B 2 (1932), 45–80.
3. J.-P. Gram, *Note sur le calcul de la fonction $\zeta(s)$ de Riemann*, Bull. Acad. Copenhague 1895, p. 303.
4. J. I. Hutchinson, *On the roots of the Riemann zeta-function*, Trans. Am. Math. Soc. **27** (1925), 49–60.
5. E. C. Titchmarsh, *The zeros of the Riemann zeta-function*, Proc. Roy. Soc. Lond., Ser. A **151** (1935), 234–255.

6. E. C. Titchmarsh, *The zeros of the Riemann zeta-function*, Proc. Roy. Soc. Lond., Ser. A **157** (1936), 261–263.
7. J. Kalpokas, J. Steuding, *On the value distribution of the Riemann zeta-function on the critical line*, Mosc. J. Comb. Number Theory **1** (2011), 26–42.
8. A. Selberg, *The zeta-function and the Riemann hypothesis*, 10. Skand. Mat. Kongr., Kobenhavn 1946 (1947), 187–200.
9. E. C. Titchmarsh, *On van der Corput's method and the zeta-function of the Riemann (IV)*, Q. J. Math., Oxf. Ser. **5** (1934), 98–105.
10. H. Bohr, E. Landau, *Beiträge zur Theorie der Riemannschen Zetafunktion*, Math. Ann. **74** (1913) 3–30.
11. A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48**:5 (1946), 89–155.
12. A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip* $0 < t < T$, Avh. Norske Vid. Akad. Oslo I 1944, No. 1, 27 p.
13. H. von Mangoldt, *Zur Verteilung der Nullstellen der Riemannschen Funktion* $\xi(t)$, Math. Ann. **60** (1905), 1–19.
14. T. S. Trudgian, *On the success and failure of Gram's Law and the Rosser Rule*, Acta Arith. **148**:3 (2011) 225–256.
15. A. Fujii, *Gram's law for the zeta zeros and the eigenvalues of Gaussian unitary ensembles*, Proc. Japan Acad., Ser. A **63** (1987), 392–395.
16. A. Selberg, *Collected Papers.* Vol. I, Springer-Verlag, Berlin etc., 1989.
17. M. A. Korolev, *The Gram law and Selberg's conjecture on the distribution of zeros of the Riemann zeta-function*, Izv. Math. **74**:4 (2010), 743–780 (translation from Izv. Ross. Akad. Nauk, Ser. Mat. **74**:4 (2010), 83–118).
18. M. A. Korolev, *On the Gram's law in the theory of the Riemann zeta function*, Izv. Math. **76**:2 (2012), 275–309 (translation from Izv. Ross. Akad. Nauk, Ser. Mat. **76**:2 (2012), 67–102.
19. M. A. Korolev, *On Karatsuba's problem related to Gram's law*, Proc. Steklov Inst. Mat. **276** (2012), 162–172.
20. J. B. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Ill. J. Math. **6**:1 (1962), 64–94.
21. A. Ghosh, *On Riemann's zeta-function–sign-changes of* $S(T)$; in: *Recent Progress in Analytic Number Theory*, Symp. Durham 1979, Vol. 1 (1981) 25–46.
22. A. A. Karatsuba, M. A. Korolev, *Behaviour of the argument of the Riemann zeta function on the critical line*, Russ. Math. Surv. **61**:3 (2006), 389–482 (translation from Usp. Mat. Nauk **61**:3 (2006), 3–92).
23. J. B. Conrey, *More than two fifths of the zeros of the Riemann zeta function are on the critical linee*, J. Reine Angew. Math. **399** (1989), 1–26.
24. A. A. Karatsuba, M. A. Korolev, *The argument of the Riemann zeta function*, Russ. Math. Surv. **60**:3 (2005), 433–488 (translation from Usp. Mat. Nauk **60**:3 (2005), 41–96).

Department of Algebra and Number Theory
Steklov Mathematical Institute
Gubkina str., 8, 119991 Moscow
Russia
hardy_ramanujan@mail.ru
korolevma@mi.ras.ru

(Received 30 01 2012)