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# $L$-PONOMAREV SYSTEM AND IMAGES OF LOCALLY SEPARABLE METRIC SPACES

**Tran Van An and Luong Quoc Tuyen**

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Abstract. We introduce the notion of an $L$-Ponomarev system $(f, M, X, \mathcal{P}_n^*)$, and give characterizations of certain msss-images (resp., mssc-images) of locally separable metric spaces. As an application, we get a new characterization of quotient msss-images (mssc-images) of locally separable metric spaces, which is helpful in solving Velichko's question (1987).

## 1. Introduction

Lin in [**15**] introduced the concept of msss-maps (resp., mssc-maps) to characterize spaces with certain $\sigma$-locally countable (resp., $\sigma$-locally finite) networks by msss-images (resp., mssc-images) of metric spaces. After that, some characterizations for certain msss-images (resp., mssc-images) of metric (or semi-metric) spaces are obtained by many authors ([**10, 13, 14**], for example).

Velichko [**26**] proved that a space $X$ is a pseudo-open $s$-image of a locally separable metric space iff $X$ is a locally separable space which is a pseudo-open $s$-image of a metric space, and posed the following interesting question about quotient and $s$-images of metric spaces.

Question 1.1. *Find a $\Phi$-property such that a space $X$ is a quotient and $s$-image of a metric and $\Phi$-space iff $X$ is a $\Phi$-space which is a quotient and $s$-image of a metric space.*

Recently, Dung gave some characterizations for certain msss-images (resp., mssc-images) of locally separable metric spaces in the class of regular and $T_1$-spaces (see in [**3, 4**]). This leads us to consider the following question.

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QUESTION 1.2. *Find a $\Phi$-property such that a space $X$ is a quotient and msss-image (mssc-image) of a metric and $\Phi$-space iff $X$ is a $\Phi$-space which is a quotient and msss-image (resp., mssc-image) of a metric space.*

In this paper, we introduce the notion of a generalized Ponomarev system $(f, M, X, \mathcal{P}_n^*)$, calling it an $L$-Ponomarev system, and then prove some statements concerning the properties of such systems corresponding to $\sigma$-locally finite and $\sigma$-locally countable Lindelöf networks. As an application, we get a new characterization of quotient msss-images (mssc-images) of locally separable metric spaces, give an affirmative answer to Question 1.2, and we get an affirmative answer to Question 2.17 from [**3**].

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $K \subset X$ and $\mathcal{P}$ be a collection of subsets of $X$, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$, $\mathcal{P}_K = \{P \in \mathcal{P} : P \cap K \neq \emptyset\}$. For a sequence $\{x_n\}$ converging to $x$ and $P \subset X$, we say that $\{x_n\}$ is *eventually* in $P$ if $\{x\} \cup \{x_n : n \geqslant m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in $P$ if some subsequence of $\{x_n\}$ is eventually in $P$.

DEFINITION 1.1. [**2, 17**] Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for every $x \in X$.

(a) $\mathcal{P}_x$ is a network at $x$.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* for $X$, if for $G \subset X$, $G$ is open in $X$ iff for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

(2) $\mathcal{P}$ is an *sn-network* (resp., *so-network*) for $X$, if every element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ (resp., sequentially open in $X$) for every $x \in X$.

DEFINITION 1.2. Let $X$ be a space and $\mathcal{P}$ be a cover of $X$.

(1) $\mathcal{P}$ is a *Lindelöf* (resp., *compact*) cover, if each element of $\mathcal{P}$ is Lindelöf (resp., compact).

(2) $X$ is an $\aleph_0$-*space*, if $X$ is a regular space with a countable cs$^*$-network.

(3) $X$ is an $H$-$\aleph_0$-*space*, if $X$ has a countable cs$^*$-network.

DEFINITION 1.3. Let $f : X \to Y$ be a map.

(1) $f$ is *weak-open* [**27**], if there exists a weak base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for $Y$, and for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each open neighborhood $U$ of $x$, $B \subset f(U)$ for some $B \in \mathcal{B}_y$.

(2) $f$ is 1-*sequence-covering* [**17**], if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that each sequence converging to $y$ is an image of some sequence converging to $x$.

(3) $f$ is 2-*sequence-covering* [**17**], if for every $y \in Y$, $x_y \in f^{-1}(y)$, and sequence $\{y_n\}$ converging to $y$ in $Y$, there exists a sequence $\{x_n\}$ converging to $x_y$ in $X$ with each $x_n \in f^{-1}(y_n)$.

(4) $f$ is an *msss-map* (resp., *mssc-map*) [**15**], if $X$ is a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each $y \in Y$,

there is a sequence $\{V_i : i \in \mathbb{N}\}$ of open neighborhood's of $y$ such that each $p_i f^{-1}(V_i)$ is separable in $X_i$ (resp., each $\mathrm{cl}(p_i f^{-1}(V_i))$ is compact in $X_i$).

Definition 1.4. For a cover $\mathcal{P}$ of a space $X$, let $(P)$ be a (certain) covering-property of $\mathcal{P}$. Let us say that $\mathcal{P}$ has *property* $\sigma$-$(P)$, if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ having the property $(P)$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$.

For some undefined or related concepts, we refer the reader to [**18**].

## 2. Main results

From now on, let us restrict the properties $(P)$ and $\alpha(P)$ to the following.

(1) $(P)$ are locally finite, locally countable.
(2) $\alpha(P)$ is mssc if $(P)$ is locally finite, and $\alpha(P)$ is msss if $(P)$ is locally countable.

Notation 2.1. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a Lindelöf network having property $\sigma$-$(P)$ for a space $X$. For each $n \in \mathbb{N}$, we put $\mathcal{P}_n^* = \{X\} \cup \mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow $\Lambda_n$ with the discrete topology. Assume that for each $x \in X$, there exists a network $\{P_{\alpha_n} : n \in \mathbb{N}\}$ at $x$ with $\alpha_n \in \Lambda_n$. Then,

$$M = \left\{\alpha = (\alpha_n) \in \prod\nolimits_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\right\}$$

is a metric space and the point $x_\alpha$ is unique in $X$ for every $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, \mathcal{P}_n^*)$ an *L-Ponomarev system*.

Remark 2.1. (1) Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a Lindelöf network of $X$, where each $\mathcal{P}_n$ having property $(P)$. Then, $\mathcal{P}$ is a Lindelöf network has property $\sigma$-$(P)$.

(2) If $(f, M, X, \mathcal{P}_n^*)$ an $L$-Ponomarev system, then $f$ is an $s$-map.

Lemma 2.1. *If $\mathcal{P}$ is a cs-network having property $\sigma$-$(P)$, then $\mathcal{P}$ is a cfp-network.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a cs-network having property $\sigma$-$(P)$ for $X$, and $K \subset V$ with $K$ is compact and $V$ is open in $X$. Since $\mathcal{P}$ is a cs-network having property $\sigma$-$(P)$, $K$ has a countable cs-network. Thus, $K$ is metrizable. By [**19**, Lemma 1.2], for each $x \in K$, there exists $P_x \in \mathcal{P}$ such that $x \in \mathrm{int}_K(P_x \cap K) \subset P_x \subset V$. By the regularity of $K$, for each $x \in K$, there exists an open neighborhood $V_x$ in $K$ such that $x \in V_x \subset \mathrm{cl}_K(V_x) \subset \mathrm{int}_K(P_x \cap K)$. Since $K$ is compact, there exists a finite subset $F$ of $K$ such that $K \subset \bigcup_{x\in F} V_x$. Thus, $\{P_x : x \in F\}$ is a cfp-cover of $K$ and $\bigcup_{x\in F} P_x \subset U$. Therefore, $\mathcal{P}$ is a cfp-network. □

Lemma 2.2. *If $X$ has a Lindelöf cs$^*$-network with property $\sigma$-$(P)$, then $X$ has a Lindelöf cs-network with property $\sigma$-$(P)$.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ be a Lindelöf cs$^*$-network having property $\sigma$-$(P)$ for $X$. Since each element of $\mathcal{P}_i$ is Lindelöf, each $\mathcal{P}_i$ is star-countable. It follows from [**22**, Lemma 2.1] that for each $i \in \mathbb{N}$, $\mathcal{P}_i = \bigcup\{\mathcal{Q}_\alpha^{(i)} : \alpha \in \Lambda_i\}$, where

$\mathcal{Q}^{(i)}_\alpha$ is a countable subfamily of $\mathcal{P}_i$ for all $\alpha \in \Lambda_i$ and $\big(\bigcup \mathcal{Q}^{(i)}_\alpha\big) \cap \big(\bigcup \mathcal{Q}^{(i)}_\beta\big) = \emptyset$ for all $\alpha \neq \beta$. For each $i \in \mathbb{N}$ and $\alpha \in \Lambda_i$, we put

$$\mathcal{R}^{(i)}_\alpha = \Big\{\bigcup \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{Q}^{(i)}_\alpha\Big\}.$$

Since each $\mathcal{R}^{(i)}_\alpha$ is countable, we can write $\mathcal{R}^{(i)}_\alpha = \{R^{(i)}_{\alpha,j} : j \in \mathbb{N}\}$. Now, for each $i, j \in \mathbb{N}$, put $\mathcal{F}^{(i)}_j = \{R^{(i)}_{\alpha,j} : \alpha \in \Lambda_i\}$, and denote $\mathcal{G} = \bigcup\{\mathcal{F}^{(i)}_j : i, j \in \mathbb{N}\}$. Then, each $R^{(i)}_{\alpha,j}$ is Lindelöf and each family $\mathcal{F}^{(i)}_j$ has property $(P)$. Now, we shall show that $\mathcal{G}$ is a cs-network. In fact, let $\{x_n\}$ be a sequence converging to $x \in U$ with $U$ is open in $X$. Since $\mathcal{P}$ is a point-countable cs$^*$-network, it follows from [**25**, Lemma 3] that there exists a finite family $\mathcal{A} \subset (\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{A} \subset U$. Furthermore, since $\mathcal{A}$ is finite and $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for all $i \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $\mathcal{A} \subset \mathcal{P}_i$. So, there exists unique $\alpha \in \Lambda_i$ such that $\mathcal{A} \subset \mathcal{Q}^{(i)}_\alpha$, and $\bigcup \mathcal{A} \in \mathcal{R}^{(i)}_\alpha$. Thus, $\bigcup \mathcal{A} = R^{(i)}_{\alpha,j}$ for some $j \in \mathbb{N}$. Hence, $\bigcup \mathcal{A} \in \mathcal{G}$, and $\mathcal{G}$ is a cs-network. It follows from Remark 2.1(1) $\mathcal{G}$ is a Lindelöf cs-network having property $\sigma$-$(P)$. $\square$

LEMMA 2.3. *Let $f : M \to X$ be a $\alpha(P)$-map, and $M$ be a locally separable metric space. Then,*

(1) *$X$ has a Lindelöf cs$^*$-network with property $\sigma$-$(P)$, if $f$ is sequentially-quotient.*
(2) *$X$ has a Lindelöf sn-network with property $\sigma$-$(P)$, if $f$ is* 1*-sequence-covering.*
(3) *$X$ has a Lindelöf so-network with property $\sigma$-$(P)$, if $f$ is* 2*-sequence-covering.*

PROOF. By [**15**, Lemma 1.2] and by the proof of (3) $\Rightarrow$ (1) in [**12**, Theorem 4], there exists a base $\mathcal{B}$ of $M$ such that $\mathcal{F} = f(\mathcal{B})$ is a network for $X$, and $\mathcal{F}$ can be expressed as $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$, where each $\mathcal{F}_n$ has property $(P)$. Since $M$ is locally separable, for each $a \in M$, there exists a separable open neighborhood $U_a$. Denote

$$\mathcal{C} = \big\{B \in \mathcal{B} : B \subset U_a, a \in M\big\}.$$

Then, $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C}$ is a separable base for $M$. If put $\mathcal{P} = f(\mathcal{C})$, then $\mathcal{P} \subset \mathcal{F}$, and it follows from Remark 2.1(1) that $\mathcal{P}$ is a Lindelöf network having property $\sigma$-$(P)$. Thus, $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ having the property $(P)$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Furthermore, we have

(1) If $f$ is sequentially-quotient, then since $\mathcal{C}$ is a base for $M$, $\mathcal{P}$ is a cs$^*$-network. Therefore, $X$ has a Lindelöf cs$^*$-network with property $\sigma$-$(P)$.

(2) If $f$ is 1-sequence-covering, then for each $x \in X$, there exists $a_x \in f^{-1}(x)$ such that each sequence converging to $x$ is an image of a sequence converging to $a_x$. Now, for each $x \in X$, we put $\mathcal{G}_x = \{f(B) : a_x \in B \in \mathcal{C}\}$, $\mathcal{G} = \bigcup\{\mathcal{G}_x : x \in X\}$. Then, $\mathcal{G} \subset \mathcal{P}$ and $\mathcal{G}$ is an sn-network. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \mathcal{G} \cap \mathcal{P}_n$. Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a Lindelöf sn-network having property $\sigma$-$(P)$ for $X$.

(3) If $f$ is 2-sequence-covering, then for each $x \in X$, we put

$$\mathcal{C}_x = \big\{B \in \mathcal{C} : B \cap f^{-1}(x) \neq \emptyset\big\},$$

and let $\mathcal{G}_x$ be the family of all finite intersections of members of $f(\mathcal{C}_x)$, and $\mathcal{G} = \bigcup\{\mathcal{G}_x : x \in X\}$. Then, $\mathcal{G} \subset \mathcal{P}$ and $\mathcal{G}$ is an so-network. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \mathcal{G} \cap \mathcal{P}_n$. Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a Lindelöf so-network having property $\sigma$-$(P)$ for $X$. □

Lemma 2.4. *Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a Lindelöf network having property $\sigma$-$(P)$ and $(f, M, X, \mathcal{P}_n^*)$ be an L-Ponomarev system. Then, the following statements hold.*

(1) *$f$ is a $\alpha(P)$-map.*
(2) *$M$ is locally separable.*
(3) *$f$ is sequence-covering compact-covering, if $\mathcal{P}$ is a cs-network.*
(4) *$f$ is 1-sequence-covering compact-covering, if $\mathcal{P}$ is an sn-network.*
(5) *$f$ is 2-sequence-covering compact-covering, if $\mathcal{P}$ is an so-network.*

Proof. (1) Similar to the proof of [**12**, Theorem 4] and [**14**, Theorem 2.1].

(2) Let $a = (\alpha_i) \in M$. Then, $\{P_{\alpha_i}\}$ is a network at some point $x_a \in X$. Thus, there exists $i_0 \in \mathbb{N}$ such that $P_{\alpha_{i_0}}$ is Lindelöf. Put

$$U_a = M \cap \Big\{(\beta_i) \in \prod\nolimits_{i\in\mathbb{N}} \Lambda_i : \beta_i = \alpha_i, i \leqslant i_0\Big\}.$$

Then, $U_a$ is an open neighborhood of $a$ in $M$. Now, for each $i \leqslant i_0$, put $\Delta_i = \{\alpha_i\}$, and for each $i > i_0$, we put $\Delta_i = \{\alpha \in \Lambda_i : P_\alpha \cap P_{\alpha_{i_0}} \neq \emptyset\}$. Then, $U_a \subset \prod_{i\in\mathbb{N}} \Delta_i$. Furthermore, since each $\mathcal{P}_i$ having property $(P)$ and $P_{\alpha_{i_0}}$ is Lindelöf, $\Delta_i$ is countable for every $i > i_0$. Thus, $U_a$ is separable, and $M$ is locally separable.

(3) Let $\mathcal{P}$ be a cs-network. Then,

(3.1) $f$ is sequence-covering. Let $S = \{x_n : n \in \mathbb{N}\}$ be a sequence converging to $x$ in $X$. Since $\mathcal{P}$ is a point-countable cs-network, we can write

$$\big\{P \in \mathcal{P} : S \text{ is eventually in } P\big\} = \{P_i : i \in \mathbb{N}\}.$$

On the other hand, since $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for all $i \in \mathbb{N}$, we can choose sequence $\{i_n\} \subset \mathbb{N}$ such that $i_n < i_{n+1}$, and $P_n \in \mathcal{P}_{i_n}$ for every $n \in \mathbb{N}$. Now, for each $j \in \mathbb{N}$, we take

$$F_{\alpha_j} = \begin{cases} P_n, & \text{if } j = i_n, \\ X, & \text{if } j \neq i_n, \end{cases}$$

and $a = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i$. Then $f(a) = x$ and $S$ is eventually in each $F_{\alpha_i}$. Now, for each $n \in \mathbb{N}$, put $B_n = \big\{(\gamma_i) \in M : \gamma_i = \alpha_i \text{ for each } i \leqslant n\big\}$. It is easy to check that $\{B_n\}$ is a decreasing neighborhood base at $a$ in $M$ and $f(B_n) = \bigcap_{i\leqslant n} P_{\alpha_i}$ for all $n \in \mathbb{N}$. Because $S$ is eventually in each $f(B_n)$, it follows from [**8**, Lemma 6] that for each $n \in \mathbb{N}$, there exists $a_n \in f^{-1}(x_n)$ such that the sequence $\{a_n\}$ converging to $a$ in $M$. Therefore, $f$ is sequence-covering.

(3.2) $f$ is compact-covering. Let $K$ be a compact subset of $X$. Since $\mathcal{P}$ is a Lindelöf cs-network having property $\sigma$-$(P)$, it follows from Lemma 2.1 that $\mathcal{P}$ is a cfp-network for $X$. Furthermore, since $\mathcal{P}_K$ is countable, we can put

$$\big\{\mathcal{Q} \subset \mathcal{P}_K : \mathcal{Q} \text{ is a finite cfp-cover of } K\big\} = \{\mathcal{Q}_i : i \in \mathbb{N}\}.$$

Since $\mathcal{Q}_n \subset \mathcal{P}$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, then we can choose a sequence $\{i_n\} \subset \mathbb{N}$ such that $i_n < i_{n+1}$, and $\mathcal{Q}_n \subset \mathcal{P}_{i_n}$ for every $n \in \mathbb{N}$. Now, we choose a sequence $\{\mathcal{A}_i\}$ as follows

$$\mathcal{A}_j = \begin{cases} \mathcal{Q}_n, & \text{if } j = i_n, \\ \{X\}, & \text{if } j \neq i_n. \end{cases}$$

Since each $\mathcal{A}_i$ is a cfp-cover for $K$, there exists a finite subfamily $\mathcal{H}_i = \{P_\alpha\}_{\alpha \in \Gamma_i}$ of $\mathcal{A}_i$ and a cover $\{F_\alpha\}_{\alpha \in \Gamma_i}$ of $K$ consisting of closed subset of $K$ satisfying that each $F_\alpha \subset P_\alpha$. Put $L = \big\{a = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset\big\}$. Then, we have

(3.2.1) $L \subset M$, and $f(L) \subset K$. Suppose $a = (\alpha_i) \in L$, then $\bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset$. Pick $x_a \in \bigcap_{i \in \mathbb{N}} F_{\alpha_i}$. Now we will show that $\{P_{\alpha_i}\}$ is a network at $x_a$ in $X$. Then, $a \in M$ and $f(a) = x_a \in K$, so $L \subset M$ and $f(L) \subset K$. Indeed, let $V$ be a neighborhood of $x_a$ in $X$. Since $K$ is a regular subspace of $X$, there exists an open neighborhood $W$ of $x_a$ in $K$ such that $\mathrm{cl}_K(W) \subset V$. Since $\mathrm{cl}_K(W)$ is a compact subset of $K$, there exists a finite collection $\mathcal{Q}'$ of $\mathcal{P}_K$ such that $\mathcal{Q}'$ is a cfp-cover of $\mathrm{cl}_K(W)$ and $\bigcup \mathcal{Q}' \subset V$. On the other hand, since $K - W$ is a compact subset of $K$ satisfying $K - W \subset X - \{x_a\}$, there exists a finite collection $\mathcal{Q}''$ of $\mathcal{P}_K$ such that $\mathcal{Q}''$ is a cfp-cover for $K - W$ and $\bigcup \mathcal{Q}'' \subset X - \{x_a\}$. Put $\mathcal{Q} = \mathcal{Q}' \cup \mathcal{Q}''$. Then, $\mathcal{Q}$ is a cfp-cover for $K$, and so $\mathcal{Q} = \mathcal{Q}_k$ for some $k \in \mathbb{N}$. But $x_a \in F_{\alpha_k} \subset P_{\alpha_k} \in \mathcal{Q}_k$, thus $P_{\alpha_k} \in \mathcal{Q}'$ and $P_{\alpha_k} \subset V$. Hence, $\{P_{\alpha_i}\}$ is a network at $x_a$ in $X$.

(3.2.2) $K \subset f(L)$. Assume that $x \in K$. For each $i \in \mathbb{N}$, pick $\alpha_i \in \Gamma_i$ such that $x \in F_{\alpha_i}$. Put $a = (\alpha_i)$, it follows that $a \in L$. By the proof of (3.2.1), $f(a) = x$. So, $K \subset f(L)$.

(3.2.3) $L$ is compact. Because each $\Gamma_i$ is finite, $\prod_{i \in \mathbb{N}} \Gamma_i$ is compact. Note that $L \subset \prod_{i \in \mathbb{N}} \Gamma_i$, we only need to prove that $L$ is closed in $\prod_{i \in \mathbb{N}} \Gamma_i$. In fact, let $a = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i - L$. Then, $\bigcap_{i \in \mathbb{N}} F_{\alpha_i} = \emptyset$. From the compactness of $K$, there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i \leqslant i_0} F_{\alpha_i} = \emptyset$. Put $W = \{(\beta_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \beta_i = \alpha_i \text{ for each } i \leqslant i_0\}$. Then, $W$ is an open subset of $\prod_{i \in \mathbb{N}} \Gamma_i$ satisfying $a \in W$ and $W \cap L = \emptyset$. This implies that $L$ is a closed subset of $\prod_{i \in \mathbb{N}} \Gamma_i$. Therefore, $L$ is a compact subset of $M$.

(4) Let $\mathcal{P}$ be an sn-network. Then, $X$ is sn-first countable. Since every sn-network is cs-network, it follows from (3) that $f$ is a sequence-covering, compact-covering map. By Remark 2.1(2) and [**1**, Proposition 2.2(1)], $f$ is 1-sequence-covering.

(5) Let $\mathcal{P}$ be an so-network. Since each so-network is a cs-network, by (3), it suffices to prove that $f$ is 2-sequence-covering.

Let $x \in X$ and $a = (\alpha_i) \in f^{-1}(x)$. It is obvious that each $P_{\alpha_i}$ is a sequential neighborhood of $x$ in $X$. For each $n \in \mathbb{N}$, put $B_n = \big\{(\gamma_i) \in M : \gamma_i = \alpha_i \text{ for each } i \leqslant n\big\}$. Then, $\{B_n\}$ is a decreasing neighborhood base of $a$ in $M$, and $f(B_n) = \bigcap_{i \leqslant n} P_{\alpha_i}$ for all $n \in \mathbb{N}$. Now, let $\{x_n\}$ be a sequence converging to $x$ in $X$. Since each $f(B_n)$ is a sequential neighborhood at $x$ in $X$, it follows from [**10**, Lemma 3.2] that for each $n \in \mathbb{N}$, there exists $a_n \in f^{-1}(x_n)$ such that the sequence $\{a_n\}$ converging to $a$ in $M$. Therefore, $f$ is 2-sequence-covering. $\square$

THEOREM 2.1. *The following are equivalent for a space $X$.*

(1) *$X$ has a Lindelöf cs$^*$-network with property $\sigma$-$(P)$;*
(2) *$X$ has a Lindelöf cfp-network with property $\sigma$-$(P)$;*
(3) *$X$ has a Lindelöf cs-network with property $\sigma$-$(P)$;*
(4) *$X$ is a sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;*
(5) *$X$ is a sequentially-quotient $\alpha(P)$-image of a locally separable metric space;*
(6) *$X$ is a sequentially-quotient $\alpha(P)$-image of a metric space, and has an so-cover consisting of $H$-$\aleph_0$-subspaces.*

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By Lemma 2.1 and Lemma 2.2.

(3) $\Rightarrow$ (4). By Lemma 2.4.

(4) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (6). Assume that (5) holds. It suffices to prove that $X$ has an so-cover consisting of $H$-$\aleph_0$-subspaces. In fact, by Lemma 2.3(1) and Lemma 2.2, $X$ has a Lindelöf cs-network $\mathcal{P}$ having property $\sigma$-$(P)$. Then, each element of $\mathcal{P}$ is an $H$-$\aleph_0$-subspace. By the proof of (2) $\Rightarrow$ (3) in [**20**, Theorem 3.4], $X$ has an so-cover consisting of $H$-$\aleph_0$-subspaces.

(6) $\Rightarrow$ (1). Let $\mathcal{O}$ be an so-cover consisting of $H$-$\aleph_0$-subspaces of $X$ and $f : M \to X$ be a sequentially-quotient $\alpha(P)$-map, where $M$ is a metric space. Similar to the proof of Lemma 2.3, there exists a base $\mathcal{B}$ of $M$ such that $\mathcal{P} = f(\mathcal{B})$ having property $\sigma$-$(P)$. Since $f$ is sequentially-quotient, $\mathcal{P}$ is a cs$^*$-network for $X$. We can assume that $\mathcal{P}$ is closed under finite intersections. Let $\mathcal{G} = \{P \in \mathcal{P} : P \subset O, O \in \mathcal{O}\}$. Then, each element of $\mathcal{G}$ is an $H$-$\aleph_0$-subspace. Hence, each element of $\mathcal{G}$ is Lindelöf. Now, we proved that $\mathcal{G}$ is a cs$^*$-network. In fact, let $L$ be a sequence converging to $x \in U$ with $U$ open in $X$. Since $\mathcal{O}$ is an so-cover for $X$, there exists $O \in \mathcal{O}$ such that $x \in O$. On the other hand, since $\mathcal{P}$ is a point-countable cs$^*$-network, it follows from [**25**, Lemma 3] that there exists a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $L$ is eventually in $\bigcup \mathcal{H} \subset U$. So, the family $\big\{\mathcal{H} \subset (\mathcal{P})_x : \mathcal{H}$ is finite and $L$ is eventually in $\bigcup \mathcal{H} \subset U\big\}$ is non-empty. Furthermore, since $(\mathcal{P})_x$ is countable, we can write $\big\{\mathcal{H} \subset (\mathcal{P})_x : \mathcal{H}$ is finite and $L$ is eventually in $\bigcup \mathcal{H} \subset U\big\} = \{\mathcal{H}_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $H_n = \bigcap_{i \leqslant n}(\bigcup \mathcal{H}_i)$. It is obvious that $L$ is eventually in each $H_n$. Now, we shall show that $H_n \subset O$ for some $n \in \mathbb{N}$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in H_n - O$. Then, $\{x_i\}$ converges to $x$. Indeed, let $x \in W$ with $W$ is open in $X$. Then, $U \cap W$ is an open neighborhood of $x$. By [**25**, Lemma 3], there exists a finite subfamily $\mathcal{Q} \subset (\mathcal{P})_x$ such that $L$ is eventually in $\bigcup \mathcal{Q}$ and $\bigcup \mathcal{Q} \subset U \cap W$. Since $\mathcal{Q}$ is a finite subfamily of $(\mathcal{P})_x$ and $L$ is eventually in $\bigcup \mathcal{Q} \subset U$, $\mathcal{Q} = \mathcal{H}_n$ for some $n \in \mathbb{N}$. Furthermore, since $x_i \in H_i$ for all $i \in \mathbb{N}$ and

$$H_i = \bigcap_{j \leqslant i} \Big(\bigcup \mathcal{H}_j\Big) \subset \bigcap_{j \leqslant n} \Big(\bigcup \mathcal{H}_j\Big) \subset \bigcup \mathcal{H}_n \subset W,$$

for all $i \geqslant n$, we get $x_i \in W$ for all $i \geqslant n$. Therefore, $\{x_i\}$ converges to $x$. Since $O$ is a sequential neighborhood of $x$, this implies that there exists $n \in \mathbb{N}$ such that $x_i \in O$ for all $i \geqslant n$. This is a contradiction to $x_i \notin O$ for all $i \in \mathbb{N}$. Thus, $H_n \subset O$ for some $n \in \mathbb{N}$.

On the other hand, since $H_n = \bigcap_{i \leqslant n} \left(\bigcup \mathcal{H}_i\right) = \bigcup \left\{ \bigcap_{i \leqslant n} F_i : F_i \in \mathcal{H}_i \right\}$, and $L$ is eventually in $H_n$, it implies that for each $i \leqslant n$, there exists $F_i \in \mathcal{H}_i$ such that $L$ is frequently in $F = \bigcap_{i \leqslant n} F_i$. Since $\mathcal{P}$ is closed under finite intersections, $F \in \mathcal{P}$. Then, $L$ is frequently in $F$, $F \subset U$ and $F \in \mathcal{G}$. Thus, $\mathcal{G}$ is a cs$^*$-network for $X$. By Remark 2.1(1), $\mathcal{G}$ is a Lindelöf cs$^*$-network having property $\sigma$-$(P)$. □

Remark 2.2. By Theorem 2.1, in case that the property $(P)$ is locally countable, we get an affirmative answer to Question 2.17 of [**3**].

By Theorem 2.1, the following corollary holds.

Corollary 2.1. *The following are equivalent for a space $X$.*

(1) *$X$ is a $k$-space with a Lindelöf cs$^*$-network having property $\sigma$-$(P)$;*
(2) *$X$ is a $k$-space with a Lindelöf cfp-network having property $\sigma$-$(P)$;*
(3) *$X$ is a $k$-space with a Lindelöf cs-network having property $\sigma$-$(P)$;*
(4) *$X$ is a sequence-covering, compact-covering, quotient $\alpha(P)$-image of a locally separable metric space;*
(5) *$X$ is a quotient $\alpha(P)$-image of a locally separable metric space;*
(6) *$X$ is a local $H$-$\aleph_0$-space and a quotient $\alpha(P)$-image of a metric space.*

Remark 2.3. By Corollary 2.1, we get an affirmative answer to the Question 1.2.

Remark 2.4. Let $\mathcal{P}$ be a network having property $\sigma$-$(P)$ for a regular space $X$. Then,

(1) If $\mathcal{P}$ is a cs$^*$-network (cfp-network; cs-network), then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is a $\aleph_0$-subspace.
(2) If $\mathcal{P}$ is an sn-network, then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is an sn-second countable subspace.
(3) If $\mathcal{P}$ is an so-network, then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is an so-second countable subspace.

By Theorem 2.1 and Remark 2.4, we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

Corollary 2.2. [**3**, Theorem 2.8], *The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally countable cs-network consisting of $\aleph_0$-subspaces;*
(2) *$X$ has a $\sigma$-locally countable cs-network consisting of cosmic subspaces;*
(3) *$X$ is a sequence-covering msss-image of a locally separable metric space.*

Corollary 2.3. [**4**, Theorem 2.1], *The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally finite cs-network consisting of $\aleph_0$-subspaces;*
(2) *$X$ has a $\sigma$-locally finite cs-network consisting of cosmic subspaces;*
(3) *$X$ is a sequence-covering mssc-image of a locally separable metric space.*

The following results hold by means of the above results.

THEOREM 2.2. *The following are equivalent for a space $X$.*

(1) *$X$ has a Lindelöf sn-network with property $\sigma$-$(P)$;*
(2) *$X$ is a 1-sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;*
(3) *$X$ is a 1-sequence-covering $\alpha(P)$-image of a locally separable metric space;*
(4) *$X$ is a 1-sequence-covering $\alpha(P)$-image of a metric, and has an so-cover consisting of $H$-$\aleph_0$-subspaces.*

COROLLARY 2.4. *The following are equivalent for a space $X$.*

(1) *$X$ has a Lindelöf weak base with property $\sigma$-$(P)$;*
(2) *$X$ is a weak-open, compact-covering $\alpha(P)$-image of a locally separable metric space;*
(3) *$X$ is a weak-open $\alpha(P)$-image of a locally separable metric space;*
(4) *$X$ is a local $H$-$\aleph_0$-space and a weak-open $\alpha(P)$-image of a metric.*

By Theorem 2.2 and Remark 2.4, we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

COROLLARY 2.5. [**3**, Theorem 2.11] *The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally countable sn-network consisting of sn-second countable subspaces;*
(2) *$X$ has a $\sigma$-locally countable sn-network consisting of cosmic subspaces;*
(3) *$X$ is a 1-sequence-covering msss-image of a locally separable metric space.*

COROLLARY 2.6. [**4**, Theorem 2.2] *The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally finite sn-network consisting of sn-second countable subspaces;*
(2) *$X$ has a $\sigma$-locally finite sn-network consisting of cosmic subspaces;*
(3) *$X$ is a 1-sequence-covering mssc-image of a locally separable metric space.*

REMARK 2.5. By Theorem 2.2, it is possible to add the prefix "compact-covering" before "1-sequence-covering" in Corollary 2.5(3) and Corollary 2.6(3).

THEOREM 2.3. *The following are equivalent for a space $X$.*

(1) *$X$ has a Lindelöf so-network with property $\sigma$-$(P)$;*
(2) *$X$ is a 2-sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;*
(3) *$X$ is a 2-sequence-covering $\alpha(P)$-image of a locally separable metric space;*
(4) *$X$ is a 2-sequence-covering $\alpha(P)$-image of a metric, and has an so-cover consisting of $H$-$\aleph_0$-subspaces.*

COROLLARY 2.7. *The following are equivalent for a space $X$.*

(1) *$X$ has a Lindelöf base with property $\sigma$-$(P)$;*
(2) *$X$ is an open, compact-covering $\alpha(P)$-image of a locally separable metric space;*

(3) *$X$ is an open $\alpha(P)$-image of a locally separable metric space;*
(4) *$X$ is a local $H$-$\aleph_0$-space and an open $\alpha(P)$-image of a metric.*

By Theorem 2.3 and Remark 2.4, we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

Corollary 2.8. [**3**, Theorem 2.14] *The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally countable so-network consisting of so-second countable subspaces;*
(2) *$X$ has a $\sigma$-locally countable so-network consisting of cosmic subspaces;*
(3) *$X$ is a 2-sequence-covering msss-image of a locally separable metric space.*

Corollary 2.9. [**4**, Theorem 2.3]*, The following are equivalent for a regular space $X$.*

(1) *$X$ has a $\sigma$-locally finite so-network consisting of so-second countable subspaces;*
(2) *$X$ has a $\sigma$-locally finite so-network consisting of cosmic subspaces;*
(3) *$X$ is a 2-sequence-covering mssc-image of a locally separable metric space.*

Remark 2.6. By Theorem 2.3, it is possible to add the prefix "compact-covering" before "2-sequence-covering" in Corollary 2.8(3) and Corollary 2.9(3).

## 3. Examples

Example 3.1. A quotient $s$-image of a locally separable metric space need not be locally separable (see [**11**, Example 9.8] or [**16**, Example 2.9.27]). Then, Question 1.1 is not true in the case $\Phi$-property is an $\aleph_0$-space (or locally separable).

Example 3.2. There exists a space $X$ with a $\sigma$-locally finite compact $k$-network (hence, $X$ has a $\sigma$-locally finite Lindelöf cs-network by Theorem 2.1), but $X$ is not locally Lindelöf (hence, $X$ has no locally countable network) (see [**24**, Example 4.1(2)]). Then,

(1) A space $X$ has a Lindelöf cs-network with property $\sigma$-$(P)$ need not have a locally countable cs-network.
(2) In Theorem 2.1(6), $X$ need not be local $\aleph_0$-space.

Example 3.3. $S_\omega$ is a Fréchet and $\aleph_0$-space, but it is not first countable. Then, it has a $\sigma$-locally finite Lindelöf cs-network. Since $S_\omega$ is not first countable, it doesn't have a $\sigma$-locally countable sn-network (or weak base).

(1) A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf cs-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) Lindelöf sn-network.
(2) A $k$-space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf cs-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) Lindelöf weak base.

EXAMPLE 3.4. There exists a $g$-second countable space $X$, but it is not Fréchet (see, [**23**, Example 2.1]). Then, $X$ has a $\sigma$-locally finite Lindelöf weak base. Since $X$ is sequential and it is not Fréchet, $X$ does not have a $\sigma$-locally countable so-network (or weak base). Therefore,

(1) A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf sn-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) so-network.
(2) A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf weak base need not have a $\sigma$-locally finite (or $\sigma$-locally countable) base.

EXAMPLE 3.5. There exists a space $X$ having a locally countable sn-network, which is not an $\aleph$-space (see [**5**, Example 2.19]). Then, $X$ has a $\sigma$-locally countable Lindelöf sn-network. Therefore,

(1) A space with a locally countable sn-network need not have a $\sigma$-locally finite Lindelöf cs-network.
(2) A space with a $\sigma$-locally countable Lindelöf sn-network need not have a $\sigma$-locally finite Lindelöf sn-network (or cs-network).
(3) A space with a $\sigma$-locally countable Lindelöf cs-network need not have a $\sigma$-locally finite Lindelöf cs-network.

EXAMPLE 3.6. Using [**7**, Example 3.1], it is easy to see that $X$ is Hausdorff, non-regular and $X$ has a countable base, but it is not a sequentially-quotient $\pi$-image of a metric space. Then, $X$ is not an $\aleph_0$-space. By Theorem 2.3, $X$ is a 2-sequence-covering (and open) mssc-image of a locally separable metric space.

(1) There exists an $H$-$\aleph_0$-space, but it is not an $\aleph_0$-space.
(2) A space with a $\sigma$-locally finite Lindelöf cs-network (or an sn-network, or an so-network) need not be a sequentially-quotient $\pi$, mssc-image (or msss-image) of a metric space.

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Department of Mathematics
Vinh University
Vinh City
Vietnam
andhv@yahoo.com

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Department of Mathematics
Da Nang University
Danang City
Vietnam
luongtuyench12@yahoo.com